

BOOK REVIEWS

The problem of moments. By J. A. Shohat and J. D. Tamarkin. (Mathematical Surveys, vol. 1.) New York, American Mathematical Society, 1943. 140 pp. \$2.25.

This book is the first in a new series sponsored by the American Mathematical Society. Each number of the series is evidently designed to collect all important existing material on a single mathematical discipline and to expound it in a readable and understandable manner. If so, then the volume under review succeeds admirably for that branch of mathematics which has recently grown up about a problem posed and solved by T. J. Stieltjes and named by him "the moment problem." The authors have set a high standard of excellence in presentation and choice of material, which may well establish the tone for the series.

The moment problem may be stated as follows: Given a sequence of real numbers μ_0, μ_1, \dots ; determine a nondecreasing function $\psi(t)$ such that

$$(1) \quad \mu_n = \int_a^b t^n d\psi(t), \quad n = 0, 1, 2, \dots$$

If $a = 0, b = \infty$, the problem is precisely as Stieltjes set it and is known as the Stieltjes problem; it is known by the names of F. Hausdorff or H. Hamburger according as $a = 0, b = 1$ or $a = -\infty, b = +\infty$. Of course the first two are special cases of the last, but it is profitable to study them separately. The case in which $\psi(t)$ is to be the Riemann integral of a positive function was studied by earlier authors, notably E. Heine and P. Tchebycheff. In spite of this it seems proper that the name of Stieltjes should be attached to the problem, for the generalized integral which he introduced, now known as the Riemann-Stieltjes integral, was indeed a very happy idea for the development of the problem. Without it finite linear combinations and integrals would have to be studied separately, and nothing like the existing elegance in the theory could possibly be attained.

The reason for the term "moment" becomes evident if one interprets $\psi(t)$ as defining a distribution of mass along the interval $a \leq t \leq b$. Then μ_0 is the total mass; μ_1 is the statical moment which measures the tendency of the segment to turn about the origin if held horizontally; μ_2 is the moment of inertia about the origin. Stieltjes defines μ_n as the n th moment, and his fundamental question

is whether the distribution of mass can be ascertained from a knowledge of all the moments. After this physical statement it is clear why one is particularly interested in nondecreasing functions $\psi(t)$. From the point of view of the pure mathematician the case when $\psi(t)$ is of bounded variation is equally interesting. Indeed this case has been studied in detail for the Hausdorff problem with suitable restrictions on the μ 's. However, for the other two problems it turns out that the equations (1) are consistent for an *arbitrary* sequence of moments if $\psi(t)$ is only required to be of bounded variation. With so slight a relation between the μ 's and $\psi(t)$ there remains little to study.

We return to the problems as originally set and state the fundamental conditions for consistency. The Hausdorff problem has a solution if and only if

$$(-1)^k \Delta^k \mu_n \geq 0, \quad k, n = 0, 1, 2, \dots$$

The sequences $(n+1)^{-1}$ and 2^{-n} have this property. Indeed they correspond to the nondecreasing functions $\psi(t) = t$ and $\psi(t) = 0, t < 1/2, \psi(t) = 1, t > 1/2$, respectively. The corresponding conditions for the Hamburger problem are that

$$(2) \quad \sum_{i=0}^n \sum_{j=0}^n \mu_{i+j} x_i x_j \geq 0, \quad n = 0, 1, 2, \dots,$$

for all real numbers x_i . Finally, for the Stieltjes problem one must add to the inequalities (2) that

$$\sum_{i=0}^n \sum_{j=0}^n \mu_{i+j+1} x_i x_j \geq 0, \quad n = 0, 1, 2, \dots$$

Having established conditions for the existence of a solution we naturally ask if the solution is unique. At first sight we might perhaps not expect uniqueness under any circumstances, for we are trying to determine a function over an interval, a point set with the power of the continuum, by the countable set of moments. However, our more familiar experience with Fourier series, where the Fourier coefficients do determine the function from which the series derives, may serve to orient us here. In fact the solution of the Hausdorff problem is unique, even if $\psi(t)$ is of bounded variation. However, in the other two cases when the interval is infinite the solution is not in general unique, and it becomes desirable to know if further conditions can be imposed on the μ 's to guarantee uniqueness. Many such sufficient conditions have been found. The most general one known, due to T. Carleman, is that the divergence of the series

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)}$$

implies the uniqueness of the solution of the Hamburger problem.

The book begins with a brief introductory chapter designed chiefly to collect the needed tools but also containing a historical sketch and a description of the main results. Here is proved a theorem on the extension of non-negative functionals which is fundamental for later work. There follow four chapters which we now describe briefly.

Chapter 1 is concerned with the fundamental theorems of consistency mentioned above. It also proves Carleman's uniqueness theorem and derives others therefrom.

Chapter 2 deals with the Hamburger problem in minute detail and contains perhaps the most profound material of the book. The approach is through the theory of continued fractions, which really was the origin of the investigation of Stieltjes. These fractions in turn are studied by use of a class of functions which are analytic in a half-plane and which have a negative imaginary part there. Necessary and sufficient conditions for uniqueness are found in terms of the continued fraction associated with the moment problem. Many details concerning the polynomials which appear in the successive approximants of this continued fraction are developed.

The third chapter discusses various modifications of the original problem, notably one mentioned above: the Hausdorff problem in which $\psi(t)$ is no longer required to be nondecreasing. Relations with the Laplace transform are considered. Explicit expansions of $\psi(t)$ in terms of the moments are obtained. This part of the book is replete with ideas and makes fascinating reading.

The fourth and final chapter concerns itself with methods of approximate quadrature. There is such a method, corresponding to each moment sequence, for approximating the integral

$$(3) \quad \int_{-\infty}^{\infty} f(t) d\psi(t).$$

Here $\psi(t)$ is any solution of the Hamburger problem. The method described may be considered as a generalization of the classical formula of mechanical quadrature due to Gauss. In fact it reduces thereto if $\psi(t) = t$ in the interval $(-1, 1)$ and is constant elsewhere. The chapter closes with a discussion of the convergence of the approximations to the integral (3).

Throughout the book the method of exposition is to proceed from the general to the particular. An effort is made to drive directly at the

most general result known and to discuss all other results as special cases. This procedure is an ideal one for a survey of the present character. As a result the reader is able to get a clear picture of the whole subject and to see in proper perspective the interrelations of the various special parts. In view of the nature of the book the authors have not included proofs of all theorems. But enough of these have been given to give the reader a real understanding of the methods. The choice of what to include and what to omit is generally in excellent taste. The reviewer would prefer to have seen included more of the proofs of the results of N. I. Achyesser and M. Krein, for their treatise on the subject is written in Russian. There are very few misprints. Those discovered by the reviewer are of a trivial nature and can be corrected in an obvious way by the reader. The book is certainly a very valuable addition to mathematical literature. The American Mathematical Society is to be congratulated on this auspicious initiation of its series of surveys.

D. V. WIDDER

Introduction to mathematical logic. Part I. By Alonzo Church. (Annals of Mathematics Studies, no. 13.) Princeton University Press, 1944. 6+118 pp. \$1.75.

This booklet contains the material of the first half of a graduate course given by the author at Princeton and is, in fact, the revision of a set of lecture notes of that course. It is hoped that Part II, covering the second half of the course, which exists now in rough draft in the form of lecture notes, will some day appear in print.

The title of the book is somewhat misleading. As the author says, this is a monograph rather than a text. Its aim is not to give a broad survey of recent developments in symbolic logic but to present formally and rigorously, and with all the latest improvements, the theory of one of the oldest branches of the subject, namely, the calculus of propositional functions. In this aim the author has succeeded admirably. The difficult points are emphasized instead of being avoided. The care and precision for which the author is noted are in evidence throughout. The end result is, for the qualified beginner, a compact presentation of an important theory which can serve also as a model of present-day standards of rigor. The book is of value to the specialist also in bringing together in one place and in one notation rigorous proofs of important basic theorems which are otherwise only to be found scattered throughout the literature.

There are four chapters, of which the first deals with the calculus