DIOPHANTINE EQUATIONS OF DEGREE[1] n

A. A. AUCOIN

In a recent issue of the National Mathematics Magazine,[2] W. V. Parker and the author obtained solutions of the Diophantine equation $F(x_1, \cdots, x_p) = G(y_1, \cdots, y_q)$, where F and G are homogeneous polynomials, with integral coefficients, of degree 3, and F is such that for a set of integers $x_i = a_i$ (not all zero), $\partial F/\partial x_i = 0$, $(i = 1, \cdots, p)$. In this paper the above is extended to functions of degree n. One type, which satisfies the conditions of the main theorem, is also solved by an entirely different method. The solutions obtained are in terms of arbitrary parameters, and they are integral for an integral choice of the parameters.

If $x_i = \alpha_i$, $y_k = \beta_k$ is a solution of the equation $f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q)$, where f and g are homogeneous polynomials, with integral coefficients, of degrees n and m respectively, and there are no integers $s > 1$, α_i', β_k' such that $\alpha_i = s^\lambda \alpha_i'$, $\beta_k = s^\mu \beta_k'$ where λ, μ are relatively prime positive integers such that $\lambda n = \mu m$, then $x_i = \alpha_i$, $y_k = \beta_k$ is said to be a primitive solution. If $x_i = \alpha_i$, $y_k = \beta_k$ is a primitive solution of the above equation, then $x_i = \alpha_i t^\lambda$, $y_k = \beta_k t^\mu$ (derived from this primitive solution), where λ, μ are any positive integers such that $\lambda n = \mu m$, is also a solution. Two solutions are said to be equivalent if they are derived from the same primitive solution.

THEOREM 1. *Let $f(x_1, \cdots, x_p)$, $g(y_1, \cdots, y_q)$ be homogeneous polynomials with integral coefficients, of degrees n and m respectively. Let $a_1, \cdots, a_p$ be integers not all zero such that the partial derivatives of f of all orders less than $n-1$ vanish[3] when $x_i = a_i$. Then every solution in integers x_i, y_k of*

$$f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q), \tag{1}$$

for which

$$\sum_{j=1}^{p} a_j \frac{\partial f}{\partial x_j} \neq 0, \tag{2}$$

is equivalent to one of the infinitude of solutions given by

$$x_i = a_i s t^{\lambda-1} + \alpha_i t^\lambda, \quad y_k = \beta_k t^\mu, \quad i = 1, 2, \cdots, p;\ k = 1, 2, \cdots, q, \tag{3}$$

[1] Presented to the Society, December 2, 1939.

[2] *On cubic Diophantine equations*, vol. 13 (1938), pp. 115–117.

[3] It follows from Euler's theorem that the function itself vanishes for this choice of x_i.

where λ, μ *are positive integers such that* $\lambda n=\mu m$, α_i *and* β_k *are arbitrary integers,*

$$(4)\qquad s=g(\beta)-f(\alpha),\qquad t=\sum_{j=1}^{p}a_j\frac{\partial f}{\partial\alpha_j},$$

and $f(\alpha)=f(\alpha_1,\cdots,\alpha_p)$, $g(\beta)=g(\beta_1,\cdots,\beta_q)$.

PROOF. By Taylor's theorem, if we set $x_i=a_is+\alpha_it$,

$$f(x_1,\cdots,x_p)=st^{n-1}\sum_{j=1}^{p}a_j\frac{\partial f}{\partial\alpha_j}+t^nf(\alpha_1,\cdots,\alpha_p).$$

Hence if x_i, y_k have the values given by (3), s and t those given by (4), (1) becomes $st^{n\lambda-1}\sum_{j=1}^{p}a_j\partial f/\partial\alpha_j+t^{n\lambda}f(\alpha)=t^{m\mu}g(\beta)$, and is satisfied identically in the α_i and β_k. Hence (3) is a solution of (1) with s and t given by (4).

Suppose $x_i=\rho_i$, $y_k=\sigma_k$ is any solution of (1) and (2). If we choose $\alpha_i=\rho_i$, $\beta_k=\sigma_k$, we have that $s=0$, and (3) becomes $x_i=\rho_it^\lambda$, $y_k=\sigma_kt^\mu$, equivalent to the given solution, since by (2), $t\neq 0$.

If $g\equiv 0$, the theorem still holds, with λ arbitrary.

COROLLARY. *The equation* $f(x)=\sum_{j=1}^{p}x_jg_j(y)+g(y)$, *where* $g_j(y)=g_j(y_1,\cdots,y_q)$ *and* $g(y)=g(y_1,\cdots,y_q)$ *are homogeneous polynomials with integral coefficients of degrees* $n-1$ *and* n, *respectively, has solutions, and every solution which is not also a solution of* $\sum_{j=1}^{p}a_j[\partial f/\partial x_j-g_j(y)]=0$ *is equivalent to one of the infinitude of solutions given by* $x_i=a_is+\alpha_it$, $y_k=\beta_kt$ *where*

$$s=g(\beta)-f(\alpha)-\sum_{j=1}^{p}\alpha_jg_j(\beta),\qquad t=\sum_{j=1}^{p}a_j\left[\frac{\partial f}{\partial\alpha_j}-g_j(\beta)\right].$$

One function of interest which satisfies the hypothesis of Theorem 1 is the function $D(x)=|a_{ij}x_{ij}|$, a determinant of order n with a_{ij} integral such that not all the a's in any row or column are zero. For this function not all the x_{ij} need be distinct. If there is any x_{ij}, say x_{pq}, which occurs only once in D, we may make the choice $x_{pq}=1$, $x_{ij}=0$ otherwise; then all the partial derivatives of all orders less than $n-1$ vanish.

In the solution the form of the expression is the same for every element except x_{pq}. This fact is illustrated by the equation $D(x)=g(y)$, the solution of which is $x_{pq}=st^{\lambda-1}+\alpha_{pq}t^\lambda$, $x_{ij}=\alpha_{ij}t^\lambda$, $(i\neq p,\ j\neq q)$, $y_k=\beta_kt^\mu$ where $s=g(\beta)-D(\alpha)$, $t=D'(\alpha)$, $D'(\alpha)$ being $D(\alpha)$ with a_{pq} the element in the pth row and qth column, and the other elements in the qth column zero.

It is not necessary, in some cases, that there be a unique element x_{pq}. If $a_{ij}=1$, for example, D may be the circulant. In this case we make the choice $x_{ij}=1$.

Another function of interest which also satisfies the condition of Theorem 1 is $P(x)=\prod_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_j$, where a_{ij} are integral, and the determinant $A=|a_{ij}|\neq 0$. For we may choose x_j, integral, so that $n-1$ of the above factors vanish and hence for this choice of x_j all partial derivatives of all orders less than $n-1$ vanish.

The next equation satisfies the hypothesis of Theorem 1, but will be solved by an entirely different method. This is given in the following theorem:

THEOREM 2. *The equation*

$$P(x) = g(y), \tag{5}$$

where $P(x)$ is given above and $g(y)$ is given in Theorem 1, *has solutions, and every solution which is not also a solution of*

$$\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}x_j = 0 \tag{6}$$

is equivalent to one of the infinitude of solutions given by

$$\begin{aligned} x_j &= t^{\lambda}(A_{nn})^{\lambda-1}A_{nn}^{(j)} + st^{\lambda-1}(A_{nn})^{\lambda-1}A_{nj}, \qquad j = 1, \cdots, n-1, \\ x_n &= st^{\lambda-1}(A_{nn})^{\lambda}, \qquad y_k = t^{\mu}(A_{nn})^{\mu}\beta_k, \end{aligned} \tag{7}$$

where A_{ij} is the cofactor of a_{ij} in[4] *A; $A_{nn}^{(j)}$ is the determinant obtained from A_{nn} by replacing the jth column by $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$; s and t are given by*

$$s = A_{nn}g(\beta) - \prod_{i=1}^{n-1}\alpha_i\sum_{j=1}^{n}a_{nj}A_{nn}^{(j)}, \qquad t = A\prod_{i=1}^{n-1}\alpha_i, \tag{8}$$

λ, μ are relatively prime positive integers such that $\lambda n=\mu m$, and the α's and β's are arbitrary integers.

PROOF. Set

$$\sum_{j=1}^{n} a_{ij}x_j = t^{\lambda}(A_{nn})^{\lambda}\alpha_i, \qquad i = 1, \cdots, n-1. \tag{9}$$

If we let $x_n=st^{\lambda-1}(A_{nn})^{\lambda}$, we may write equation (9) in the form

[4] Since $A\neq 0$, there is a minor of order $n-1$ which does not vanish. Without loss of generality, we may choose the notation so that $A_{nn}\neq 0$.

$\sum_{j=1}^{n-1} a_{ij}x_j = t^\lambda (A_{nn})^\lambda \alpha_i - st^{\lambda-1}a_{in}(A_{nn})^\lambda$. Solving this system of equations we get

$$(10)\qquad x_j = t^\lambda (A_{nn})^{\lambda-1} A_{nn}^{(j)} + st^{\lambda-1}(A_{nn})^{\lambda-1}A_{nj}, \qquad j = 1, \cdots, n-1.$$

It follows then that

$$(11)\qquad \sum_{j=1}^{n} a_{nj}x_j = t^{\lambda-1}(A_{nn})^{\lambda-1}\left[t\sum_{j=1}^{n-1} a_{nj}A_{nn}^{(j)} + sA\right].$$

If we let $y_k = t^\mu (A_{nn})^\mu \beta_k$, then by (9) and (11), (5) becomes

$$(12)\qquad t^{n\lambda-1}(A_{nn})^{n\lambda-1}\prod_{i=1}^{n-1}\alpha_i\left[t\sum_{j=1}^{n-1} a_{nj}A_{nn}^{(j)} + sA\right] = t^{m\mu}(A_{nn})^{m\mu}g(\beta),$$

and since $\lambda n = \mu m$, (12) is identically satisfied in the α's and β's if s and t are given by (8). Hence (7) forms a solution of (5) with s and t given by (8).

Suppose now that $x_j = \rho_j$, $y_k = \sigma_k$ is any solution of (5). If we choose $\alpha_i = \sum_{j=1}^{n} a_{ij}\rho_j$, $\beta_k = \sigma_k$, we have[5]

$$\begin{aligned} t &= A\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j, \\ s &= A_{nn}g(\sigma) - \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j \sum_{k=1}^{n-1} a_{nk}[\rho_k A_{nn} - \rho_n A_{nk}] \\ &= A_{nn}\prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\rho_j - \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j \sum_{k=1}^{n-1} a_{nk}[\rho_k A_{nn} - \rho_n A_{nk}] \\ &= \prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j\left[A_{nn}\sum_{j=1}^{n} a_{nj}\rho_j - A_{nn}\sum_{j=1}^{n-1} a_{nj}\rho_j + \rho_n\sum_{j=1}^{n-1} a_{nj}A_{nj}\right] \\ &= \rho_n A\prod_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\rho_j = \rho_n t. \end{aligned}$$

Hence

$$\begin{aligned} x_j &= t^\lambda (A_{nn})^{\lambda-1}[\rho_j A_{nn} - \rho_n A_{nj}] + (A_{nn})^{\lambda-1}t^\lambda \rho_n A_{nj} = (tA_{nn})^\lambda \rho_j, \\ x_n &= (tA_{nn})^\lambda \rho_n, \qquad\qquad y_k = (tA_{nn})^\mu \sigma_k, \end{aligned}$$

which is equivalent to the given solution provided $x_j = \rho_j$, $y_k = \sigma_k$ is not a solution of (6). We may find, however, values of x_j which satisfy (6) and these values, together with $y_k = 0$, afford additional solutions of (5).

[5] $A_{nn}^{(j)}$ becomes $\rho_j A_{nn} - \rho_n A_{nj}$ when α_i is replaced by $\sum_{j=1}^{n} a_{ij}\rho_j$.

By the above method we may show the following consequence.

COROLLARY. *The equation* $P(x)=\sum_{j=1}^{n}x_jg_j(y)+g(y)$, *where* g_j *and* g *are the functions of the corollary to Theorem* 1, *has solutions, and every solution not also a solution of* $\sum_{j=1}^{n}A_{nj}g_j(y)-A\prod_{i=1}^{n-1}\sum_{j=1}^{n}a_{ij}x_j=0$ *is given by* $x_j=A_{nn}^{(j)}t+A_{nj}s$, $(j=1,\cdots,n-1)$, $x_n=A_{nn}s$, $y_k=A_{nn}\beta_kt$, *where*

$$s=\prod_{i=1}^{n-1}\alpha_i\sum_{j=1}^{n}a_{nj}A_{nn}^{(j)}-\sum_{j=1}^{n-1}A_{nn}^{(j)}g_j(\beta)-A_{nn}g(\beta),$$

$$t=\sum_{j=1}^{n}A_{nj}g_j(\beta)-A\prod_{i=1}^{n-1}\alpha_i.$$

The final theorem treats an equation which satisfies the hypothesis of Theorem 1, but is reduced to an equivalent problem and then solved.

THEOREM 3. *The equation*

$$f(x)\sum_{j=1}^{p}d_jx_j=R(y), \tag{13}$$

where $f(x)$ *satisfies the conditions of Theorem* 1, *and* $R(y)=R(y_1,\cdots,y_q)$ *is a homogeneous polynomial with integral coefficients of degree* $n-1$, *has solutions; and every solution which is not also a solution of*

$$f(x)\sum_{j=1}^{p}a_j\frac{\partial f}{\partial x_j}\left[\sum_{j=1}^{p}a_j\frac{\partial f}{\partial x_j}\sum_{k=1}^{p}d_kx_k-f(x)\sum_{j=1}^{p}d_ja_j\right]=0 \tag{14}$$

is equivalent to one of the infinitude of solutions given by

$$x_i=a_is+\alpha_it,\qquad y_k=\beta_kt \tag{15}$$

where

$$\begin{aligned} s&=A^{n-1}[\lambda(AD-BC)]^{n-2}[D\lambda^2-BR(\mu)],\\ t&=A^{n-1}[\lambda(AD-BC)]^{n-2}[AR(\mu)-C\lambda^2],\\ \beta_k&=A^2[\lambda(AD-BC)]^2\mu_k,\end{aligned} \tag{16}$$

and $A=\sum_{j=1}^{p}a_j\partial f/\partial\alpha_j$, $B=f(\alpha)$, $C=\sum_{j=1}^{p}d_ja_j$, $D=\sum_{j=1}^{p}d_j\alpha_j$, *the* α_j, λ, μ_k *being arbitrary integers.*

PROOF. If we let x_i, y_k have the values given by (15), (13) becomes, after dividing out the factor[6] t^{n-1},

$$(As+Bt)(Cs+Dt)=R(\beta). \tag{17}$$

[6] It will be shown later that $t\neq 0$.

By Theorem 2, the solutions of the equation (17) are given by (16).

If $x_i=\rho_i$, $y_k=\sigma_k$ is any solution of (13) and we choose $\alpha_i=\rho_i$, $\mu_k=\sigma_k$, $\lambda=f(\rho)$, we have that $s=0$ and the solution becomes $x_i=\rho_i K^{n-1}$, $y_k=\sigma_k K^{n+1}$, where $K=A\lambda(AD-BC)$, which is equivalent to the given solution provided $K\neq 0$; that is, provided $x_i=\rho_i$, $y_k=\sigma_k$ is not a solution of (14). It will be noted that if $K\neq 0$, then $t\neq 0$.

LOUISIANA STATE UNIVERSITY

A MULTIPLE NULL-CORRESPONDENCE AND A SPACE CREMONA INVOLUTION OF ORDER $2n-1$[1]

EDWIN J. PURCELL

PART I. A NULL-SYSTEM $(1, mn, m+n)$ BETWEEN THE PLANES AND POINTS OF SPACE $(m, n=1, 2, 3, \cdots)$

1. **Introduction.** Consider a curve δ_m of order m having $m-1$ points in common with a straight line d, and a curve δ_n' of order n having $n-1$ points in common with a straight line d', $(m, n=1, 2, 3, \cdots)$. It is assumed for the present that neither δ_m nor d intersects either δ_n' or d'.

In general, through any point P of space there passes one ray ρ which intersects δ_m once and d once, and one ray ρ' which intersects δ_n' once and d' once; ρ and ρ' determine a plane π, the null-plane of P. Conversely, a plane π determines m rays ρ_i and n rays ρ_j' lying in it which intersect, a ray ρ with a ray ρ', in mn points, the null-points of the plane π.

Any point α in general position determines a ray ρ. As α describes a line l, the plane π of ρ and l contains n rays ρ', which intersect l in n points β; conversely, any point β on l determines a ray ρ' which determines with l the plane π, and π contains m rays ρ which intersect l in m points α—one being the original α. Thus an (m, n) correspondence is set up among the points of l with valence zero; there are $m+n$ coincidences and therefore $m+n$ points on any line l whose null-planes contain l.

2. **Planes whose null-points behave peculiarly.** We can obtain the last result by another method; this will yield additional information about planes whose null-points behave peculiarly.

Let a plane π turn about a line l as axis. A ruled surface will be generated by the m rays ρ_i lying in π. This surface is of order $m+1$; δ_m is a onefold curve on the surface and d is an m-fold line. Another

[1] Presented to the Society, December 2, 1939.