ON A THEOREM OF H. CARTAN

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As an application of the Galois theory of skew fields, H. Cartan[1] obtained recently the following theorem: If K is a skew field of finite rank over its center C, the only skew fields H, $C \subseteq H \subseteq K$, which are mapped into themselves by every inner automorphism of K are K and C.

I give a very short and direct proof removing at the same time the finiteness assumption, In fact, we have:

THEOREM. *If H is a skew field contained in the skew field K, and if every inner automorphism of K maps H into itself, then H is either K, or H belongs to the center of K.*

PROOF. If $a \in K$, $b \in H$, the assumption shows that an equation

$$ba = ab_1 \tag{1}$$

with $b_1 \in H$ holds (for $a=0$ this is true with $b_1=b$). Also,

$$b(1 + a) = (1 + a)b_2 \tag{2}$$

with $b_2 \in H$. On subtracting (1) from (2), we find

$$b - b_2 = a(b_2 - b_1).$$

If a does not lie in H, this implies $b_2 = b_1$ and hence $b = b_2$. Then $b_1 = b$, that is, $ba = ab$. Every element a of K which does not belong to H commutes therefore with every element of H.

Suppose that H does not belong to the center of K. There exists an element b of H and an element c of K such that

$$bc \neq cb. \tag{3}$$

The remark above shows that $c \in H$. If $H \neq K$, there exist elements a outside of H in K. Then $a+c$ does not belong to H either. Hence $a+c$ and a both commute with $b \in H$,

$$b(a + c) = (a + c)b, \ ba = ab.$$

These two equations are not consistent with (3), and the theorem is proved.

The same argument applies under much weaker assumptions. For instance, it is sufficient to assume that K is a (not necessarily associa-

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[1] C. R. Acad. Sci. Paris vol. 224 (1947) pp. 249–251.

tive) ring, H a subring of K and that (α) H has a 1-element; (β) the equation $xh=h_1$ with $h, h_1 \in H$, $x \in K$, $h \neq 0$ implies that $x \in H$, (γ) for every $a \in K$, $b \in H$, there exists an element b_1 in H with $ba=ab_1$. If $H \neq K$, it follows that every element of H commutes with every element of K.

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A THEOREM ON INTEGRAL SYMMETRIC MATRICES[1]

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Though the following theorem yields important results in the theory of quadratic forms, its statement and proof are independent of such theory and seem to possess significance in their own right.

THEOREM. *Let A and B be symmetric integral nonsingular matrices with respective dimensions n and m $(n>m)$ and S an n by m matrix of rank m with rational elements such that s is the* l.c.m. *of the denominators and $S^TAS=B$. Then there is an n by n matrix T with rational elements the prime factors of whose denominators all divide s, whose determinant is* 1 *and which takes A into an integral matrix A_0 which represents B integrally, that is, $U^TA_0U=B$ for some integral matrix U.*

To prove this we first, for brevity's sake, define an *s-matrix* or *s-transformation* to be one with rational elements the prime factors of whose denominators all divide s. Then write $R=sS$, and, by elementary divisor theory, determine unimodular matrices P and Q such that

$$PRQ = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = s\begin{bmatrix} S_1 \\ 0 \end{bmatrix} = sS'$$

where R_1 is the diagonal matrix $r_1 \dotplus r_2 \dotplus \cdots \dotplus r_m$, $\dotplus$ denotes direct sum, r_i divides r_{i+1} for $i=1, 2, \cdots, m-1$ and S' and S_1 are defined by the equations. Write $r_i/s=u_i/s_i$ where the latter fraction is in lowest terms and $s_i>0$. Then s_i is divisible by s_{i+1} and hence s_i is prime to u_j for $j \leqq i$.

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