A NONTOPOLOGICAL 1-1 MAPPING ONTO E^3

BY KENNETH WHYBURN

Communicated by L. Zippin, November 8, 1964

It is well known that any 1-1 mapping from one compact Hausdorff space onto another is a homeomorphism. It is also an easy consequence of the Brouwer theorem on invariance of domain that any 1-1 mapping from one Euclidean space onto another is topological. Recently considerable interest has been expressed by several authors in the degree to which the rather stringent requirement that the domain space be Euclidean can be relaxed in the latter case. This paper gives an example which strongly indicates that very little if any relaxation can be allowed.

We will exhibit a 1-1 mapping which is not a homeomorphism from a connected, locally connected and locally compact metric space onto an open polyhedral 3-cell embedded in E^3. This is contrary to a result supposedly proven by V. V. Proizvolov[1] asserting that any 1-1 mapping from a connected paracompact space onto E^n is a homeomorphism. The domain space of this example is simply a polyhedral subset of E^3 on which the mapping is piecewise linear.

The domain space A of our mapping is an open cube with four vertical triangular columns rising out of the top, with their bases symmetrically about a square. The hypotenuse of each triangle covers half of one side of the square with one of its ends at a vertex of the square. The remainder of the triangle and its interior lie outside the square. Note that neither the square nor the line segments bounding the base triangles of the columns are in our space. In fact the only piece of the boundary that is in the space is one collar around each column starting part way up the column and extending up to but not including the top. The line segments bounding this collar are also not in the space (see Figure 1).

This space may be described analytically as

$$
\begin{aligned}
&\{-4 < z < 0,\ -2 < x < 2,\ -2 < y < 2\} \\
&\quad \cup \bigcup \big[\{t_2 > 1, t_2 - t_1 - 1 < 0, t_2 + t_1 - 2 < 0, 0 \leqq z \leqq 2^{1/2}\} \\
&\quad \cup \{t_2 \geqq 1,\ t_2 - t_1 - 1 \leqq 0,\ t_2 + t_1 - 2 \leqq 0,\ 2^{1/2} < z < 1 + 2^{1/2}\}\big], \\
&\qquad\qquad (t_1, t_2) \in \{(x, y), (-x, -y), (y, -x), (-y, x)\}.
\end{aligned}
$$

[1] Dokl. Akad. Nauk SSSR **151** (1963), 1286–1287; English transl., Soviet Math. Dokl. **4** (1963), 1194.

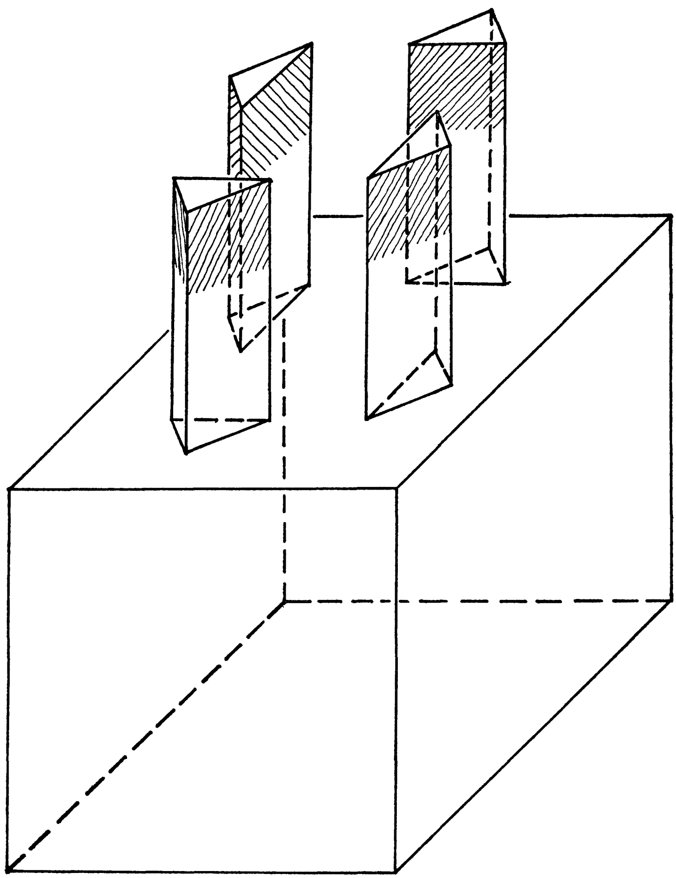

FIGURE 1. Range space.

The range space B is the same open cube as in A with a volcano sitting on the square together with the base triangles described in the domain space. This volcano has the bottom of its crater at the center of the square and has four peaks in its rim with passes between successive ones as you go around the crater. There is one peak over each quadrant of the square (see Figure 2).

The analytical description is

$$\begin{aligned}&\{-4 < z < 0,\ -2 < x < 2,\ -2 < y < 2\}\\ &\cup \bigcup \left[\{z - t_1 - t_2 + 1 < 0, z + t_1 + t_2 - 2 < 0, 0 < t_1 < 1, t_2 \geqq \tfrac{1}{2}, z \geqq 0\}\right.\\ &\cup \left.\{z - t_1 - t_2 < 0, z + t_1 - t_2 - 1 < 0, z \geqq 0, 0 < t_1 < 1, 0 \leqq t_2 < \tfrac{1}{2}\}\right],\\ &\qquad\qquad (t_1, t_2) \in \{(x, y), (-x, -y), (y, -x), (-y, x)\}.\end{aligned}$$

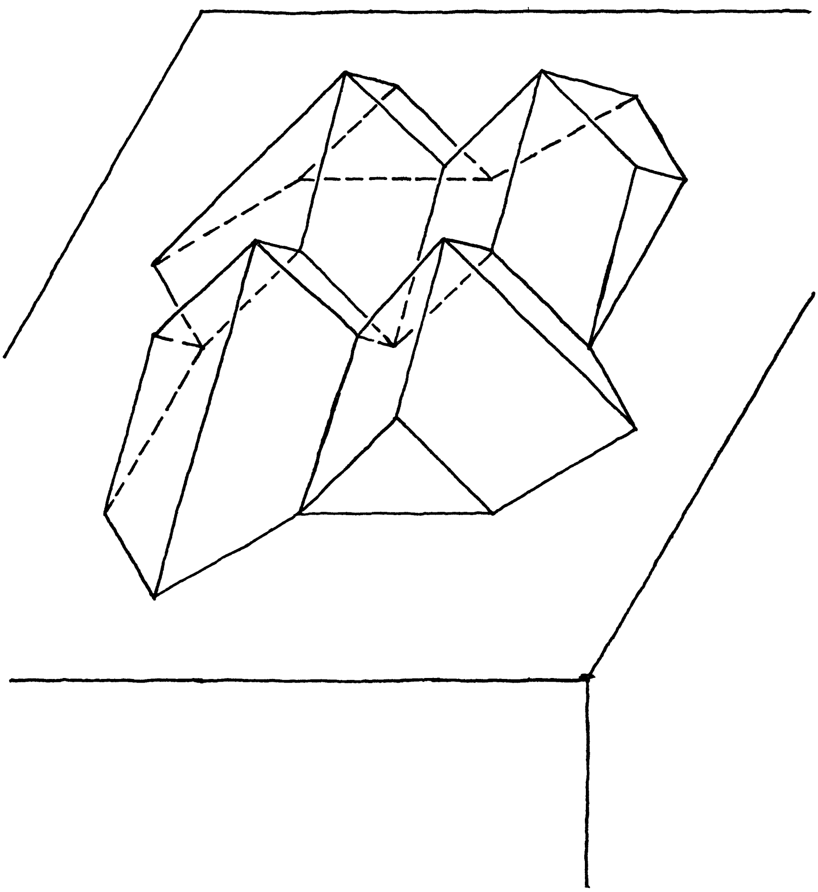

FIGURE 2. Domain space.

The mapping of our example is from the space A to the space B which is homeomorphic with E^3. It results in pushing each column through an angle of 45° with respect to the (x, y)-plane toward the coordinate plane to which one of its faces is parallel. Then it is folded down toward the (x, y)-plane through an angle of 90° along the image of $z=1/2^{1/2}$ and finally it is folded along the image of $z=2^{1/2}$ back up through an angle of 45° so that what was the face parallel to a coordinate plane in the domain is now the bottom face and for $z>2^{1/2}$ this bottom face of the collar lies in the (x, y)-plane. Thus each column forms an arch leaving out a solid wedge under it and then is reconnected to the block along one face of its collar. The wedge under the arch formed by each column is filled in by the collared section of the column which follows it by 90° (see Figure 3).

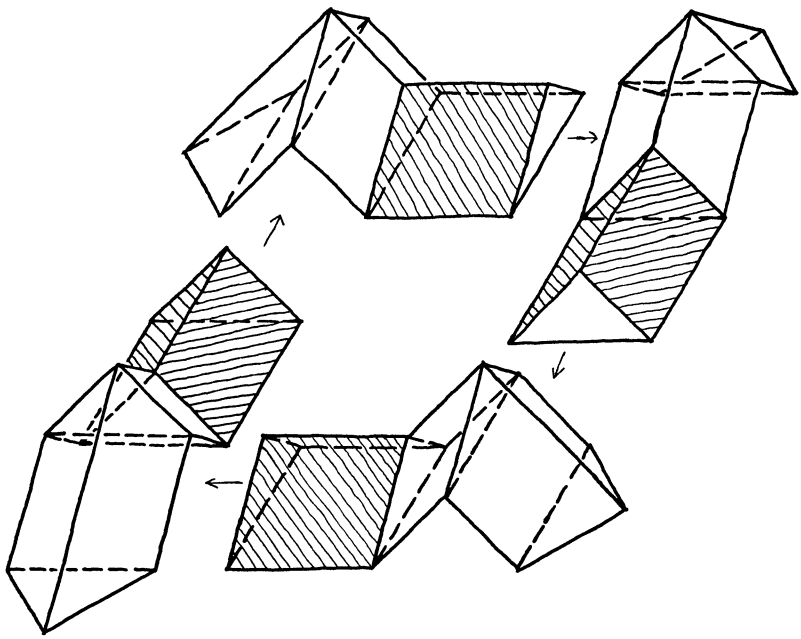

FIGURE 3. Mapping.

This map is given analytically

for $x > 0,\ y > 0$ by

$$(x, y, z) \to \left[x, y - \left(2^{1/2}y - \frac{2^{1/2}}{2}\right)z, \left(2^{1/2}y - \frac{2^{1/2}}{2}\right)z\right],$$

for $0 \leqq z \leqq 1/2^{1/2}$,

$$(x, y, z) \to \left[x, 1 - \frac{z}{2^{1/2}}, y - \frac{z}{2^{1/2}}\right],$$

for $1/2^{1/2} < z \leqq 2^{1/2}$, and

$$(x, y, z) \to (x, 2^{1/2} - z, y - 1)$$

for $2^{1/2} < z < 1 + 2^{1/2}$;

for $x, < 0,\ y > 0$ by

$$(x, y, z) \to \left[\left(-2^{1/2}x - \frac{2^{1/2}}{2}\right)z + x, y, \left(-2^{1/2}x - \frac{2^{1/2}}{2}\right)z\right],$$

for $0 \leqq z \leqq 1/2^{1/2}$,

$$(x, y, z) \to \left[\frac{z}{2^{1/2}} - 1, y, -x - z/2^{1/2}\right],$$

for $1/2^{1/2} < z \leqq 2^{1/2}$, and

$$(x, y, z) \to (z - 2^{1/2}, y, -x - 1)$$

for $2^{1/2} < z < 1 + 2^{1/2}$;

for $x < 0$, $y < 0$ by

$$(x, y, z) \to \left[x, y + \left(-2^{1/2}y - \frac{2^{1/2}}{2}\right)z, \left(-2^{1/2}y - \frac{2^{1/2}}{2}\right)z\right],$$

for $0 \leqq z \leqq 1/2^{1/2}$,

$$(x, y, z) \to \left[x, -\frac{z}{2^{1/2}} - 1, -y - \frac{z}{2^{1/2}}\right],$$

for $1/2^{1/2} < z \leqq 2^{1/2}$, and

$$(x, y, z) \to (x, z - 2^{1/2}, -y - 1),$$

for $2^{1/2} < z < 1 + 2^{1/2}$

for $x > 0$, $y < 0$ by

$$(x, y, z) \to \left[-\left(2^{1/2}x - \frac{2^{1/2}}{2}\right)z + x, y, \left(-2^{1/2}x - \frac{2^{1/2}}{2}\right)z\right],$$

for $0 \leqq z \leqq 1/2^{1/2}$

$$(x, y, z) \to \left[1 - \frac{z}{2^{1/2}}, y, x - \frac{z}{2^{1/2}}\right],$$

for $1/2^{1/2} < z \leqq 2^{1/2}$, and

$$(x, y, z) \to (2^{1/2} - z, y, x - 1)$$

for $2^{1/2} < z < 1 + 2^{1/2}$;

and for all x and y with $z < 0$,

$$(x, y, z) \to (x, y, z)$$

CORNELL UNIVERSITY