REMARKS ON A PAPER BY ZEEV NEHARI

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In the preceding paper Zeev Nehari has proved some interesting inequalities for the Schwarzian derivative of a univalent (=schlicht) function. Thus *if* $f(z)$ *is univalent in the unit circle, then*

$$| \{f(z), z\} | \leqq 6[1 - |z|^2]^{-2} \tag{1}$$

while if

$$| \{f(z), z\} | \leqq 2[1 - |z|^2]^{-2}, \tag{2}$$

then $f(z)$ *is univalent for* $|z| < 1$. The object of the present note is to show that 2 *is the best possible constant* in (2) in the following sense:

For every $C > 2$ *there exists a function* $f(z)$ *such that for* $|z| < 1$ *we have* (i) $f(z)$ *is holomorphic,* (ii) $f(z)$ *takes on the value one infinitely often, and* (iii) $| \{f(z), z\} | \leqq C[1 - |z|^2]^{-2}$ *with equality for real values of* z.

An explicit example of such a function is given by

$$f(z) = \left(\frac{1 - z}{1 + z}\right)^{\gamma i}, \tag{3}$$

where γ is a real constant, $f(0) = 1$, and $C = 2(1 + \gamma^2)$.

In view of the background of the problem, the following approach is natural. Let F_a denote the family of fractional linear transforms with constant coefficients of the quotient of two linearly independent solutions of the differential equation,

$$\frac{d^2y}{dz^2} + a(1 - z^2)^{-2}y = 0, \tag{4}$$

where a is an arbitrary parameter. If $f(z) \in F_a$, then

$$\{f(z), z\} = 2a(1 - z^2)^{-2}.$$

If one function $f(z)$ of F_a is univalent in the unit circle, then they all are. Let us determine the region U of the a-plane such that if $a \in U$, then the functions of F_a are univalent for $|z| < 1$. By Nehari's Theorem I the region $|a| \leqq 1$ belongs to U and (1) makes it plausible that U is contained in $|a| \leqq 3$.

Equation (4) has elementary solutions. The indicial equations at

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the singular points $z = \pm 1$ are identical,

$$\rho^2 - \rho + \tfrac{1}{4}a = 0,$$

and if its two roots are α and β, a simple calculation shows that

$$y(z) = (1 - z)^\alpha (1 + z)^\beta$$

is a solution. If $a \neq 1$, $\alpha \neq \beta$ and $y(-z)$ is a second linearly independently solution. The family F_a is generated by their quotient

$$f(z) = \left(\frac{1 - z}{1 + z}\right)^\delta, \qquad \delta = \beta - \alpha = (1 - a)^{1/2} = \mu + i\nu, \tag{5}$$

where $f(0) = 1$ and the square root is $+1$ for $a = 0$.

It remains to determine when $f(z)$ is univalent in $|z| < 1$ or, what is equivalent, when t^δ is univalent in the right half-plane. A necessary and sufficient condition that for every choice of v the equation $t^\delta = v$ shall have at most one solution with $|\arg t| < \frac{1}{2}\pi$ is given by $2|\mu| \geqq \mu^2 + \nu^2$. This implies that either $|\delta - 1| \leqq 1$ or $|\delta + 1| \leqq 1$. Since $a = 1 - \delta^2$, we see that the boundary of U is the cardioid

$$a = -2e^{i\phi} - e^{2i\phi}, \qquad -\pi < \phi \leqq \pi, \tag{6}$$

and U consists of the interior and boundary of the cardioid. We note that the largest circle with center at the origin contained in U is $|a| = 1$ and that $|a| \leqq 3$ in U with equality only for $a = -3$.

Every point a outside of U leads to a family of functions which are not univalent in the unit circle. In particular, for every real γ, $\gamma \neq 0$, the point $a = 1 + \gamma^2$ is outside of U. Thus the function (3) is not univalent for $|z| < 1$ and a simple calculation shows that it takes on the value one infinitely often in the unit circle.

In conclusion, we observe that the boundary point $a = -3$ of U is of considerable interest. One of the functions of F_{-3} is $4z(1+z)^{-2}$ which is the extremal function of Koebe's mapping problem.

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