PRODUCTS OF PSEUDO CELLS

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Communicated by Deane Montgomery, July 30, 1962

By a *pseudo n-cell* is meant a contractible compact combinatorial n-manifold with boundary (whose boundary is not necessarily an $(n-1)$-sphere). Poenaru [6] and Mazur [5] gave the first examples of pseudo 4-cells which are not topological 4-cells, and Curtis [4] has shown that, for each $n \geqq 4$, there exists a pseudo n-cell which is not a topological n-cell because its boundary fails to be simply connected. By a *homotopy n-cell* is meant a pseudo n-cell whose boundary is the $(n-1)$-sphere S^{n-1}. It follows from the generalized Poincaré conjecture and the generalized Schoenflies theorem that a homotopy n-cell is a topological n-cell if $n \geqq 5$ [4].

The following consequence of theorems of Brown and Stallings generalizes results of Curtis [4], who has shown that the cartesian product of a pseudo n-cell and an interval is the topological $(n+1)$-cell I^{n+1} if $n \geqq 5$, and Andrews [1], who has shown that the product of a homotopy 3-cell with I^3 and the product of a homotopy 4-cell with I^2 are both I^6.

THEOREM. *If M^p and N^q are pseudo cells of positive dimensions p and q respectively, with $p+q \geqq 6$, then*[2] $M^p \times N^q = I^{p+q}$.

COROLLARY. *If $n \geqq 8$, then I^n is the product of two combinatorial manifolds with boundary, neither of which is a topological cell.*

The following lemma is perhaps well known, but it does not seem to have appeared in print.

LEMMA. *If C^n is a compact n-manifold with boundary such that* Int $C^n = E^n$ *(euclidean n-space) and* $B =$ Bd $C^n = S^{n-1}$, *then* $C^n = I^n$.

PROOF. By Brown's result that the boundary of a manifold is collared [3], there is a homeomorphism h of $B \times [0, 1]$ into C^n such that $h(x, 0) = x$ if $x \in B$. Then, by the generalized Schoenflies theorem [2], the collared $(n-1)$-sphere $h(B \times 1/2)$ bounds a closed n-cell A in Int $C^n = E^n$. Hence $C^n = A \cup h(B \times [0, 1/2])$ is a closed n-cell.

PROPOSITION. *If C^n is a compact combinatorial n-manifold with boundary, $n \geqq 6$, with* Int $C^n = E^n$, *then* $C^n = I^n$.

PROOF. By the Lemma it suffices to show that the boundary B of C^n is an $(n-1)$-sphere. By the generalized Poincaré conjecture, it is therefore sufficient to show that $\pi_i(B)$ is trivial for $0 \leqq i < n-1$ [7; 9].

[1] This research was supported by N.S.F. Contract G-11665.

[2] Equality here denotes topological equivalence.

Let D be an $(i+1)$-cell whose boundary is the i-sphere T, $0 \leqq i < n-1$, and let h be a continuous map of T into B. By Brown's theorem [3], there is a homeomorphism f of $B\times[0, 1]$ into C^n such that $f(x, 0)=x$ for each $x\in B$. Let A be a tame n-cell in Int $C^n=E^n$ such that $f(B\times 1)\subset \text{Int } A$. Choose $\epsilon>0$ so small that $f(B\times\epsilon)\subset \text{Int } C^n - A$, and let $s(x)=f(x, \epsilon)$ be the natural homeomorphism of B onto $f(B\times\epsilon)$. Now Int $C^n-A\subset f(B\times[0, 1])$ and Int C^n-A is homeomorphic to $S^{n-1}\times E^1$, so that $\pi_i(\text{Int } C^n-A)$ is trivial if $0\leqq i<n-1$. Consequently the map sh of T into Int C^n-A can be extended to a continuous map g of D into Int C^n-A. Now let r be the deformation retraction of $f(B\times[0, 1])$ onto B defined by $r(f(x, t))=x\in B$. Then the continuous map rg of D into B is an extension of the map h of T into B. Therefore B is a combinatorial homotopy sphere of dimension $n-1\geqq 5$, and is therefore homeomorphic to S^{n-1} [7; 9].

PROOF OF THEOREM. Since M^p and N^q are pseudo cells, Int M^p and Int N^q are contractible open combinatorial manifolds (when given infinite triangulations). By Stallings' result to the effect that the product of two contractible open combinatorial manifolds is a Euclidean space if the sum of their dimensions is greater than four, it follows that $\text{Int }(M^p\times N^q)=\text{Int } M^p\times \text{Int } N^q$ is homeomorphic to E^{p+q} [8]. By the above Proposition it now follows that $M^p\times N^q = I^{p+q}$.

As a corollary to this proof, it follows that, if the 4-dimensional Poincaré conjecture is true, then the product of a homotopy 3-cell with I^2 and the product of a pseudo 4-cell with I^1 are both homeomorphic to I^5.

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