treated very briefly, and some knowledge of various special operations is presupposed. A familiarity with the preceding volumes would not be sufficient preparation for the intelligent reading of the present one.

VIRGIL SNYDER.

A SYNOPTIC COURSE FOR TEACHERS.

Elementarmathematik vom höheren Standpunkte aus. Von F. KLEIN. Teil I: *Arithmetik, Algebra, Analysis.* Vorlesung gehalten im Wintersemester 1907–08. 5 + 590 pp. Teil II: *Geometrie.* Vorlesung gehalten im Sommersemester 1908. 6 + 515 pp. Ausgearbeitet von E. HELLINGER. Autogr. Leipzig, in Kommission bei B. G. Teubner, 1908–09.

THE volumes under review contain a course of lectures intended for prospective teachers of mathematics in the secondary schools of Germany. The objects of the course and the reasons for giving it are so well stated in the introduction to the first volume and are of such vital interest in their application to conditions in our own country, that it seems desirable to quote at length.

"In recent years" — thus does Professor Klein begin his first lecture — "a widespread interest has developed among university teachers of mathematics and the natural sciences regarding the proper training of teachers for our secondary schools. This movement is of quite recent date; for a long period previously our universities were concerned exclusively with the higher science without any reference to the needs of the secondary schools and, in fact, without attempting to bring about a connection with secondary mathematics. But what is the result of such a practice? The young student at the outset of his university work is brought face to face with problems that do not serve to remind him of what he has previously studied and naturally he proceeds to forget all of it quickly and thoroughly. On the other hand, if after leaving the university he enters upon his work as a teacher, he is required to give instruction in the established courses in elementary mathematics and, as he is unable without assistance to bring his new work into relation with his advanced mathematics, he soon adopts the old traditional methods and his university studies become merely a more or less pleasant memory which has no influence on his teaching.

"At the present time the attempt is being made to destroy this *double discontinuity* which has certainly been in the interest neither of the secondary schools nor of the university. This is to be done, on the one hand by infusing into our secondary school courses new ideas in keeping with the modern development of our science and of culture in general . . ., on the other hand by giving due consideration in our universities to the needs of the prospective teacher. One of the most important means to this end, it seems to me, is a synoptic course of lectures of the kind I begin today. . . . My purpose will be throughout to exhibit *the mutual connection between the problems of the various branches of mathematics* which is not always sufficiently emphasized in the special courses devoted to them, as well as to emphasize their relations to the problems of elementary mathematics. Thereby I hope to make easier for you what I should like to designate as the real purpose of your university study of mathematics; viz., that *you may be able in a large measure to draw inspiration for your teaching from the great body of knowledge that has here been presented to you.*"

It is true also in this country that university courses in mathematics are generally given without direct reference to the needs of those who expect to teach the subject in our secondary schools. The result is that described by Professor Klein: the mathematics studied in the university has little or no influence on the teaching of elementary mathematics. An increasing number of prospective high school teachers are taking advanced courses and advanced degrees at our universities in preparation for their future work. As a result it seems probable that at no distant date a master's degree at least will be required of candidates for positions on the faculties of our best high schools. This is of course as it should be. But it imposes on our universities the duty of offering these men and women carefully planned courses of study that will give them a thoroughly practical and helpful preparation for their later work. The problems connected herewith have received but little attention hitherto in this country. In Germany, on the other hand, they have been widely discussed during the past decade. These discussions culminated in the report of the German commission of 1907 which contains a comprehensive plan of study for the prospective teacher of mathematics or the natural sciences.*

*Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte, Leipzig, 1908, pp. 264–306.

This report is of great interest to us in this country and might well serve as a basis for the discussions of an American commission instructed to consider these questions.

The desirability of such a synoptic course as part of the training of a teacher is so obvious that it seems highly probable that courses of this character will be given at our own universities as soon as they begin seriously to consider the needs of those who are preparing to teach. In considering the volumes under review, which give us in detail what their distinguished author believes should form the content of such a course, it seems desirable, therefore, to keep in mind their possible use as a basis for courses of this kind in America.

From what has been said of the nature of the course, it will already be clear that the prospective teacher is to take this course in his last year at the university. It is assumed that he has previously taken courses in analytic geometry, calculus, differential equations, descriptive and projective geometry, the theory of numbers, the theory of curves and surfaces, and the theory of functions, in addition to courses in mechanics, experimental and theoretical physics, chemistry, philosophy, pedagogy, logic and psychology.* Can we expect one of our students to cover this ground, assuming him to spend one year in graduate work, without making his undergraduate course too one-sided? I think we can. Many of our students now cover the work in analytic geometry and calculus, in chemistry and in experimental physics by the end of their sophomore year. The junior year might then bring one semester each (three hours weekly) of differential equations, solid analytic geometry, the theory of numbers, and descriptive geometry, and a full year course in analytic mechanics. The senior year should then contain a full year course in the theory of functions of a complex variable and a full year course in theoretical physics. The work in philosophy, logic, psychology, and pedagogy may be distributed through the junior and senior years. The year of graduate work might then be devoted to curves and surfaces, projective geometry and a synoptic course of the kind now before us. This outline would seem to leave an adequate amount of time during the four undergraduate years for general cultural courses. Many desirable modifications of this outline will suggest themselves. My present purpose is accomplished if I have shown that without im-

* This list is taken from the report of the German commission referred to above.

pairing his undergraduate course along general cultural lines a student can prepare himself along the lines of the plan laid down by the German commission in the time usually required for the obtaining of a master's degree.

We may now turn to the consideration of the contents of the volumes under review. The limitations of space (and of time) prevent a detailed description. We must content ourselves with a general survey, to which may be added the discussion of one or two features which seem to merit special attention. The first volume is divided into three main parts devoted respectively to "Arithmetik" (i. e., the theory of numbers in a broad sense), algebra, and analysis. In some 80 pages the author considers in order the operations with the natural numbers, the fundamental laws to which these operations are subject, the logical foundations of arithmetic (as we use the term in this country, i. e., the German Rechnen), the technique of numerical calculation with the natural numbers (the thoroughly practical point of view is here emphasized by the detailed description of the mechanism of a calculating machine), the successive extensions of the number concept by the introduction of the negative, fractional, and irrational numbers. There follows a section of some 40 pages on the theory of numbers proper, treating in particular the properties of prime numbers, decimal fractions, continued fractions, Fermat's last theorem, cyclotomic problems, and closing with an elementary proof that the regular inscribed polygon of seven sides cannot be constructed with a ruler and compass. Then follows a section of about 40 pages on ordinary and higher complex numbers, with a more detailed consideration of quaternions. This completes the first main part.

It seems desirable at this point to say a word regarding the author's method throughout this volume. In accordance with his avowed purpose of exhibiting everywhere the mutual relations of the various branches of mathematics we have throughout the first volume a wealth of geometric illustrations. We have geometric interpretations of the fundamental laws of algebra (with a vigorous protest against the practice of using such interpetations as proofs in cases where they do not apply), Klein's beautiful geometric representation of a continued fraction, a geometric formulation of the problem of the pythagorean numbers and its extension to that of Fermat's theorem already mentioned. We have already spoken of the application of analysis to prove the impossibility of constructing with ruler and compass an inscribed regular heptagon. We are given also the usual geometric representation of complex numbers. Finally the multiplication of quaternions is interpreted as rotation and stretching in space. True to his other avowed object, the author also gives at the end of the discussion of each topic what he conceives to be its relation to elementary mathematics and its bearing on the problems confronting the teacher. In the latter particular the method of the first volume differs essentially from that of the second, where all matter relating to the teaching of the elementary branches is reserved for the end and is discussed at length in an appendix. This arrangement is made desirable, to some extent, by the sequence of topics adopted. But the reviewer is inclined to believe that the plan of the first volume is the more effective. Finally, in this connection we must not fail to call attention to the prominent place given throughout every discussion to the history of the subject under consideration. The historical setting is made an essential feature of the presentation and contributes greatly to a clear understanding and the stimulating effect of the whole.

In order to prepare his hearers further in this direction, the author here interposes a general survey of the modern development of mathematics. It is a mere sketch, but it is a masterpiece. In some twenty pages he calls attention first to two fundamental tendencies in the growth of our science. The first tendency has for its object the development of a given branch of mathematics for its own sake and with its own methods. If this tendency alone obtained, mathematics would appear as a group of distinct theories which may show here and there incidental points of contact, but which have no organic unity. The second tendency on the other hand has for its object precisely to wipe out the boundaries between the various so-called branches, and conceives of mathematics as a unified whole, in which the results and methods of every branch are common property of the whole. From this point of view the two fundamental divisions of analysis and geometry become fused into a single whole in which every theorem of either may be regarded as a theorem of the other. It seems to us that the author might have been even more emphatic on this point than he is, a subject to which we will return presently in the discussion of his treatment of the modern developments of the foundations of the science. After calling attention to these two tendencies, the author sketches briefly the history of mathematics from ancient to modern times with reference to them. This discussion, if printed, would hardly fill more than half a dozen ordinary octavo pages; nevertheless, the author is able to give a remarkably vivid picture of the successive stages in the development of mathematics.

After this interlude, he takes up the second main part of his lectures. In his discussion of algebra he confines himself exclusively to the theory of equations, in particular to the use of graphical and in general of geometric methods in this theory. The first part of the discussion is devoted to the description of certain graphical methods for the solution of real equations with real roots. By the use of the principle of duality two methods of representation are obtained, according as the coordinates are interpreted as point or as line coordinates. This discussion contains much of interest also to the student of analytic geometry and forms an excellent example of the interplay of two branches of mathematics when applied to a special problem. The second part of the treatment of the theory of equations is devoted to the consideration of the equation from the point of view of the theory of functions of a complex variable, the representation by means of conformal mapping on two spheres occupying the central position.

The third main part of the volume, that on analysis, begins with a discussion of the logarithm. After sketching the origin and development of the theory in historical sequence which brings out clearly why it is that the irrational number e should appear as the base of the system of "natural" logarithms, professor Klein concerns himself in detail with the difficulties attendant on the introduction of logarithms in secondary instruction. This is followed by an exposition of how he would like to see this subject treated in elementary courses. His plan contemplates the definition of the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$ and the ordinate $x = a$; *i. e.*, by the relation

$$\log x = \int_1^x \frac{dx}{x}.$$

Whatever may be said of the practicability of this procedure in Germany, there can be little doubt that it is utterly impracticable in this country, at least with the present organization of our secondary school curricula. With us, logarithms are needed first as an aid in numerical computation in our courses in trigonometry. At the time when it thus becomes necessary to introduce logarithms, the pupil is not even familiar with the hyperbola, to say nothing of his total ignorance of the notion of an integral. However much we may be in sympathy with the movement looking toward an early introduction of the notion of function, graphical representation, and of the elements of the infinitesimal calculus, it appears to the reviewer quite impossible to make these concepts all familiar to a pupil by the time it first becomes desirable to consider logarithms. Judging by a recent German review, conditions in that country do not seem to be very different from our own in this regard.* The reviewer, moreover, is of the opinion that nothing is gained by an attempt to introduce the natural logarithms before the regular course in the calculus. As to the best method of introducing logarithms, when they become desirable as an aid to computation, opinions may differ. The chief pedagogical difficulty at this time is to make clear to the pupil that corresponding to every positive real number there is a logarithm to a given positive base, say 10. The reviewer has found that the following plan works well in practice: Supposing the base to be a, the successive powers

$$(1) \qquad \cdots, a^{-3},\ a^{-2},\ a^{-1},\ 1,\ a,\ a^2,\ a^3,\ \cdots$$

and the corresponding exponents

$$(2) \qquad \cdots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \cdots$$

form a geometrical and an arithmetical progression, with the property that to the product of any two numbers of the former corresponds the sum of the corresponding numbers of the latter. Moreover, this property is readily seen to hold for any geometrical progression and any arithmetical progression, provided the number 1 is in the former and corresponds to the number 0 in the latter. If then any number of geometric means be inserted between two numbers of (1) and the same number of arithmetic means be inserted between the corresponding terms of (2), the two sequences of numbers will still have the fundamental property mentioned. It remains only to show that by carrying

* Cf. C. Färber, *Archiv der Mathematik und Physik*, vol. 15 (July, 1909), p. 75.

this process far enough, any given positive number may be approximated to as closely as we please in the sequence (1).

This is the only place, it should be added, in the two volumes in which it appears that one of Professor Klein's suggested pedagogical reforms is impracticable in this country. The section on the logarithm closes with a consideration of this function from the point of view of the theory of functions of a complex variable. Naturally the exponential function is disposed of in the same connection.

There follows an extended section on the trigonometric functions (and also of the hyperbolic functions) mainly from the point of view of the theory of functions of a complex variable. The applications of the trigonometric functions also receive considerable attention, especially as regards the theory of small oscillations and (more extensively) the representation of an arbitrary function by Fourier series. This third and last main part on analysis closes with a treatment of the infinitesimal calculus as such. The sequence of topics is again historical, the comment critical. Some twenty pages (a little less than half of this section) is devoted to Taylor's theorem. Throughout there is much emphasis on geometric representation; the connection of the calculus with the theory of finite differences and the problems of interpolation is clearly exhibited, and many interesting comments on pedagogical questions are inserted.

The first volume closes with an appendix in which the author proves the transcendental character of e and π, and then gives a beautifully clear account of the fundamental ideas in the theory of sets (Mengenlehre).

In the second volume the author sets himself the problem of giving a survey of the whole field of geometry. As has already been said all pedagogical questions are here reserved for separate discussion in an appendix. The method is almost exclusively analytic, the applications are very largely in the field of mechanics. As in the first volume, there are also here three main parts: Part I: The simplest geometric forms; Part II: Geometric transformations; Part III: Systematic development and the foundations of geometry.

Part I begins with a discussion of the well known expressions of length, area, and volume as determinants, in which the significance of the algebraic sign of these expressions receives particular attention. This is followed by a detailed discussion of Grassmann's systematic use of determinants for the definition of geometric quantities. We have here a very readable exposition of the comparatively little known ideas that Grassmann developed in his Aüsdehnungslehre. The notions here developed are at once coordinated with fundamental concepts in mechanics. Throughout also the history of the subject is emphasized. Vector analysis, Plücker's line coordinates, n dimensional geometry are among the topics introduced. As a detail we may mention that Poncelet is given the whole credit as the discoverer of the principle of duality, while Gergonne, who would seem to deserve as much credit in this discovery, is not even mentioned.

Part II discusses in order and with considerable detail the affine and the projective transformations, with many interesting applications to geometry and in particular to the problems of perspective drawing. Here the author makes a strong plea that every teacher of mathematics should be familiar with the principles of descriptive geometry, a plea with which the reviewer is in full sympathy. This is followed by a briefer discussion of higher transformations, the transformations by reciprocal radii, and the several methods in use in the construction of geographical maps receiving special attention. The most general continuous point transformations are then briefly considered in connection with problems of analysis situs. After a section of less than twenty pages devoted to the dualistic and contact transformations, the author devotes some thirty pages to the introduction and use of imaginary elements into geometry, von Staudt's classical work in this respect receiving detailed consideration.

Part III, finally, itself falls into two parts. The first of these is devoted to the systematic organization of the field of geometry. Klein's well-known use of the group concept as a means of geometric classification is of course fundamental. On the analytic side the geometry associated with a given group then becomes the theory of invariants for this group. We are, therefore, given an extended survey of this theory and its applications to geometry. This section closes with the special developments regarding the characterization of the affine and metric geometries within the general projective geometry. The second part is devoted to the logical foundations of geometry. In many respects this is the most interesting part of the volume; in one fundamental particular, however, it is the least satisfying. The discussion of the two methods of building up the ordinary metric geometry on the one hand by making the group of rigid displacements fundamental, on the other by choosing as undefined the notions of distance, angle, and congruence is beautifully clear. The discussion of the foundations as they appear in Euclid's Elements that follows, with its critical comment and the clear description in historical sequence of the developments to which the problems involved gave rise, forms a remarkably stimulating bit of reading.

What appears to us as unsatisfactory in Professor Klein's treatment is due to his point of view. This applies as well to his treatment of the foundations of analysis. This point of view is summed up in the statement that Professor Klein can see no virtue in the purely formal and abstract treatment of these questions. He expresses himself very clearly on this point on several occasions. To regard the undefined elements as mere symbols devoid of meaning (except such as is implied in the explicit assumptions concerning them) and to regard the unproved propositions as mere arbitrary assumptions appears to Professor Klein as "the death knell of all science" (volume 2, page 384). The reviewer is obliged to take emphatically a diametrically opposite view. We should say this formal point of view has given to science a new and powerfully vital principle. It is without doubt, so it seems to us, the most powerful unifying principle in our science today. We will try briefly to justify these assertions. The formal logical consequences of a set of assumptions concerning certain undefined symbols constitute what we will here call an *abstract theory*. If any concrete meaning is assigned these symbols for which all the assumptions appear to be satisfied, we obtain what we will call a *concrete representation* of the abstract theory. From the point of view of the foundations such a concrete representation is regarded as a consistency proof of the assumptions. We are in full accord with what Professor Klein has to say as to the logical limitations of such a "proof" and the important metaphysical problems to which the formal point of view gives rise (volume 1, pages 33–36); we are glad that he has insisted on these matters so emphatically. However, our present interest is in *the application of the formal treatment outside of the field of the logical foundations*. A given abstract theory may have many different concrete representations. These different concrete representations are then unified by the underlying abstract theory. From this point of view geometry and analysis are coextensive, a fact which in a course of lectures of the kind here considered we might have expected to see strongly emphasized.

Professor Klein would probably take issue with us right here. To him geometry is essentially intuitional (anschaulich); if it is not intuitional it is not geometry, he would probably say. That he holds this point of view is abundantly shown. He insists for example that geometric methods are less precise and rigorous than analytic methods, a statement which is certainly true only under the crude intuitional view of geometry. But this is only a quibble of words. Professor Klein is of course justified in using the word geometry to apply only to that which is "anschaulich" (he does speak of n dimensional geometry, however), if he so desires. Whatever we call it, however, there is such a thing as an abstract theory underlying geometry, which to avoid circumlocution we call geometry. The methods of this abstract geometry are just as precise and rigorous as those of (abstract) analysis, no more and no less. The unifying power of this point of view, to mention only one example, is seen in the fact that projective geometry and certain tactical configurations (triple systems, etc.), with only a finite number of elements appear as concrete representations of the same abstract theory. The formal point of view is of the greatest advantage also in the consideration of imaginary elements in geometry. After describing von Staudt's method of introducing complex elements into geometry, whereby certain involutions are taken to represent complex points and calling attention to the fact that the whole theory of complex projective geometry could be built up on this basis, Professor Klein remarks (volume 2, page 270): "In most cases, however, the application of this geometric interpretation (of complex elements) would, in spite of its theoretical advantages, be attended with so many complications that one may well be satisfied with its theoretical possibility and that one will return to the naiver point of view to the effect that a complex point is a set of imaginary coordinates with which one may in a certain way operate as with the real points." From the abstract point of view a complex point is just the same sort of a point in complex geometry as a real point; in fact this distinction is entirely meaningless, until a chain has been selected as the real chain. It is unnecessary to multiply examples. The advantages of a formal treatment are briefly expressed as follows: If an abstract theory is developed concerning a set of mere symbols, then this theory applies to *every* concrete representation of these symbols for which the fundamental assumptions are satisfied. These concrete representations of a given abstract theory may be many and varied in character; the abstract theory serves to unify them all.

It would seem that a clear understanding of this point of view is of particular importance to the prospective teacher. It is now well recognized that a knowledge of the foundations of mathematics is essential in the best preparation of the teacher, and the abstract point of view, if not absolutely necessary, greatly facilitates a clear understanding of the problems here presented. That the non-euclidean geometries serve, though not as conveniently, to describe the properties of our intuitional space, is merely due to the fact that the points, lines, etc., of the latter may be regarded as satisfying all the assumptions lying at the basis of each of these geometries.

The volume closes with a very interesting account of the movements toward reform in the teaching of elementary geometry as they have developed during the past years in England, France, Italy, and Germany. The work as a whole is a remarkable example of the distinguished author's mastery of the art of clear and stimulating exposition. We sincerely hope that it will have a wide influence, also in America, in arousing an active interest in a more serviceable preparation of our teachers.

J. W. YOUNG.

CORRECTION.

The following misprints occur in page 122 of Dr. Onnen's paper in the December number of the BULLETIN:

Lines 4–5. *For* . . . dividing n times by any integer a . . . *read* . . . dividing n times by a any integer. . . .

Line 6 from the bottom. *For* . . . an integral value for n . . . *read* . . . an integer. . . .

NOTES.

THE opening (January) number of volume 11 of the *Transactions of the American Mathematical Society* contains the following papers: "Theorems on simple groups," by H. F. BLICHFELDT; "Infinite discontinuous groups of birational