ON A PROBLEM OF MAX A. ZORN

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1. Introduction. Max A. Zorn has proved[1] the following theorem.

THEOREM. *If every substitution $x=at$, $y=bt$ in which a and b are complex numbers transforms $\sum a_{ij}x^iy^j$ into a power series with a non-vanishing radius of convergence, the series $\sum |a_{ij}x^iy^j|$ converges for sufficiently small $|x|$ and $|y|$.*

He has also suggested the following problem. If $\sum a_{ij}x^iy^j$ is a power series which is transformed by every substitution of convergent power series $\sum_1^\infty a_it^i$ and $\sum_1^\infty b_it^i$ with real coefficients for x and y into a convergent power series in t, is the double series $\sum a_{ij}x^iy^j$ convergent? The answer is yes. In fact, Zorn's theorem itself holds even when the coefficients a and b are restricted to take only real values. We can obtain a proof quite directly by Zorn's method, if we use an estimate for the coefficients of homogeneous polynomials in real variables.

2. Homogeneous polynomials in real variables. We shall prove a lemma which may easily be extended to the case of many variables.

LEMMA. *Let $P(x, y)=\sum_{i+j=n} a_{ij}x^iy^j$ be a homogeneous polynomial in real variables. If $|P(x, y)|\leqq M$ for $|x-x_0|\leqq 2\delta$, $|y-y_0|\leqq 2\epsilon$, then $|a_{ij}\delta^i\epsilon^j|\leqq M$.*

PROOF. Set $x=x_0+\delta(\xi+\xi^{-1})$. Then $\xi^nP(x, y)=\sum a_{ij}\,\xi^j(\xi x)^iy^j$ is a polynomial in ξ whose absolute value does not exceed M when ξ moves on the unit circle of the Gaussian plane. By Cauchy's inequality of function theory, and considering the coefficients of ξ^k in $\xi^nP(x, y)$, we have

$$\left|\sum_{j=0}^{k} a_{ij}c_iy^j\right| \leqq M,$$

where $0\leqq k\leqq n$, $i+j=n$, and c_i is the coefficient of ξ^{k-j} in $(\xi x)^i$.

Again set $y=y_0+\epsilon(\eta+\eta^{-1})$ and apply the Cauchy inequality to the constant term of $\eta^k\sum_{j=0}^k a_{ij}c_iy^j$. We have

$$|a_{lk}c_l\epsilon^k| \leqq M,$$

where $l+k=n$ and c_l equals δ^l by definition. This completes our proof.

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[1] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 791–792.

3. **Proof of Zorn's theorem in the real case.** Now we can follow Zorn's method directly.

PROOF. Let $P_n(x, y) = \sum_{i+j=n} a_{ij}x^i y^j$. The set D of vectors (x, y) for which $\sum P_n(x, y)$ converges is of the second category. For every vector is in[2] mD for some positive integer m. If D were of the first category, the set mD and therefore the two-dimensional Euclidean space would be the same character.

By virtue of the continuity of the functions P_n there will exist a square $|x-x_0| \leqq 2p$, $|y-y_0| \leqq 2p$, $p>0$ and an M such that $|P_n(x, y)| \leqq M$ holds in the square for all n. Then by our lemma $|a_{ij}p^{i+j}| \leqq M$, Hence for $|x|$, $|y| \leqq p/2$, we have

$$| a_{ij}x^i y^j | \leqq M/2^{i+j}$$

which establishes the absolute convergence of the double series.

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[2] mD is the set of (mx, my) where $(x, y) \in D$.