SUPERPOSABILITY PROPERTIES OF NATURALLY METRIZED GROUPS[1]

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Let p and q be elements of an additively written abelian group G. The unordered pair of elements $(p-q, q-p)$ is written $pq=|p-q|$ and is called the "distance" of p and q. This distance has the properties of symmetry $(pq=qp)$ and vanishing ($pq=|0|$ if and only if $p=q$). The group bearing this distance is called naturally metrized.

Two subsets of G are *congruent* provided there is a distance-preserving mapping of one onto the other. A congruence of G with itself is a *motion* of G. Two subsets of G are *superposable* if one is mapped onto the other by a motion of G. G has the property of *free mobility* if any congruence between any two subsets of G can be induced by a motion of G.[2]

It was shown by Menger[3] that any two congruent pairs of elements of G are superposable. In this note it is shown that G has the property of free mobility.

If $t\in G$ the mapping $x\rightarrow x\oplus t$ of G onto itself is the translation induced by t. The reflection in O of G is the mapping $x\rightarrow -x$ of G onto itself. Clearly all translations and the reflection in O are congruences. Denote by Γ the group of motions of G obtained by adjoining the reflection in O to the group of all translations of G.

THEOREM. *Let $S\subset G$ and α be a mapping of S onto $T\subset G$. A necessary and sufficient condition that there be a motion $\mu\in\Gamma$ which induces α over S is that $ab=a\alpha b\alpha$ for all $a, b\in G$, that is, α is a congruence.*

PROOF.[4] The necessity is obvious. Suppose that α is a congruence.

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[2] The concepts of superposability and free mobility *vis-à-vis* congruence have recently assumed new importance in certain of the characterization problems of metric geometry (cf. Garrett Birkhoff, *Metric foundations of geometry*. I, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 465–492). These notions, logically equivalent in euclidean, spherical, and hyperbolic spaces, are quite distinct, for example, in elliptic space where they have been extensively studied in a recent publication (cf. L. M. Blumenthal, *Congruence and superposability in elliptic space*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 431–451).

[3] K. Menger, *Beiträge zur Gruppentheorie* I. *Über einer Abstand im Gruppen*, Math. Zeit. vol. 33 (1931) pp. 396–418.

[4] The author is indebted to the referee for this more concise form of the original proof.

Since Γ contains all translations and the group of translations is transitive over G, we may assume without loss of generality that $0 \in S$ and that $0\alpha = 0$. Denote by S' the subset of S which is invariant under α. We distinguish two cases:

Case 1. $x \in S'$ implies $2x = 0$.

Suppose $w \in S$, $w \notin S'$. Then $w0 = w\alpha 0\alpha = w\alpha 0$. Since $w \notin S'$, $w \neq w\alpha$. However, $(w, -w) = (w\alpha, -w\alpha)$ so that $w\alpha = -w$. Apply the reflection in O over G. S' is invariant under the reflection and for all other elements of S, $-w = w\alpha$ so that the reflection coincides with α over S.

Case 2. There exists $t \in S'$ so that $2t \neq 0$.

Suppose $w \in S$. As in case 1, $w\alpha$ is either w or $-w$. If $w\alpha = w$, $w \in S'$. Suppose then that $w\alpha = -w$. Since α is a congruence $wt = w\alpha t\alpha = -wt$ ($t \in S'$ and hence invariant under α). Hence $w - t = w + t$ or $w - t = -w - t$. The first equation is impossible since $2t \neq 0$. Therefore $w - t = -w - t$ and $w = -w = w\alpha$ so that $w \in S'$. This shows that the identity mapping of G (translation induced by 0) coincides with α over S.

The generalization of superposability theorems to non-abelian groups is not fruitful (although such groups have been studied from the metric viewpoint)[5] since in these studies it is shown that the reflections of such a group are motions if and only if the group is either abelian or is a Hamiltonian 2-group.[6]

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[5] Cf. O. Taussky, Math. Ann. vol. 108 (1933) pp. 615–620 and Quart. J. Math. Oxford Ser. vol. 12 (1941) pp. 64–67.

[6] A Hamiltonian 2-group is a group in which the order (period) of each element is a power of 2.