REPRESENTATION OF NUMBERS IN TERNARY QUADRATIC FORMS

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We employ integral quaternions $t=t_0+t_1i_1+t_2i_2+t_3i_3$, where the coordinates t_i range over rational integers, while the i_1, i_2, i_3, satisfy the multiplication table

$$i_1^2 = i_2^2 = -2,\quad i_3^2 = -3,\quad i_2i_3 = 2i_1 - i_2 = (\overline{i_3i_2}),$$
$$i_3i_1 = -i_1 + 2i_2 = (\overline{i_1i_3}),\qquad i_1i_2 = -1 + i_3 = (\overline{i_2i_1}),$$

and $\bar{t}=t_0-t_1i_1-t_2i_2-t_3i_3$ is the conjugate to t. The norm $N(t)$ of t is $t\bar{t}=\bar{t}t=t_0^2+2t_1^2+2t_2^2+2t_1t_2+3t_3^2$. The norm of a product of two quaternions equals the product of their norms, and $\overline{vt}=\bar{t}\bar{v}$ for any two quaternions. The associative law $rs\cdot t=r\cdot st$ holds.

The quaternary quadratic $Q=t_0^2+2t_1^2+2t_2^2+2t_1t_2+3t_3^2$ has determinant 9, the g.c.d. of the literal coefficients of the adjoint to Q is 3, and the second concomitant of Q represents no residues 1 modulo 3, and as there is only one form of determinant 9 with these properties in Charve's table* of reduced quaternary quadratic forms, Q belongs to a genus of one class. Since Q represents 1 for two values of $t_0, \cdots, t_3$, we have,† a proper quaternion being defined as one having coprime coordinates, the following theorem:

THEOREM 1. *A proper quaternion $v=v_0+v_1i_1+v_2i_2+v_3i_3$ whose norm is divisible by a positive integer m prime to 6 has exactly two right-divisors (left-divisors) t and $-t$ of norm m.*

Every proper pure quaternion $s=s_1i_1+s_2i_2+s_3i_3$ of norm km^2 is of form $\bar{t}at$ where $N(a)=k$ and $N(t)=m$. For, $s=vt$ where $N(t)=m$ by Theorem 1; $\bar{s}=-s=\bar{t}\bar{v}$, and $\bar{t}$ is a left-divisor of s. Hence, since $N(v)=km$, $\bar{t}$ is a left-divisor of the proper quaternion v, $v=\bar{t}a$. Hence $s=\bar{t}at$, $N(a)=k$. Clearly a is pure since $\bar{t}\bar{a}t=-\bar{t}at$, $\bar{a}=-a$.

THEOREM 2. *Consider the equation $24n+1=x_1^2+2x_2^2+2x_3^2-2x_2x_3$. If $24n+1=m^2$, $(m>0)$, then all proper solutions are of type A if $m\equiv 1$* (mod 4) *but of type B if $m\equiv 3$* (mod 4), *where*

$$A:\ x_1\equiv \pm 1 \pmod{12},\qquad B:\ x_1\equiv \pm 5 \pmod{12}.$$

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* L. Charve, Comptes Rendus de l'Académie des Sciences, vol. 96 (1883), p. 773.

† G. Pall, *On the factorization of generalized quaternions*, submitted to the Duke Mathematical Journal.

If $24n+1$ is not a square, there are equally many solutions of each type A and B.

A proof for the case in which $24n+1=m^2$ follows. Consider

$$h = x_1^2 + 2x_2^2 + 2x_3^2 - 2x_2x_3. \tag{1}$$

Then

$$3h = 3x_1^2 + 2(-x_2 - x_3)^2 + 2(-x_2 - x_3)(2x_2 - x_3) + 2(2x_2 - x_3)^2.$$

Put $h=m^2$, and let $x=i_1(-x_2-x_3)+i_2(2x_2-x_3)+i_3x_1$ represent a solution (x_1, x_2, x_3) of (1); then all proper pure quaternions x are given by $x=\bar{t}at$, $N(t)=m$, $N(a)=3$, and from the latter condition we must have $a=\pm i_3$. Expanding $\bar{t}at$ gives $x_1=3t_3^2-2t_1^2-2t_2^2-2t_1t_2+t_0^2$ where only $a=i_3$ is considered since $a=-i_3$ merely changes the sign of x_1 which leaves A and B unaltered.

Thus $x_1\equiv m \pmod 4$. Since $(m, 3)=1$, $x_1\equiv 1 \pmod 3$; hence when $m\equiv 1 \pmod 4$, $x_1\equiv \pm 1 \pmod{12}$, a solution of type A, but if $m\equiv 3 \pmod 4$, then $x_1\equiv \pm 5 \pmod{12}$, a solution of type B.

The case for $h=24n+1$ not a square will now be considered. Let x be a representative solution of (1) under this condition. We can choose an odd prime p such that simultaneously

$$(-3h \mid p) = 1, \qquad p \equiv 11 \pmod{12}. \tag{2}$$

By the first equation in (2) we can choose x_0 so that $3x_0^2+h\equiv 0 \pmod p$. Then by Theorem 1, $3x_0+x$ has exactly two right-divisors $\pm t$ of norm p, say

$$3x_0 + x = ut, \qquad N(t) = p.$$

Then

$$tx\bar{t} = py, \qquad \text{where} \qquad y = tu - 3x_0, \tag{3}$$

and y represents another solution of (1). If t is replaced by $-t$, y is unchanged.

We shall prove that x and y are in opposite classes A and B in view of the second equation of (2), and as multiplication by p does not alter A or B, it will suffice to show that x and $tx\bar{t}$ are solutions of opposite types.

Setting $tx\bar{t}=i_1(-y_2-y_3)+i_2(2y_2-y_3)+i_3y_1$ and expanding gives

$$y_1 = x_1(t_0^2 + 3t_3^2 - 2t_2^2 - 2t_1^2 - 2t_1t_2) + x_2(6t_2t_3 + 4t_1t_0 + 2t_2t_0) + x_3(-6t_2t_3 - 6t_3t_1 + 2t_2t_0 - 2t_1t_0).$$

In (1), x_2 and x_3 are always even, and thus

$$y_1 \equiv 3x_1 \pmod 4, \qquad y_1 \equiv x_1 \pmod 3$$

as $N(t) \equiv 11 \pmod{12}$. Hence y represents a solution of type opposite to x.

We can now establish the (1, 1) correspondence. We employ the preceding process for a fixed p, with x_0 for a solution of type A, but $-x_0$ for a solution of type B. Hence if our process carries x, a solution of type A, into y, then y is carried into x. For from (3)

$$3(-x_0) + y = (-\bar{u})\bar{t}, \qquad N(\bar{t}) = p.$$

Then, $\bar{t}yt = px$, $x = ut - 3x_0$. Further, two distinct solutions of one type cannot correspond to the same solution of the other type.

Application of other types of quaternions furnishes arithmetical proofs of the following additional results:

For representation of $24n+1$ in $x_1^2+3x_2^2+3x_3^2$ the A and B relations are

$$A: \; x_1 \equiv \pm 1 \pmod{12}, \; x_2 \text{ and } x_3 \equiv 0 \pmod 4,$$
$$x_1 \equiv \pm 5 \pmod{12}, \; x_2 \text{ and } x_3 \equiv 2 \pmod 4,$$

$$B: \; x_1 \equiv \pm 1 \pmod{12}, \; x_2 \text{ and } x_3 \equiv 2 \pmod 4,$$
$$x_1 \equiv \pm 5 \pmod{12}, \; x_2 \text{ and } x_3 \equiv 0 \pmod 4.$$

For $2(24n+1) = 3x_1^2 + x_2^2 + x_3^2$

A: (0; 1, 1), (0; 7, 7), (0; 11, 5), (4; 5, 5), (4; 11, 11), (4; 7, 1),
B: (4; 1, 1), (4; 7, 7), (4; 11, 5), (0; 5, 5), (0; 11, 11), (0; 7, 1),

where each triplet $(x_1;\, x_2,\, x_3)$ lists the least absolute residues $x_1 \pmod 8$, $x_2 \pmod{24}$, $x_3 \pmod{24}$ in a definite order.

For either form if $24n+1 = m^2$, $(m>0)$, all solutions are of type A if $m \equiv 1 \pmod 6$, but of type B if $m \equiv 5 \pmod 6$. But there are equally many solutions of each type if $24n+1$ is not a square.

These results were proved in the writer's thesis at McGill University, 1938.

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