BOOK REVIEWS

Lectures on invariant subspaces. By Henry Helson. Academic Press, New York, 1964. 11+130 pp. $5.00

The problem of invariant subspaces is this: does every operator on a non-trivial Hilbert space have a non-trivial invariant subspace? (Explanations: "operator" means bounded linear transformation; "Hilbert space" means complete complex inner-product space; "subspace" means closed linear manifold; "non-trivial", for Hilbert spaces, means of dimension greater than 1; and "non-trivial", for subspaces, means distinct from both $\{0\}$ and the whole space.) The question is thought to be important by some mathematicians and interesting by most; it could be argued that an answer to it (whether yes or no) would be a large step toward a general structure theory for operators on Hilbert spaces. The chief value of the question, however, as of all clearly formulated, unsolved, yes-or-no questions in mathematics, is that of a catalyst and a touchstone. As a catalyst it has precipitated valuable related questions and answers; as a touchstone it has served to measure the extent to which those questions and answers have advanced the theory as a whole.

Helson's book is concerned with the problem of invariant subspaces, some of its special cases, some of its generalizations, and some of the techniques that have yielded partial answers. It is a timely book and will surely be a useful one; it is a highly personal book and a difficult one for all but the specialist; and it is a beautiful book, well conceived and well executed.

There can be little doubt that the subject is currently of interest to many mathematicians. Brodskiĭ [4], Brodskiĭ and Livshits [5], de Branges [6], Kalisch [13], Sakhnovich [18], and Schwartz [19] are actively studying the "subdiagonalization" of operators with "small" imaginary parts. (This list of references is intended to be representative, not exhaustive.) The paper of Foiaş and Sz.-Nagy [9] on the existence of non-trivial invariant subspaces for operators A such that neither A^n nor A^{*n} tends strongly to 0 at any non-zero vector has just appeared. Bernstein and Robinson [2] (see also [11]) have just generalized the Aronszajn-Smith theorem for compact operators [1] to operators that are algebraic over the algebra of compact operators. Some of these results have become known too late to be treated by Helson, and none of them appears to be of central interest to him; Schwartz's work is acknowledged with no more than a reference to the bibliography, and the work of Aronszajn and Smith is not listed even there.

What the book does treat is the circle of ideas clustering around Beurling's theorem [**3**]. That theorem characterizes, in a good and usable sense, all invariant subspaces of one special operator (the unilateral shift of multiplicity 1). Though the result is relatively recent, and its extensions and generalizations are still being studied, its roots are deep in classical analysis. One classical theorem that Beurling's theorem is closely connected with is the theorem of F. and M. Riesz [**16**] on the vanishing of the boundary values of functions in the Hardy class H^2. Beurling referred to the work of F. and M. Riesz and leaned on it; after Beurling's paper appeared it became clear that one could go in the other direction and derive the Riesz result from Beurling's. The proof of Theorem 1 in Helson's book is the ultimate distillation of this insight. The statement is the qualitative version of the Riesz theorem, and the proof is maximally simple and elegant. The proof does not use Beurling's theorem; all it uses is one simple idea that occurs in the geometric approach to that theorem. It can now be said, with the usual acuity of hindsight, that the F. and M. Riesz theorem is a geometric triviality and that, for its qualitative version at any rate, the hard analysis of F. and M. Riesz is completely unnecessary.

The Beurling theorem itself occurs early (Theorem 3) with a clean modern statement and a simple geometric proof. A corollary (Theorem 4) is the factoring of functions in H^2 into "inner" and "outer" functions. Beurling's treatment of this subject was complicated; Helson's is startlingly simple.

At this point there begins a rather long analytic section. The startling simplicity of the theorem about outer functions (based on the geometric definition of an outer function as a cyclic vector for the unilateral shift) is paid for by proving that the geometric definition is equivalent to Beurling's analytic one. Beurling's theorem may be regarded as a theorem about L^2, and, as such, it is susceptible of an L^p generalization; this is looked into. Beurling's theorem may also be regarded as a theorem about the discrete semigroup of non-negative integers, and, as such, it is susceptible of a generalization to the continuous semigroup of non-negative real numbers; this was first done by Lax [**14**] and is here expounded by Helson.

After the analysis, back to geometry. Beurling's theorem may be regarded as a theorem about the unilateral shift of multiplicity 1, and, as such, it is susceptible of a generalization to shifts of higher multiplicity. Equivalently, and this is the point of view preferred by

Helson, Beurling's theorem on H^2 for numerical-valued functions can be generalized to a theorem about spaces like H^2 that consist of vector-valued functions. The purpose of the work is to generalize as much as possible of the numerical case, and it is surprising how much is possible. The known techniques along these lines are not yet perfect. Sometimes they are quite general, but more often they apply only when the multiplicity (the dimension of the value space) is finite.

Since the total number of non-trivial operators whose invariant subspaces are completely known is very small (three?), Beurling's results and their generalizations would be significant even if unilateral shifts were nothing more than a very special case of the general theory. They are, however, much more than that. It was Rota [17] who first pointed out that unilateral shifts play the role of universal operators, in the sense that every operator that satisfies a mild size condition is similar to a part of the adjoint of a shift. ("Part" means restriction to an invariant subspace.) This result was recently improved by de Branges and Rovnyak [7]; with an even weaker hypothesis they get a much sharper conclusion (unitary equivalence instead of similarity). (The reference here is to Theorem 1 of the paper by de Branges and Rovnyak, and to that theorem only. The reader who verifies this reference should see [8] also.)

In view of these results, the general problem of invariant subspaces reduces to this: does every non-trivial part of the adjoint of a unilateral shift have a non-trivial invariant subspace? The high point of Helson's book (Theorem 16) is the exposition of the solution of this problem (affirmative) for shifts of finite multiplicity. This exposition is extremely valuable. For the shift of multiplicity 1 it is a consequence of the full (quantitative, measure-theoretic) power of the Beurling theory. For shifts of higher multiplicity the necessary techniques are so scattered in the literature that, although workers in the field have believed the result for some time, they would have been hard pressed to cite chapter and verse for the proof.

The personal character of the book is visible, in part, in the rather arbitrary choice of some topics and omission of others. The style is charming, and very much in the first person singular; whether it is at times too much so is a matter of individual taste. The organization is informal. As the title indicates, the book is divided into lectures (eleven of them). The arrangement and the display of the material make it easy to learn the statements of the main results by casually riffling the pages, and, at the same time, the neat and careful proofs make the book useful for someone who wants to verify the details.

It is, however, not an easy book to find something in; there is no index.

The preface says that "the book is written for a graduate student who knows a little, but not necessarily very much, about analytic functions and about Hilbert space". Despite this claim, a close study of the book requires very much more ammunition than that; the reader must be an expert in many phases of both modern and classical analysis. Thus the function-theoretic structure of inner functions is called "known and familiar" and dismissed with a reference to Hoffman's book [12]. Among the things that the reader must know are conjugate functions, Banach spaces and their weak topologies, the Riesz representation theorem for bounded linear functionals on spaces of continuous functions, Stone's theorem on the representations of unitary groups, and the Hardy-Littlewood theorem on rearrangements of functions.

Theorem 2 is (easily equivalent to) the statement that multiplications form a maximal abelian algebra of operators on L^2 of the unit circle; it is called "a famous theorem of Wiener". The result is certainly well known by now, and has routinely been a part of courses on Hilbert space for many years. Is the attribution correct? No reference is given. Along the same lines, Theorem 9 is the generalization of Beurling's theorem to shifts of arbitrary multiplicity; it is called Lax's theorem. Lax proved the result for shifts of finite multiplicity only. For shifts of infinite multiplicity he first raised it as an unsolved problem [14], and later informally stated it without proof [15]. The first published proof of the theorem appears to be the one in [10]. As for misprints and other such minor blemishes, the book has some, but apparently they are very few and very small. On p. 5, line 12 from the bottom, it is not (1.8) that can be inferred from the argument, but the complex conjugate of (1.8). On p. 38, line 9 from the bottom, $\{T_t\}$ should be replaced by $\{V_t\}$.

In sum: the book has its faults (and what book does not?), but the mathematical world is much better off with it than it would be without it (and how many books can one say that about?).

References

1. N. Aronszajn and K. T. Smith, *Invariant subspaces of completely continuous operators*, Ann. of Math. (2) 60 (1954), 345–350.

2. A. R. Bernstein and A. Robinson, *Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos*, Pacific J. Math. 15 (1965), (to appear).

3. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949), 239–255.

4. M. S. Brodskiĭ, *Triangular representations of some operators with completely con-*

tinuous imaginary part, Soviet Math. Dokl. **1** (1961), 952–955.

5. M. S. Brodskiĭ and M. S. Livshits, *Spectral analysis of non-self-adjoint operators and intermediate systems*, Uspehi Mat. Nauk **13** (1958), 3–85. (Russian)

6. L. de Branges, *Invariant subspaces of nonselfadjoint transformations*, Bull. Amer. Math. Soc. **69** (1963), 587–590.

7. L. de Branges and J. Rovnyak, *The existence of invariant subspaces*, Bull. Amer. Math. Soc. **70** (1964), 718–721.

8. ———, *Correction to "The existence of invariant subspaces"*, Bull. Amer. Math. Soc. **71** (1965), 396.

9. C. Foiaş and B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert.* IX. *Factorisations de la fonction caractéristique. Sous espaces invariantes*, Acta Sci. Math. (Szeged) **25** (1964), 283–316.

10. P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.

11. ———, *Invariant subspaces of polynomially compact operators*, Pacific J. Math. **15** (1965), (to appear).

12. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

13. G. K. Kalisch, *A functional analysis proof of Titchmarsh's theorem on convolution*, J. Math. Anal. Appl. **5** (1962), 176–183.

14. P. D. Lax, *Translation invariant spaces*, Acta Math. **101** (1959), 163–178.

15. ———, *Translation invariant spaces*, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, pp. 299–306, Macmillan, New York, 1961.

16. F. Riesz and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès de Mathématiciens Scandinaves, Stockholm, 1916; pp. 27–44.

17. G. C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472.

18. L. A. Sakhnovich, *A study of the "triangular form" of non-selfadjoint operators*, Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 4 (11), 141–149.

19. J. T. Schwartz, *Subdiagonalization of operators in Hilbert space with compact imaginary part*, Comm. Pure Appl. Math. **15** (1962), 159–172.

P. R. HALMOS