ON SEQUENCES OF POLYNOMIALS AND THE DISTRIBUTION OF THEIR ZEROS

OTTO SZÁSZ

The first results on this subject are due to Laguerre (1882); they were generalized to a remarkable degree by Pólya and in a joint paper by Lindwart and Pólya. I quote the following theorems [2].[1]

THEOREM 1. *If a sequence of polynomials*

$$P_n(z) = 1 + \sum_1^n c_{n\nu} z^\nu = \prod_\nu (1 - z z_{n\nu}^{-1}) \tag{1}$$

converges uniformly in a circle $|z| < R$, *and if for some integer* k

$$\sum_1^n |z_{n\nu}|^{-k} < M, \qquad M \text{ independent of } n, \tag{2}$$

then the sequence (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is the product of a function of genus at most* $k-1$ *and of* $e^{\gamma z^k}$, γ *a constant.*

THEOREM 2. *If the sequence* (1) *converges uniformly in a circle* $|z| < R$, *and if the roots* $z_{n\nu}$ *lie in the half-plane* $\Re z \geqq 0$ *for each* n, *then the sequence* (1) *converges uniformly in every finite domain to an entire function* $F(z)$ *which is at most of genus* 2, *and the roots* z_ν *of* $F(z)$ *satisfy* $\sum |z_\nu|^{-2} < \infty$.

While in Theorem 1 the assumption of uniform convergence could be replaced by convergence at infinitely many points with a finite limit point and by boundedness of the sequences: $|c_{n1}|, \cdots, |c_{n\,k-1}|$, $n = 1, 2, \cdots$, the deduction of Theorem 2 required uniform convergence in $|z| < R$. We give here a new proof for Theorem 2 with a weaker hypothesis assuming instead of uniform convergence only convergence at infinitely many points in some finite domain and boundedness of the sequences $|c_{n1}|$, $|c_{n2}|$. We further generalize the assumption on the location of the zeros (following a similar remark of Weisner [5]), assuming only that the zeros of $P_n(z)$ lie in a half-plane containing the origin on its boundary, but otherwise varying with n. Finally we extend the results to certain sequences of entire functions.

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

LEMMA 1. *Given n complex numbers $\xi_1, \xi_2, \cdots, \xi_n$, which lie in a half-plane*: $\Re e^{i\theta}\xi_\nu \geqq 0$ *for some* θ. *Suppose further*

$$\left|\sum_1^n \xi_\nu\right| \leqq a, \qquad \left|\sum_1^n \xi_\nu^2\right| \leqq b; \tag{3}$$

then

$$\sum_1^n |\xi_\nu|^2 \leqq 2a^2 + b.$$

For a similar statement see [**4**, vol. 1, p. 90, problem 37]. For the proof let first $\theta = 0$, $\xi_\nu = u_\nu + iv_\nu$, so that $u_\nu \geqq 0$; it then follows that

$$\sum_1^n u_\nu^2 \leqq \left(\sum u_\nu\right)^2 \leqq a^2.$$

Also from (3)

$$\left|\sum (v_\nu^2 - u_\nu^2)\right| \leqq b,$$

hence

$$\sum v_\nu^2 \leqq b + a^2, \qquad \sum |\xi_\nu|^2 = \sum u_\nu^2 + \sum v_\nu^2 \leqq 2a^2 + b.$$

In the general case let $\xi_\nu = e^{-i\theta}\eta_\nu$, then $\Re\eta_\nu \geqq 0$, $\left|\sum\eta_\nu\right| = \left|\sum\xi_\nu\right| \leqq a$, $\left|\sum\eta_\nu^2\right| = \left|\sum\xi_\nu^2\right| \leqq b$, and from the previous result

$$\sum |\eta_\nu|^2 = \sum |\xi_\nu|^2 \leqq 2a^2 + b.$$

LEMMA 2. *For any complex* z

$$|(1 - z)e^z| \leqq e^{r^2} \quad \textit{for} \quad |z| \leqq r.$$

Elementary calculus yields easily this inequality.

LEMMA 3. *If a polynomial*

$$P(z) = \sum_0^n c_\nu z^\nu, \qquad c_0 \neq 0,\ c_n \neq 0,$$

has all its roots in a half-plane $\Re e^{i\theta}z \geqq 0$, *then*

$$|P(z)| \leqq |c_0| \exp\left(r\left|\frac{c_1}{c_0}\right| + 3r^2\left(\left|\frac{c_1}{c_0}\right|^2 + \left|\frac{c_2}{c_0}\right|\right)\right) \quad \textit{for } |z| \leqq r.$$

Let

$$P(z) = c_0\prod_1^n\left(1 - \frac{z}{z_\nu}\right), \qquad P(z)P(-z) = c_0^2\prod_1^n\left(1 - \frac{z^2}{z_\nu^2}\right),$$

so that

$$(4) \qquad -c_0\sum \frac{1}{z_\nu} = c_1, \qquad -c_0^2\sum \frac{1}{z_\nu^2} = 2c_0c_2 - c_1^2.$$

The numbers $1/z_\nu = \xi_\nu$ lie in the half-plane $\Re e^{-i\theta}z \geqq 0$, hence Lemma 1 and (4) yield

$$(5) \qquad \sum \frac{1}{|z_\nu|^2} \leqq 3\left|\frac{c_1}{c_0}\right|^2 + 2\left|\frac{c_2}{c_0}\right|.$$

Now

$$P(z) = c_0 \exp\left(-z\sum \frac{1}{z_\nu}\right)\prod\left(1 - \frac{z}{z_\nu}\right)e^{z/z_\nu},$$

and using Lemma 2 we have

$$|P(z)| \leqq |c_0| \exp\left(r\left|\frac{c_1}{c_0}\right| + r^2\left(-3\left|\frac{c_1}{c_0}\right|^2 + 2\left|\frac{c_2}{c_0}\right|\right)\right) \quad \text{for } |z| \leqq r.$$

This proves Lemma 3. An immediate consequence is the following theorem:

THEOREM I. *Given a sequence of polynomials*

$$(6) \qquad P_n(z) = \sum_{\nu=0}^{m} c_{n\nu}z^\nu, \qquad m = m_n \uparrow \infty,\ c_{n0} \neq 0,\ c_{nm} \neq 0;$$

suppose that the roots of $P_n(z)$ *lie in a half-plane* $\Re e^{i\theta_n}z \geqq 0$, *and suppose that for some constants* α_0, α_1,

$$(7) \qquad 0 < \alpha_0 \leqq |c_{n0}| \leqq \alpha_1, \quad |c_{n1}| \leqq \alpha_1, \quad |c_{n2}| \leqq \alpha_1 < \infty, \quad \textit{for all } n.$$

Then the sequence (6) *is uniformly bounded in any circle* $|z| \leqq r$; *in fact*

$$(8) \qquad |P_n(z)| \leqq \alpha_1 \exp\left(r\frac{\alpha_1}{\alpha_0} + 3r^2\left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0}\right)\right).$$

The roots $z_{n\nu}$, $\nu = 1, 2, \cdots, m$ of $P_n(z)$ satisfy the inequality

$$(9) \qquad \sum_{\nu=1}^{m} \frac{1}{|z_{n\nu}|^2} \leqq 3\left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0}\right).$$

This inequality follows from (5) and (7).

Another way of getting Theorem I is by applying a theorem of Hermite and Biehler [**4**, vol. 1, p. 88, problem 25].

Using well known results (see for example [**3**, pp. 21–30] or Vitali's convergence theorem), it follows from Theorem I that every subsequence of (6) contains a subsequence convergent everywhere, and uniformly in every circle $|z| < r$. Thus the sequence (6) has either one or several limit functions $F(z)$, and $F(z)$ is an entire function of the form (cp. (8) and (9))

$$F(z) = e^{\gamma_0+\gamma_1 z+\gamma_2 z^2}\prod\left(1 - \frac{z}{z_\nu}\right)e^{z/z_\nu}, \qquad \sum \frac{1}{|z_\nu|^2} < \infty. \tag{10}$$

In particular $F(z)$ is unique if the sequence (6) converges at infinitely many points with a finite limit point. A necessary and sufficient condition for this case is (see for a similar situation [**1**, §§1 and 2]) that each of the limits

$$\lim_{n\to\infty} c_{n\nu} = c_\nu \qquad \nu = 0, 1, 2, \cdots, \tag{11}$$

exists. In this case (7) can be replaced by $c_0 \neq 0$. For (11) we can write:

$$\lim_{n\to\infty} P_n^{(\nu)}(0) = \nu! c_\nu, \qquad \nu = 0, 1, 2, \cdots,$$

exists. It then follows from our result that the limits

$$\lim_{n\to\infty} P_n^{(\nu)}(z) = F^{(\nu)}(z), \qquad \nu = 0, 1, 2, \cdots,$$

exist uniformly in any finite domain.

An immediate corollary of our result is this theorem.

THEOREM II. *If a formal power series $\sum c_\nu z^\nu$ has infinitely many partial sums $s_n(z) = \sum_0^n c_\nu z^\nu$, such that the roots of $s_n(z)$ lie in a half-plane $\Re e^{i\theta_n} z \geqq 0$, then the power series represents an entire function of the form* (10).[2]

A linear transformation enables us to shift the role of the point $z=0$ in our results to any point z_0 in the plane. Thus let $w = z - z_0$, $P_n(z) = P_n(w+z_0) = Q_n(w)$. If the roots of $P_n(z)$ lie in a half-plane $\Re e^{i\theta_n}(z - z_0) \geqq 0$, then the roots of $Q_n(w)$ lie in a half-plane $\Re e^{i\theta_n} w \geqq 0$. If further

$$\lim_{n\to\infty} P_n^{(\nu)}(z_0) = \nu! c_\nu(z_0) \text{ exists for all } \nu, \text{ and } c_0(z_0) \neq 0, \tag{12}$$

then

[2] In this connection I refer to a paper by E. Benz, *Über lineare, verschiebungstreue Funktional operationen und die Nullstellen ganzer Funktionen*, Comment. Math. Helv. vol. 7 (1935) pp. 243–289, which was pointed out to me by the referee.

$$\lim_{n\to\infty} Q_n^{(\nu)}(0) = \nu ! c_\nu(z_0).$$

We thus get this theorem:

THEOREM III. *If the sequence of polynomials* (6) *is such that the roots of* $P_n(z)$ *lie in a half-plane* $\Re e^{i\theta_n}(z-z_0) \geqq 0$, *if moreover* (a) *either* (12) *holds, or* (b) $0<\beta_0 \leqq |P_n(z_0)| \leqq \beta_1$, $|P_n'(z_0)| \leqq \beta_1$, $|P_n''(z_0)| \leqq \beta_1$ *for some constants* β_0, β_1, *and for all* n *and* $\lim_{n\to\infty} P_n(z)$ *exists at infinitely many points in a finite domain, then* $P_n(z)$ *converges uniformly in every finite domain to an entire function of the form* (10).

Instead of polynomials we may consider more generally primitive entire functions of genus 0, thus

$$(13)\quad F_n(z) = \sum_{\nu=0}^{\infty} c_{n\nu} z^\nu = c_{n0} \prod_\nu \left(1 - \frac{z}{z_{n\nu}}\right), \qquad \sum_\nu \frac{1}{|z_{n\nu}|} < \infty,$$
$$n = 1, 2, 3, \cdots .$$

Theorem I remains true if we replace P_n by F_n, assuming accordingly that the roots of $F_n(z)$ lie in a half-plane $\Re e^{i\theta_n} z \geqq 0$ and that (7) holds. To prove this we note that

$$c_{n1} = - c_{n0} \sum_\nu \frac{1}{z_{n\nu}}, \qquad - c_{n0}^2 \sum \frac{1}{z_{n\nu}^2} = 2 c_{n0} c_{n2} - c_{n1}^2,$$

hence

$$\left| \sum_\nu z_{n\nu}^{-1} \right| \leqq \frac{\alpha_1}{\alpha_0}, \qquad \left| \sum_\nu z_{n\nu}^{-2} \right| \leqq 2\frac{\alpha_1}{\alpha_0} + \frac{\alpha_1^2}{\alpha_0^2} .$$

Lemma 1 evidently applies to absolutely convergent series, and yields

$$\sum_\nu |z_{n\nu}|^{-2} \leqq 3\frac{\alpha_1^2}{\alpha_0^2} + 2\frac{\alpha_1}{\alpha_0} .$$

The corresponding extension of Lemma 3 yields

$$|F_n(z)| \leqq c_0 \exp\left\{ r\frac{\alpha_1}{\alpha_0} + 3r^2\left(\frac{\alpha_1^2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_0}\right)\right\} \qquad \text{for } |z| \leqq r;$$

thus the sequence (13) is uniformly bounded in a given circle $|z| \leqq r$. The rest of the argument is the same as in the case of polynomials.

A similar reasoning holds for a more general class of entire functions. We assume that $F_n(z)$ is of the form

$$(14)\qquad F_n(z) = \sum_\nu c_{n\nu} z^\nu = c_{n0} e^{\gamma_n z} \prod_\nu (1 - z z_{n\nu}^{-1}) e^{z z_{n\nu}^{-1}},$$

where $\Re e^{i\theta_n} z_{n\nu} \geqq 0$, $\sum_\nu |z_{n\nu}|^{-2} < \infty$. We assume furthermore that the series $\sum_\nu (\Re e^{-i\theta_n} z_{n\nu}^{-1}) = \beta_n$ converges and that $\beta_n \leqq \beta$ for all n. Assuming (7) we shall prove that the sequence (14) is uniformly bounded in any circle $|z| < r$.

We have

$$\frac{F_n'(z)}{F_n(z)} = \gamma_n + \sum_\nu \left(\frac{1}{z - z_{n\nu}} + \frac{1}{z_{n\nu}} \right),$$

$$F_n''(z) F_n(z) - (F_n'(z))^2 = -F_n^2(z) \sum_\nu \frac{1}{(z - z_{n\nu})^2},$$

and, putting $z = 0$,

$$c_{n1} = c_{n0}\gamma_n, \qquad 2c_{n0}c_{n2} - c_{n1}^2 = -c_{n0}^2 \sum_\nu z_{n\nu}^{-2}.$$

Now from (7)

$$|\gamma_n| \leqq \frac{\alpha_1}{\alpha_0}, \qquad \left| \sum_\nu z_{n\nu}^{-2} \right| \leqq \left(\frac{\alpha_1}{\alpha_0} \right)^2 + 2\frac{\alpha_1}{\alpha_0};$$

furthermore putting $e^{-i\theta_n} z_{n\nu}^{-1} = u_{n\nu} + i v_{n\nu}$, we have $u_{n\nu} \geqq 0$, and (as in Lemma 1)

$$\sum_\nu |z_{n\nu}|^{-2} = \sum_\nu |e^{-i\theta_n} z_{n\nu}^{-1}|^2 = 2\sum_\nu u_{n\nu}^2 - \sum_\nu \Re(e^{-i\theta_n} z_{n\nu}^{-1})^2$$
$$\leqq 2\beta^2 + \left(\frac{\alpha_1}{\alpha_0} \right)^2 + 2\frac{\alpha_1}{\alpha_0}.$$

Thus, using Lemma 2, we have

$$|F_n(z)| < \alpha_1 \exp\left\{ \frac{\alpha_1}{\alpha_0} r + \left(2\beta^2 + 2\frac{\alpha_1}{\alpha_0} + \frac{\alpha_1^2}{\alpha_0^2} \right) r^2 \right\} \qquad \text{for } |z| < r,$$

which proves our assertion.

If we assume that $\sum_\nu z_{n\nu}^{-1} = \xi_n$ converges, and put $\gamma_n = -\xi_n$, then the assumption on the β_n is superfluous.

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UNIVERSITY OF CINCINNATI

A NOTE ON SEPARATION AXIOMS AND THEIR APPLICATION IN THE THEORY OF A LOCALLY CONNECTED TOPOLOGICAL SPACE

J. W. T. YOUNGS

In a recent paper [1][1] G. E. Albert and the author attempt a comprehensive study of a locally connected (1.8) topological space from the point of view of Peano space theory [2]. Cyclic elements are defined (2.15) and are themselves found to be locally connected and topological (2.29). Moreover, it is shown that under proper and very natural topologization (3.3) the class of all cyclic elements (the hyperspace) becomes a locally connected topological space (3.3 and 3.8). In fact, this hyperspace has no nondegenerate[2] cyclic elements (3.17).

For the purposes of this note it is the concept of a hereditary class of spaces which is important (4.1). A subclass $\mathcal{H}$ of the class $\mathfrak{X}$ of all locally connected topological spaces is hereditary if, whenever X is a space of the class $\mathcal{H}$: (1) each true cyclic element (2.15) of X is a member of $\mathcal{H}$; and (2) the hyperspace X_h is in $\mathcal{H}$. (It should be remembered that the first condition is the one required of a class for it to be cyclicly reducible in the classical Peano space theory.)

The problem is to define small hereditary classes (4.1). In fact, though there is a smallest hereditary class, an intrinsic definition of it is lacking (4.2–4.5).

In this connection the main results are that: (1) the class of all locally connected T_0-spaces is a hereditary class (4.10); (2) the class of all locally connected T_1-spaces is *not* a hereditary class (4.1).

It is the purpose of this note: (1) to define, by means of a separation axiom, a new hereditary class; (2) to place this separation axiom

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[1] Numbers in brackets refer to the bibliography; numbers in parentheses to appropriate paragraphs in [1].

[2] A set is degenerate if it is vacuous or contains but a single point.