A NOTE ON S-SPACES

E. G. BEGLE

An S-space is a normal topological space in which each covering by open sets has a refinement which is star-finite, that is, each set of the refinement meets only a finite number of sets of the refinement. Thus a compact (=bicompact) space is an S-space, and an S-space is paracompact [1].[1]

In this note we discuss cartesian products in which one of the factors is an S-space. We show that if the other factor is compact, then the product is an S-space, and the dimension of the product does not exceed the sum of the dimensions of the factors. However, if both factors are S-spaces, the product need not be an S-space.

THEOREM. *Let A be an n-dimensional S-space and B an m-dimensional compact space. Then $A \times B$ is an S-space and* $\dim (A \times B) \leqq n+m$.

By the dimension of a space we mean, of course, the Lebesgue dimension (cf. [2, p. 206]).

Let $\mathfrak{W}_0$ be an arbitrary covering of $A \times B$. We are going to construct a locally-finite cell complex, D, with $\dim D \leqq n+m$, a mapping f of $A \times B$ onto D, and a covering $\mathfrak{Y}$ of D such that $f^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{W}_0$.

Let a be any point of A. Each point of $a \times B$ is contained in an open set of the form $U \times V$, U open in A, V open in B, such that $U \times V$ is contained in an open set of $\mathfrak{W}_0$. For a fixed point $a \in A$, the set of all such V's is a covering of B, and hence a finite number of them, say $V_{a,1}, V_{a,2}, \cdots, V_{a,k(a)}$, form a covering $\mathfrak{V}_a$ of B. Let U_a be the intersection of the corresponding U's.

The collection of all such sets U_a is a covering of A. Hence there is a star-finite refinement $\mathfrak{U}$ of this covering whose order is no more than $n+1$. We may also assume [2, p. 210] that $\mathfrak{U}$ is *normal*, that is, that there is a mapping ϕ of A onto $N(\mathfrak{U})$ such that each open set of $\mathfrak{U}$ is the inverse image, under ϕ, of the star of a vertex of $N(\mathfrak{U})$.

We form a covering $\mathfrak{W}$ of $A \times B$ as follows: each set U of $\mathfrak{U}$ is contained in some U_a, and with each U_a is associated a covering $\mathfrak{V}_a$ of B. Form the product of U with each set of $\mathfrak{V}_a$. The totality of these products forms $\mathfrak{W}$, and by construction, $\mathfrak{W}$ is a refinement of $\mathfrak{W}_0$.

Let θ be the mapping of $A \times B$ onto $N(\mathfrak{U}) \times B$ defined by $\theta(a \times b) = \phi(a) \times b$, where ϕ is the above mapping of A onto $N(\mathfrak{U})$. Each ele-

Received by the editors May 15, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

ment of $\mathfrak{W}$ is thus mapped by θ onto an open set of $N(\mathfrak{U})\times B$, so $\mathfrak{X}=\theta(\mathfrak{W})$ is a covering of $N(\mathfrak{U})\times B$.

Now let u_1 be a fixed vertex of $N(\mathfrak{U})$ and let S_1 be the closed star of u_1, and, inductively, let S_i be the closed star of S_{i-1}. Let $T_1=S_1$ and let T_i, $i>1$, be the closure of S_i-S_{i-1}, and let $R_i=T_{i-1}\cap T_i$.

Now $\bigcup_{i=1}^{\infty} S_i$ is a connected set which is both open and closed in $N(\mathfrak{U})$ and hence is a component of $N(\mathfrak{U})$. Since the constructions we make below can be made independently in each component, we may assume without loss of generality that $\bigcup_{i=1}^{\infty} S_i=N(\mathfrak{U})$.

Each vertex u of $N(\mathfrak{U})$ corresponds to an open set U of $\mathfrak{U}$ and, as above, each U is contained in a set U_a to which there corresponds a covering $\mathfrak{V}_a$ of B. Let $\mathfrak{V}_1'$ be a finite covering of B which is a common refinement of each $\mathfrak{V}_a$ which corresponds to a vertex of T_1. Let $\mathfrak{V}_1$ be a normal finite covering of B which is of order not greater than $m+1$ and which is a star-refinement of $\mathfrak{V}_1'$, that is, each set consisting of an element $\mathfrak{V}_1$ together with all the elements of $\mathfrak{V}_1$ which meet it is in an element of $\mathfrak{V}_1'$.

In general, having obtained $\mathfrak{V}_{i-1}$, we obtain $\mathfrak{V}_i$ as follows: let $\mathfrak{V}_i'$ be a common finite refinement of $\mathfrak{V}_{i-1}$ and of each $\mathfrak{V}_a$ which corresponds to a vertex of T_i. Let $\mathfrak{V}_i$ be a normal finite covering of B, of order not greater than $m+1$, which is a star-refinement of $\mathfrak{V}_i'$.

Let C_i be the finite cell-complex $T_i\times N(\mathfrak{V}_i)$. Since $\mathfrak{V}_i$ is a refinement of $\mathfrak{V}_{i-1}$, there is a projection π_i, a simplicial mapping, of $N(\mathfrak{V}_i)$ into $N(\mathfrak{V}_{i-1})$. For each i, identify the subcomplex $R_i\times N(\mathfrak{V}_i)$ of $T_i\times N(\mathfrak{V}_i)$ with the subcomplex $R_i\times\pi_iN(\mathfrak{V}_i)$ of $T_{i-1}\times N(\mathfrak{V}_{i-1})$. The result of these identifications is the cell-complex D. Since $N(\mathfrak{U})$ is at most n-dimensional, and $N(\mathfrak{V}_i)$, for each i, is at most m-dimensional, the highest possible dimension for a cell of D is $n+m$.

Since each $\mathfrak{V}_i$ is normal, there is a corresponding mapping ζ_i of B onto $N(\mathfrak{V}_i)$. Let ζ be the transformation of $N(\mathfrak{U})\times B$ onto D defined by setting $\zeta(p\times b)=p\times\zeta_i(b)$ for $p\in T_i-R_i$. Since each π_i is continuous, so is ζ. Now $f=\zeta\theta$ is a mapping of $A\times B$ onto D.

To construct the covering $\mathfrak{Y}$ of D, let u be any vertex of $N(\mathfrak{U})$. Then u is in some R_i. Let v be a vertex of $N(\mathfrak{U}_{i-1})$, and consider $u\times v$ as a vertex of C_{i-1}. Let K be the star, in C_{i-1}, of $u\times v$. Then consider u as a vertex of T_i, and let $v_1, v_2, \cdots, v_s$ be all the vertices of $N(\mathfrak{U}_i)$ which are mapped onto v by π_i. Let L be the union of the stars of $u\times v_1, \cdots, u\times v_s$ in C_i. Then the set $K\cup L$ of $C_{i-1}\cup C_i$ becomes, after the identifications made in defining D, an open set of D containing $u\times v$. The collection of all such sets constitutes the covering $\mathfrak{Y}$. Since each $\mathfrak{V}_i$ is a star-refinement of $\mathfrak{V}_{i-1}$, it is easy to see that $\zeta^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{X}$ and hence that $f^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{W}$.

It is now easy to finish the proof of the theorem. First we make a barycentric subdivision of D, thus obtaining a simplicial complex E. Let e_1 be a vertex of E, and let $\overline{S}_i$ have the same meaning for E as S_i has for $N(\mathfrak{U})$ above. Next we subdivide $\overline{S}_2$ simplicially until the star of each vertex in the induced subdivision of $\overline{S}_1$ is contained in some element of $\mathfrak{Y}$. Then we subdivide $\overline{S}_3$ simplicially, without introducing any new vertices in $\overline{S}_1$, until each vertex of the induced subdivision of $\overline{S}_2$ has its star contained in some element of $\mathfrak{Y}$.

Continuing in this fashion, all of E is subdivided in such a way that each cell of D is divided into a finite number of simplexes.

Now let $\mathfrak{Z}$ be the covering of D by the stars of the vertices of the subdivision of E. By construction, $\mathfrak{Z}$ is a refinement of $\mathfrak{Y}$. Since each cell of D is of dimension at most $n+m$, the same is true of E and of its subdivision. Hence, order $\mathfrak{Z} \leqq n+m+1$. Clearly $\mathfrak{Z}$ is star-finite. Hence $f^{-1}(\mathfrak{Z})$ is a star-finite covering of $A \times B$, of order not greater than $n+m+1$, and a refinement of $\mathfrak{W}_0$, which proves the theorem.

To show that the product of two S-spaces need not be an S-space, we appeal to an example, constructed by Sorgenfrey [**4**], of a paracompact space whose product with itself is not paracompact. It is only necessary to observe that this space is actually an S-space, as is easily seen by an inspection of his proof.

Finally, we remark that Hemmingsen [**3**] has shown that the dimension theorem holds for the product of two compact spaces, and Dieudonné [**1**] has shown that the product of a compact space and a paracompact space is paracompact. Thus, the only unsettled question in this direction is that concerning the dimension of the product of a compact and paracompact space. It is clear that the method used above cannot be used in this case.

Bibliography

1. J. Dieudonné, *Une généralization des espaces compacts*, J. Math. Pures Appl. (9) vol. 23 (1944) pp. 65–76.

2. C. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. vol. 69 (1947) pp. 200–242.

3. E. Hemmingsen, *Some theorems in dimension theory for normal spaces*, Duke Math. J. vol. 13 (1946) pp. 495–504.

4. R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 631–632.

Yale University