THE UNIVERSALITY OF FORMAL POWER SERIES FIELDS*

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In a recent paper,† André Gleyzal has constructed ordered fields consisting of certain "transfinite real numbers" and has established the interesting result that any ordered field can be considered as a subfield of one of these transfinite fields. These fields prove to be identical with fields of formal power series in which the exponents are allowed to range over a suitable ordered abelian group. Such fields were first introduced by Hahn,‡ while they have been analyzed in terms of generalized valuations by Krull.§

Gleyzal applied his construction of transfinite real numbers not only to the case when the coefficient field consisted of real numbers, but also to suitable fields of characteristic p. He conjectured that this construction should yield a "universal" field of characteristic p. We show here that Krull's technique can be used to establish Gleyzal's conjecture.

1. **Formal power series.** If K is any field and Γ any ordered abelian group (its order may be non-archimedean), we form all power series $x=\sum a_\alpha t^\alpha$ with coefficients a_α in K, exponents α in Γ, and summed over a subset N of elements α from Γ which is normally ordered by the given linear order in Γ. Such a series could also be written as

$$(1) \qquad x = a_{\alpha_1}t^{\alpha_1} + a_{\alpha_2}t^{\alpha_2} + \cdots + a_{\alpha_\rho}t^{\alpha_\rho} + \cdots ,$$

summed over all ordinal numbers ρ less than a fixed σ, and with exponents $\alpha_1 < \alpha_2 < \cdots < \alpha_\rho < \cdots$ increasing monotonically. The product of two formal powers t^α and t^β is defined as $t^{\alpha+\beta}$, where $\alpha+\beta$ is the sum in the group Γ. On this basis, the usual formal definitions of multiplication and addition make the set of all series (1) a field, which we denote by $K\{t^\Gamma\}$.

Hahn also gave a similar construction for a non-archimedean ordered group from a given ordered group G (say an additive group of

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† A. Gleyzal, *Transfinite numbers*, Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 581–587.

‡ H. Hahn, *Über die nichtarchimedischen Grössensysteme*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Vienna, section IIa, vol. 116 (1907), pp. 601–653.

§ W. Krull, *Allgemeine Bewertungstheorie*, Journal für die reine und angewandte Mathematik, vol. 167 (1931), pp. 160–196.

real numbers) and any ordered set B. One simply chooses for each b in B a "basis" element e_b, and then forms all sums $\alpha=\sum g_b e_b$, with g_b in G, taken over the indices b of a subset M of B which is well ordered by the given order of B. The formal sum of two such expressions α is again a similar expression. Under the usual lexicographic ordering, all these expressions constitute an ordered group $G\{e_B\}$.

In particular, one may start with the ordered (additive) group R of all real numbers, construct the group $\Gamma=R\{e_B\}$ and thence the field $K=R'\{t^\Gamma\}$, where R' is the real number field. This particular power series field is isomorphic to Gleyzal's field of "transfinite real numbers* with base order B."

Hahn also showed how the group Γ could be redistilled from the ordered field $K=R'\{t^\Gamma\}$ by considering each element in Γ as a set of all elements in K of the same "order of magnitude." If this process is applied again, the orders of magnitude in Γ yield the original† ordered set B.

2. **Algebraic closure.** A *root group* Γ is one with the property that for every integer n and every α in Γ there is a γ in Γ with $n\gamma=\alpha$. The particular group $R\{e_B\}$ is a root group.

THEOREM 1. *The power series field $K\{t^\Gamma\}$ is algebraically closed if the coefficient field K is algebraically closed and if the ordered abelian group Γ of exponents is a root group.*

PROOF. In the power series field $S=K\{t^\Gamma\}$ we introduce a valuation V by setting $V(x)=\alpha_1$ if $a_{\alpha_1}t^{\alpha_1}$ is the first nonvanishing term in the power series (1) for x. In this valuation, Γ is the value group and K the field of residue classes. Furthermore,‡ S is maximal with respect to this valuation, in the sense that any proper extension $S'>S$ to which the valuation V has been extended must either have a larger value group or a larger residue class field than S.

Suppose now that S is not algebraically closed, so that S has a proper finite normal extension N. Certainly S is (algebraically) perfect, so that N/S is separable. The valuation V of S can be extended to N by the usual methods, for S is (topologically) perfect§ with respect to V. The ordinary Newton polygon construction shows that

* To establish the isomorphism, observe that Gleyzal's product forms Π_μ form a multiplicative group isomorphic to $R\{e_B\}$, and that his transfinite real numbers or sum forms are manipulated exactly as are the formal series (1).

† This twofold order of magnitude process yields essentially Gleyzal's base order for K.

‡ Krull, loc. cit., Theorem 26.

§ Krull, loc. cit., §9 and Theorem 27.

each element c of N has a value of the form α/n, with α in Γ. Since Γ is a root group, $\alpha/n \in \Gamma$, and Γ is thus the value group of N. On the other hand, the residue class field of N must be an algebraic extension of the algebraically closed residue class field K of S. Thus N presents a proper extension of S in which neither value group nor residue class field is extended, contrary to the maximal property of S.

3. **Universal fields.** We call a field F *universal* if every other field F' which has the same cardinal number and the same characteristic as F is isomorphic to a subfield of F.

THEOREM 2. *A nondenumerable field F is universal if and only if it contains an algebraically closed subfield F_0 which has the same cardinal number as F.*

PROOF. We need only provide a map of any F' into F. As in the Steinitz theory, let T_0, T' be respectively transcendence bases for F_0, F' over the prime subfield P. By the axiom of choice, the cardinal number of T_0 is some aleph. Since P is at most denumerable, the usual computations with cardinal numbers then show that $P(T_0)$ and its algebraic extension F_0 must have the same cardinal number as does T_0. Since F_0 and F' have the same cardinal number, we conclude that T_0 and T' have the same cardinal number, so we can set T_0 and T' into one-to-one correspondence. This correspondence gives an isomorphism of $P(T')$ to $P(T_0)$. Because F is algebraically closed, this isomorphism can be extended* from $P(T')$ to its algebraic extension F'. We have proved F_0 and hence F universal.

THEOREM 3. *If the ordered abelian root group Γ contains an element different from 0, while the coefficient field K is algebraically closed, then the power series field $K\{t^\Gamma\}$ is universal.*

This field $K\{t^\Gamma\}$ includes all ordinary power series in t^α, with $\alpha \neq 0$ in Γ, hence has at least the power of the continuum. Theorem 3 thus follows from Theorems 1 and 2. In particular, this theorem yields relatively simple universal fields whose cardinal numbers are that of the continuum; for example, the fields $K\{t^\Gamma\}$ with K the absolutely algebraic algebraically closed field of any characteristic and Γ the group of all rational numbers. This theorem also includes Gleyzal's conjectured fields, where K is absolutely algebraic and Γ is any one of the groups $R\{e_B\}$ of §1.

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* B. L. van der Waerden, *Moderne Algebra*, §60.