GROUPS TRANSITIVE ON THE n-DIMENSIONAL TORUS

DEANE MONTGOMERY[1] AND HANS SAMELSON

In this note we denote by G a compact connected Lie group. We shall be interested in the situation where G acts as a topological transformation group[2] on a space E. Such a group is called effective if the identity is the only element of G which leaves every point of E fixed. If G is transitive on E, that is, for any two points x and y of E there is an element g of G such that $g(x)=y$, then E is called a homogeneous space or a coset space of G. Our purpose is to prove the following theorem:

THEOREM. *If a compact connected Lie group G is transitive and effective on a space E homeomorphic with an n-dimensional torus (topological product of n circles), then G is isomorphic with the n-dimensional toral group T_n (direct product of n circle groups) and no element of G except the identity leaves any point of E fixed.*

We use a method of proof which has some similarity to a method we have used in studying groups transitive on spheres.[3]

Let H' be a compact, connected, simply connected Lie group, let T_l be an l-dimensional toral group, and let N be a finite normal subgroup of the direct product $H' \times T_l$ such that G is continuously isomorphic to the factor-group $(H' \times T_l)/N$.[4] Let H' go into H by the homomorphism obtained by factoring with respect to N and let T_l go into K. The group K is also an l-dimensional toral group, and H and K are subgroups of G which span G or generate G. The elements of H commute with the elements of K, in fact K is a central subgroup of G.

Let x be an arbitrarily chosen point of E and let H_x, K_x, and G_x be, respectively, the subgroups of H, K, and G which leave x fixed. Let K^x be the subgroup of K consisting of those elements k such that $k(x)$ is in the orbit $H(x)$. The orbit $K^x(x)$ is the intersection of $H(x)$ and $K(x)$. It can be seen that if $y=g(x)$ then $K_y=gK_xg^{-1}$ and $H_y=gH_xg^{-1}$. Since K is a central subgroup we see that $K_y=K_x$.

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[1] Guggenheim fellow.

[2] For the theory of topological groups and Lie groups needed see Pontrjagin, *Topological groups*, Princeton 1939. For definitions and results concerning topological transformation groups see Zippin, *Transformation groups*, Lectures in Topology, Ann Arbor 1941 pp. 191–221.

[3] See a paper by us which is forthcoming.

[4] For the existence of these groups see Pontrjagin, loc. cit. pp. 282–285. The group H' is the direct product of the simple Lie groups there mentioned.

Because G is transitive the last remark shows us that every element of K_x leaves every point of E fixed, and this together with the fact that G is effective implies that K_x contains only the identity element. The orbit $K(x)$ is therefore homeomorphic to K and consequently l, the dimension of K, must be less than or equal to n, the dimension of E.

We denote by $\mathcal{H}_1(S)$ the one-dimensional homology group with rational coefficients of the space S. If a compact connected Lie group is mapped in the natural way on one of its orbits (namely by considering the orbit as a coset space of the group) then the one-dimensional homology group of the group manifold is mapped *onto* the one-dimensional homology group of the orbit. This follows from the fact that the mapping can be carried out in two steps, the first being a fibering of the group with respect to a connected subgroup, and the second being a finite covering. In both steps the one-dimensional homology groups (with rational coefficients) are mapped *onto* the one-dimensional homology groups of the respective spaces.

We now apply the above remark to the group G and the orbit E. The homology group $\mathcal{H}_1(E)$ is an n-dimensional vector space, that is the first Betti number of E is n. Therefore the first Betti number of G is at least n, and from this we see that the first Betti number of $H' \times T_l$ must be at least n. However the first Betti number of $H' \times T_l$ is l and we see that l is greater than or equal to n.

The above results show that l equals n. It follows that the orbit $K(x)$ is the whole of E which means that K is transitive on E. We see therefore that $K^x(x) = H(x)$. Since $K^x(x)$ is homeomorphic to K^x and since $H(x)$ is connected it follows that K^x is connected. Therefore K^x as a connected subgroup of a toral group must itself be a toral group of some dimension greater than zero or it must contain only the identity element. But from the fact that $\mathcal{H}_1(H) = 0$ it follows that $\mathcal{H}_1[H(x)] = 0$. Hence $\mathcal{H}_1(K^x) = 0$ and K^x contains only the identity element, and $H(x) = x$. The point x was chosen arbitrarily so that the equation $H(x) = x$ holds for every x in E. Because G is effective this means that H contains only the identity element and that $G = K$ which proves the theorem.

The same proof applies if instead of assuming that E is an n-dimensional torus we merely assume that it is an n-dimensional space the first Betti number of which is n. It then follows in view of the above proof that E is a torus. If we drop the assumption that G is effective the "effective group" will be a toral group of dimension n as before.

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THE INSTITUTE FOR ADVANCED STUDY