LINEAR AND BILINEAR $T(b)$ THEOREMS À LA STEIN

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ABSTRACT. In this work, we state and prove versions of the linear and bilinear $T(b)$ theorems involving quantitative estimates, analogous to the quantitative linear $T(1)$ theorem due to Stein.

1. INTRODUCTION

The impact of the classical Calderón-Zygmund theory permeates through analysis and PDEs. Nowadays, both the linear and multilinear aspects of this theory are well understood and continue to be intertwined with aspects of analysis that are beyond their reach, such as those considering the bilinear Hilbert transform.

Two fundamental results in the linear theory from the 1980s are the celebrated $T(1)$ theorem of David and Journé [4] and $T(b)$ theorem of David, Journé, and Semmes [5]. Both results were strongly motivated by the study of the Cauchy integral on a Lipschitz curve and the related Calderón commutators. Their gist lies in understanding the boundedness of a singular operator via appropriate simpler testing conditions.

In the $T(1)$ theorem, one needs to test a singular operator and its transpose on the constant function 1. If both the operator and its transpose were $L^\infty \to BMO$ bounded, then by duality and interpolation [6], the operator would be bounded on L^2. The remarkable aspect of the $T(1)$ theorem is that one does not need to test the operator on the whole L^∞, but just on one special element in it. Going back to the Cauchy integral operator associated to a Lipschitz function A, it turns out that it is not necessarily easy to test the operator on 1. It is, however, much easier to test the operator on the L^∞ function $1 + iA'$. Thus, as the name suggests, the $T(b)$ theorem extends the $T(1)$ theorem by replacing the constant function 1 with a suitable L^∞ function b or, to be more precise, by replacing 1 with two suitable functions b_0 and b_1 in L^∞ on which we test an operator and its transpose. The bilinear Calderón-Zygmund theory has its own versions of the $T(1)$ and $T(b)$ theorems, such as those proved by Grafakos and Torres [8] and by Hart [12], respectively. See Theorems D and E below.

In this work, we revisit the $T(b)$ theorem, both in the linear and bilinear settings, through the lens of a gem due to Stein [15]. We are alluding to his formulation of the $T(1)$ theorem involving quantitative estimates for a singular operator and its transpose when tested now on normalized bump functions. Our goal is to prove

that an analogous natural formulation à la Stein can be given for the $T(b)$ theorems in the linear and bilinear settings. We note that, while for the sake of clarity in our presentation we have chosen to delineate the linear and bilinear settings, a unified discussion is certainly possible under the encompassing more general multilinear setting.

2. Linear Calderón-Zygmund theory

In this section, we consider a linear singular operator T a priori defined from $\mathcal{S}$ into $\mathcal{S}'$ of the form

$$(2.1) \qquad T(f)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy.$$

Here, we assume that, away from the diagonal $\Delta = \{(x,y) \in \mathbb{R}^{2d} : x = y\}$, the distributional kernel K of T coincides with a function that is locally integrable on $\mathbb{R}^{2d} \setminus \Delta$. The formal transpose T^* of T is defined similarly with the kernel K^* given by $K^*(x,y) := K(y,x)$.

Definition 2.1. A locally integrable function K on $\mathbb{R}^{2d} \setminus \Delta$ is called a (linear) *Calderón-Zygmund kernel* if it satisfies the following conditions:

  (i) For all $x, y \in \mathbb{R}^d$, we have $|K(x,y)| \lesssim |x-y|^{-d}$.
  (ii) There exists $\delta \in (0, 1]$ such that

$$(2.2) \qquad |K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \lesssim \frac{|x-x'|^\delta}{|x-y|^{d+\delta}}$$

  for all $x, x', y \in \mathbb{R}^d$ satisfying $|x-x'| < \frac{1}{2}|x-y|$.

We say that a linear singular operator T of the form (2.1) with a Calderón-Zygmund kernel is a *linear Calderón-Zygmund operator* if T extends to a bounded operator on L^{p_0} for some $1 < p_0 < \infty$. It is well known [14] that if T is a linear Calderón-Zygmund operator, then it is bounded on L^p for all $1 < p < \infty$. Hence, in the following, we restrict our attention to the L^2-boundedness of such linear operators. We point out that the Calderón-Zygmund operator T is also $L^\infty \to BMO$ bounded. Here, BMO denotes the space of functions of *bounded mean oscillation*, which we now recall.

Definition 2.2. Given a locally integrable function f on $\mathbb{R}^d$, define the *BMO-seminorm* by

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - \operatorname*{ave}_Q f| dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ and

$$\operatorname*{ave}_Q f := \frac{1}{|Q|} \int_Q f(x)dx.$$

Then, we say that f is of bounded mean oscillation if $\|f\|_{BMO} < \infty$ and we define $BMO(\mathbb{R}^d)$ by

$$BMO(\mathbb{R}^d) := \big\{ f \in L^1_{\text{loc}}(\mathbb{R}^d) : \|f\|_{BMO} < \infty \big\}.$$

2.1. **Classical linear $T(1)$ and $T(b)$ theorems.** In this subsection, we provide a brief discussion of the classical $T(1)$ and $T(b)$ theorems proved in [4] and [5], respectively. In order to do so, we need to define a few more notions.

Definition 2.3. We say that a function $\phi \in \mathcal{D}$ is a *normalized bump function* of order M if $\operatorname{supp} \phi \subset B_0(1)$ and $\|\partial^\alpha \phi\|_{L^\infty} \leq 1$ for all multi-indices α with $|\alpha| \leq M$.

Here, $B_x(r)$ denotes the ball of radius r centered at x. Given $x_0 \in \mathbb{R}^d$ and $R > 0$, we set

$$(2.3) \qquad \phi^{x_0,R}(x) = \phi\Big(\frac{x - x_0}{R}\Big).$$

Definition 2.4. We say that a linear singular integral operator $T : \mathcal{S} \to \mathcal{S}'$ has the *weak boundedness property* if there exists $M \in \mathbb{N} \cup \{0\}$ such that we have

$$(2.4) \qquad \big|\langle T(\phi_1^{x_1,R}), \phi_2^{x_2,R}\rangle\big| \lesssim R^d$$

for all normalized bump functions ϕ_1 and ϕ_2 of order M, $x_1, x_2 \in \mathbb{R}^d$, and $R > 0$.

We note that it suffices to verify (2.4) for $x_1 = x_2$; see [11]. The statement of the $T(1)$ theorem of David and Journé [4] is the following.

Theorem A ($T(1)$ theorem). *Let $T : \mathcal{S} \to \mathcal{S}'$ be a linear singular integral operator with a Calderón-Zygmund kernel. Then, T can be extended to a bounded operator on L^2 if and only if*

    (i) *T satisfies the weak boundedness property,*
   (ii) *$T(1)$ and $T^*(1)$ are in BMO.*

Since T is a priori defined only in $\mathcal{S}$, the expressions $T(1)$ and $T^*(1)$ are, of course, not well defined and need to be interpreted carefully. The same comment applies to the corresponding theorems in the bilinear setting.

The main concept needed in extending the $T(1)$ theorem to the $T(b)$ theorem is that of para-accretive functions.

Definition 2.5. We say that a function $b \in L^\infty$ is *para-accretive*[1] if there exists $c_0 > 0$ such that, for every cube Q, there exists a subcube $\widetilde{Q} \subset Q$ such that

$$(2.5) \qquad \frac{1}{|Q|}\bigg|\int_{\widetilde{Q}} b(x)dx\bigg| \geq c_0.$$

It follows from (2.5) that

$$(2.6) \qquad |\widetilde{Q}| \geq \frac{c_0}{\|b\|_{L^\infty}}|Q|.$$

In particular, the function 1 is automatically para-accretive. It is also worth pointing out that the definition of para-accretivity in Definition 2.5 is not the same as the one used in the classical $T(b)$ theorem of David, Journé, and Semmes [5]. The notion of para-accretivity stated here is borrowed from [10, 12]; for a similar definition in which cubes are replaced by balls, see Christ's monograph [2]. The two definitions of para-accretivity are nevertheless equivalent. Since this natural observation seems to be missing from the literature, for the convenience of the reader we have included its proof in the appendix.

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[1]An extra condition that $b^{-1} \in L^\infty$ is sometimes included in the definition of para-accretivity. This, however, is not necessary. Indeed, it follows from (2.6) and the Lebesgue differentiation theorem that $|b(x)| \geq c_0$ almost everywhere. In particular, we have $b^{-1} \in L^\infty$.

Before giving a meaning to operators to which the $T(b)$ theorem applies, we need one more definition.

Definition 2.6. Given $0 < \eta \leq 1$, let C^η be the collection of all functions from $\mathbb{R}^d \to \mathbb{C}$ such that $\|f\|_{C^\eta} < \infty$, where the C^η-norm is given by

$$\|f\|_{C^\eta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}.$$

We also denote by C_0^η the subspace of all compactly supported functions in C^η.

Definition 2.7. Let b_0 and b_1 be para-accretive functions. A linear singular operator $T : b_1 C_0^\eta \to (b_0 C_0^\eta)'$ is called a *linear singular integral operator of Calderón-Zygmund type associated to b_0 and b_1* if T is continuous from $b_1 C_0^\eta$ into $(b_0 C_0^\eta)'$ for some $\eta > 0$ and there exists a Calderón-Zygmund kernel K such that

$$\langle T(M_{b_1} f), M_{b_0} g \rangle = \int_{\mathbb{R}^{2d}} K(x, y) b_1(y) f(y) b_0(x) g(x) dx dy,$$

for all $f, g \in C_0^\eta$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. Here, M_b denotes the operation of multiplication by b.

With these preparations, we are now ready to state the classical $T(b)$ theorem [5].

Theorem B ($T(b)$ theorem). *Let b_0 and b_1 be para-accretive functions. Suppose that T is a linear singular integral operator of Calderón-Zygmund type associated to b_0 and b_1. Then, T can be extended to a bounded operator on L^2 if and only if the following conditions hold:*

    (i) *$M_{b_0} T M_{b_1}$ satisfies the weak boundedness property,*
    (ii) *$M_{b_0} T(b_1)$ and $M_{b_1} T^*(b_0)$ are in BMO.*

In the special case when b_0 and b_1 are accretive[2] and $Tb_1 = T^* b_0 = 0$, the $T(b)$ theorem was independently proved by McIntosh and Meyer [13].

Remark 2.8. In [5], condition (ii) of Theorem B is stated slightly differently; it was assumed that $T(b_1), T^*(b_0) \in BMO$. We note that this is just a matter of notation. For example, the condition $T(b_1) \in BMO$ in [5] means that that there exists $\beta \in BMO$ such that

$$\langle T(b_1), f \rangle = \langle \beta, f \rangle \quad \text{for all mean-zero } f \in b_0 C_0^\eta.$$

This is clearly equivalent to

$$(2.7) \qquad \langle T(b_1), b_0 f \rangle = \langle \beta, b_0 f \rangle \quad \text{for all } f \in C_0^\eta \text{ such that } \int b_0 f dx = 0.$$

Here, we used the fact that $b_0 f \leftrightarrow f$ is a one-to-one correspondence since b_0 is para-accretive and thus, in particular, is bounded away from zero almost everywhere. In Theorem B, we followed the notation from [12] to signify the fact that the condition indeed depends on both b_0 and b_1, and what we mean by condition (ii) in Theorem B is precisely the statement (2.7). See also Theorem E below in the bilinear setting.

Lastly, note that, as in the $T(1)$ theorem, expressions $M_{b_0} T(b_1)$ and $M_{b_1} T^*(b_0)$ are not a priori well defined, and thus some care must be taken.

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[2]A function $b \in L^\infty$ is called accretive if there exists $\delta > 0$ such that $\operatorname{Re} b \geq \delta$ for all $x \in \mathbb{R}^d$. Note that an accretive function is para-accretive.

2.2. **Formulations of the $T(1)$ and $T(b)$ theorems à la Stein.** There is another formulation of the $T(1)$ theorem due to Stein [15] in which conditions (i) and (ii) in Theorem A are replaced by the quantitative estimate (2.8) involving normalized bump functions.

Theorem C ($T(1)$ theorem à la Stein)**.** *Let T be as in Theorem* A*. Then, T can be extended to a bounded operator on L^2 if and only if there exists $M \in \mathbb{N} \cup \{0\}$ such that we have*

$$(2.8) \qquad \|T(\phi^{x_0,R})\|_{L^2} + \|T^*(\phi^{x_0,R})\|_{L^2} \lesssim R^{\frac{d}{2}}$$

for any normalized bump function ϕ of order M, $x_0 \in \mathbb{R}^d$, and $R > 0$.

By viewing the expressions $T(\phi^{x_0,R})$ and $T^*(\phi^{x_0,R})$ as $T(1 \cdot \phi^{x_0,R})$ and $T^*(1 \cdot \phi^{x_0,R})$, it is natural to extend this result by replacing the constant function 1 by para-accretive functions b_0 and b_1. This is the first result of our paper.

Theorem 1 ($T(b)$ theorem à la Stein)**.** *Let T, b_0, and b_1 be as in Theorem* B*. Then, T can be extended to a bounded operator on L^2 if and only if there exists $M \in \mathbb{N} \cup \{0\}$ such that the following two inequalities hold for any normalized bump function ϕ of order M, $x_0 \in \mathbb{R}^d$, and $R > 0$:*

$$(2.9) \qquad \|T(b_1 \phi^{x_0,R})\|_{L^2} \lesssim R^{\frac{d}{2}},$$

$$(2.10) \qquad \|T^*(b_0 \phi^{x_0,R})\|_{L^2} \lesssim R^{\frac{d}{2}}.$$

We present the proof of Theorem 1 in Section 4.

As an application of this result, one could recover the well-known fact that the commutator of a pseudodifferential operator with symbol in the Hörmander class $S_{1,0}^1$ and the multiplication operator of a Lipschitz function a is bounded on L^2. Indeed, suppose that for all $x, \xi \in \mathbb{R}^d$ and all multi-indices α, β we have

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim (1 + |\xi|)^{1-|\beta|},$$

and let

$$T_\sigma(f)(x) = \int_{\mathbb{R}^d} \sigma(x,\xi) f(\xi) e^{ix \cdot \xi} \, d\xi$$

be the corresponding pseudodifferential operator. Also, given a such that $\partial a / \partial x_j \in L^\infty(\mathbb{R}^d)$ for $1 \leq j \leq d$, let

$$[T_\sigma, M_a] = T_\sigma(af) - aT(f)$$

be the commutator of T_σ and the multiplication operator M_a. It is straightforward to check that the kernel of $[T_\sigma, M_a]$ is Calderón-Zygmund and, by a similar computation to the one in [15, pp. 309-310], (2.9) and (2.10) hold as well, thus proving $[T_\sigma, M_a] : L^2 \to L^2$.

3. Bilinear Calderón-Zygmund theory

Next, we turn our attention to the bilinear setting and consider the corresponding extensions of the results in Section 2. Namely, we consider a bilinear singular operator T a priori defined from $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}'$ of the form

$$(3.1) \qquad T(f,g)(x) = \int_{\mathbb{R}^{2d}} K(x,y,z) f(y) g(z) \, dy \, dz,$$

where we assume that, away from the diagonal $\Delta = \{(x, y, z) \in \mathbb{R}^{3d} : x = y = z\}$, the distributional kernel K coincides with a function that is locally integrable on $\mathbb{R}^{3d} \backslash \Delta$. The formal transposes T^{*1} and T^{*2} are defined in an analogous manner with the kernels K^{*1} and K^{*2} given by $K^{*1}(x, y, z) := K(y, x, z)$ and $K^{*2}(x, y, z) := K(z, y, x)$.

Definition 3.1. A locally integrable function K on $\mathbb{R}^{3d} \setminus \Delta$ is called a (bilinear) *Calderón-Zygmund kernel* if it satisfies the following conditions:

(i) For all $x, y, z \in \mathbb{R}^d$, we have

$$|K(x, y, z)| \lesssim \big(|x - y| + |x - z|\big)^{-2d}.$$

(ii) There exists $\delta \in (0, 1]$ such that

$$(3.2) \qquad |K(x, y, z) - K(x', y, z)| \lesssim \frac{|x - x'|^\delta}{\big(|x - y| + |x - z|\big)^{2d + \delta}}$$

for all $x, x', y, z \in \mathbb{R}^d$ satisfying $|x - x'| < \frac{1}{2} \max\big(|x - y|, |x - z|\big)$. Moreover, we assume that the formal transpose kernels K^{*1} and K^{*2} also satisfy the regularity condition (3.2).

We say that a bilinear singular operator T of the form (3.1) with a bilinear Calderón-Zygmund kernel is a *bilinear Calderón-Zygmund operator* if T extends a bounded operator on $L^{p_0} \times L^{q_0}$ into L^{r_0} for some $1 < p_0, q_0 < \infty$ with $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{r_0} \leq 1$.

Similarly to the linear case, the crux of the bilinear Calderón-Zygmund theory is contained in the fact that if T is a bilinear Calderón-Zygmund operator, then it is bounded on $L^p \times L^q$ into L^r for all $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$ (with the appropriate statements at the endpoints); see Grafakos and Torres [8]. Therefore, the main question is to prove that there exists at least one triple (p_0, q_0, r_0) with $1 < p_0, q_0 < \infty$ and $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{r_0} \leq 1$ such that T is bounded from $L^{p_0} \times L^{q_0}$ into L^{r_0}.

The weak boundedness property for bilinear singular operators has a similar flavor as the one in the linear case.

Definition 3.2. We say that a bilinear singular integral operator $T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}'$ has the (bilinear) *weak boundedness property* if there exists $M \in \mathbb{N} \cup \{0\}$ such that we have

$$(3.3) \qquad \big|\langle T(\phi_1^{x_1, R}, \phi_2^{x_2, R}), \phi_3^{x_3, R}\rangle\big| \lesssim R^d$$

for all normalized bump functions ϕ_1, ϕ_2, ϕ_3 of order M, $x_1, x_2, x_3 \in \mathbb{R}^d$, and $R > 0$.

Remark 3.3. It follows from [1, Lemma 9] that it suffices to verify (3.3) for $x_1 = x_2 = x_3$.

3.1. **Bilinear $T(1)$ and $T(b)$ theorems.** We now state the bilinear $T(1)$ theorem in the form given by Hart [11].

Theorem D (Bilinear $T(1)$ theorem). *Let $T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}'$ be a bilinear singular integral operator with a standard Calderón-Zygmund kernel. Then, T can be*

extended to a bounded operator on $L^p \times L^q \to L^r$ for all $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ if and only if

   (i) *T satisfies the weak boundedness property,*
   (ii) *$T(1,1)$, $T^{*1}(1,1)$, and $T^{*2}(1,1)$ are in BMO.*

We chose this formulation since it closely follows the statement of the classical linear $T(1)$ theorem given in the previous section. Further, note that Theorem D is equivalent to the formulation of Grafakos and Torres [8]; see also Christ and Journé [3].

Next, we turn our attention to the bilinear version of the $T(b)$ theorem.

Definition 3.4. Let b_0, b_1, and b_2 be para-accretive functions. A bilinear singular operator $T : b_1 C_0^\eta \times b_2 C_0^\eta \to (b_0 C_0^\eta)'$ is called a *bilinear singular integral operator of Calderón-Zygmund type associated to b_0, b_1, and b_2* if T is continuous from $b_1 C_0^\eta \times b_2 C_0^\eta$ into $(b_0 C_0^\eta)'$ for some $\eta > 0$ and there exists a bilinear Calderón-Zygmund kernel K such that

(3.4)

$$\langle T(M_{b_1} f_1), M_{b_2} f_2), M_{b_0} f_0 \rangle = \int_{\mathbb{R}^{3d}} K(x,y,z) b_0(x) f_0(x) b_1(y) f_1(y) b_2(z) f_2(z) dx dy dz,$$

for all $f_0, f_1, f_2 \in C_0^\eta$ with $\operatorname{supp} f_0 \cap \operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$.

Hart [12] proved the following result.

Theorem E (Bilinear $T(b)$ theorem). *Let b_0, b_1, and b_2 be para-accretive functions. Suppose that T is a bilinear singular integral operator of Calderón-Zygmund type associated to b_0, b_1, and b_2. Then, T can be extended to a bounded operator on $L^p \times L^q \to L^r$ for all $p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ if and only if*

   (i) *$M_{b_0} T\big(M_{b_1}(\cdot), M_{b_2}(\cdot)\big)$ satisfies the weak boundedness property,*
   (ii) *$M_{b_0} T(b_1, b_2)$, $M_{b_1} T^{*1}(b_0, b_2)$, and $M_{b_2} T^{*2}(b_1, b_0)$ are in BMO.*

As in Theorem B, we used the notation such as $M_{b_0} T(b_1, b_2) \in BMO$ rather than $T(b_1, b_2) \in BMO$ to signify the fact that each of the three statements in condition (ii) of Theorem E involves b_0, b_1 and b_2. See Remark 2.8.

3.2. **Formulation of the bilinear $T(b)$ theorem à la Stein.** As in the linear setting, we consider the formulation after Stein (Theorem C) involving quantitative estimates. In the following, we only state and prove the formulation after Stein in the context of the bilinear $T(b)$ theorem. The corresponding version for the bilinear $T(1)$ theorem follows by setting $b_0 = b_1 = b_2 = 1$; this result already appears in [8].

Theorem 2 (Bilinear $T(b)$ theorem à la Stein). *Let b_0, b_1, and b_2 be para-accretive functions. Suppose that T is a bilinear singular integral operator of Calderón-Zygmund type associated to b_0, b_1, and b_2. Then, T can be extended to a bounded operator on $L^p \times L^q \to L^r$ for all $p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ if and only if there exists $M \in \mathbb{N} \cup \{0\}$ such that we have*

(3.5) $$\|T(b_1 \phi^{x_1, R}, b_2 \phi^{x_2, R})\|_{L^2} \lesssim R^{\frac{d}{2}},$$

(3.6) $$\|T^{*1}(b_0 \phi^{x_0, R}, b_2 \phi^{x_2, R})\|_{L^2} \lesssim R^{\frac{d}{2}},$$

(3.7) $$\|T^{*2}(b_1 \phi^{x_1, R}, b_0 \phi^{x_0, R})\|_{L^2} \lesssim R^{\frac{d}{2}}$$

for any normalized bump function ϕ of order M, $x_0, x_1, x_2 \in \mathbb{R}^d$, and $R > 0$.

We prove this result in Section 5.

4. Proof of Theorem 1

Suppose that T is bounded on L^2. Let ϕ be a normalized bump function. Then, given any $x_0 \in \mathbb{R}^d$ and $R > 0$, we have

$$\|T(b_1\phi^{x_0,R})\|_{L^2} \lesssim \|b_1\|_{L^\infty}\|\phi^{x_0,R}\|_{L^2} \lesssim R^{\frac{d}{2}}.$$

This proves (2.9). Condition (2.10) follows from a similar computation.

Next, we assume that conditions (2.9) and (2.10) hold. It suffices to show that conditions (2.9) and (2.10) imply conditions (i) and (ii) in Theorem B.

We first prove condition (i) in Theorem B. Let ϕ_1 and ϕ_2 be normalized bump functions of order 0. Then, it follows from (2.9) and (2.3) that we have

$$\left|\left\langle M_{b_0}TM_{b_1}(\phi_1^{x_1,R}), \phi_2^{x_2,R}\right\rangle\right| \lesssim \|b_0\|_{L^\infty}\|T(b_1\phi_1^{x_1,R})\|_{L^2}\|\phi_2^{x_2,R}\|_{L^2} \lesssim R^d$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $R > 0$. This proves the weak boundedness property of $M_{b_0}TM_{b_1}$.

Next, we prove condition (ii) in Theorem B. In the following, we only show $M_{b_0}T(b_1) \in BMO$, assuming (2.9). The proof of $M_{b_1}T^*(b_0) \in BMO$ follows from (2.10) in an analogous manner.

We first recall from [5] how to extend the definition of T to $b_1C_b^\eta$, where $C_b^\eta := C^\eta \cap L^\infty$. Denote by $\{b_0C_0^\eta\}_0$ the subspace of mean-zero functions in $b_0C_0^\eta$. Given $f \in b_1C_b^\eta$ and $g \in \{b_0C_0^\eta\}_0$, let $\psi \in C_0^\eta$ with $0 \le \psi \le 1$ and $\psi \equiv 1$ in a neighborhood of $\operatorname{supp} g$. Then, we define the action of $T(f)$ on g by

$$\langle T(f), g\rangle := \langle T(f\psi), g\rangle + \langle T(f(1-\psi)), g\rangle$$

$$(4.1) \qquad = \langle T(f\psi), g\rangle + \int_{\mathbb{R}^{2d}} \left[K(x,y) - K(x_0,y)\right]f(y)(1-\psi(y))g(x)dxdy.$$

Note that this definition is independent of the choice of ψ. Here, the last equality in (4.1) holds for any $x_0 \in \operatorname{supp} g$.

Let $\phi \in C_0^\infty$ with $0 \le \phi \le 1$ such that $\phi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\operatorname{supp}\phi \subset B_0(1)$. Let $\phi_R(x) = \phi(R^{-1}x)$. Then, $T(b_1\phi_R)$ converges to $T(b_1)$ in the weak-$*$ topology of $(\{b_0C_0^\eta\}_0)'$. Namely, for all $g \in \{b_0C_0^\eta\}_0$, we have

$$(4.2) \qquad\qquad \langle T(b_1), g\rangle = \lim_{R\to\infty} \langle T(b_1\phi_R), g\rangle.$$

Indeed, letting $\psi \in C_0^\infty$ such that $\psi \equiv 1$ on $\operatorname{supp} g$ as before, we have

$$(4.3) \qquad \langle T(b_1\phi_R), g\rangle = \langle T(b_1\psi\phi_R), g\rangle + \langle T(b_1(1-\psi)\phi_R), g\rangle.$$

First, note that

$$(4.4) \qquad\qquad \langle T(b_1\psi\phi_R), g\rangle = \langle T(b_1\psi), g\rangle$$

for all sufficiently large R. In view of (2.2), it follows from Lebesgue dominated convergence theorem that

$$\lim_{R\to\infty} \langle T(b_1(1-\psi)\phi_R), g\rangle = \lim_{R\to\infty} \int_{\mathbb{R}^{2d}} K(x,y)b_1(y)(1-\psi(y))\phi_R(y)g(x)dydx$$

$$= \lim_{R\to\infty} \int_{\mathbb{R}^{2d}} \left[K(x,y) - K(x_0,y)\right]b_1(y)(1-\psi(y))\phi_R(y)g(x)dydx$$

$$= \int_{\mathbb{R}^{2d}} \left[K(x,y) - K(x_0,y)\right]b_1(y)(1-\psi(y))g(x)dydx$$

$$(4.5) \qquad = \langle T(b_1(1-\psi)), g\rangle,$$

where $x_0 \in \operatorname{supp} g$. Then, (4.2) follows from (4.1), (4.3), (4.4), and (4.5).

Suppose now that we have

$$(4.6) \qquad \|T(b_1 \phi_R)\|_{BMO} \lesssim 1,$$

uniformly in $R > 0$. Then, by Banach-Alaoglu theorem with $BMO = (H^1)'$, there exists a sequence $\{R_j\}_{j=1}^{\infty}$ such that $T(b_1 \phi_{R_j})$ converges in the weak-$*$ topology to some function β in BMO. Namely,

$$(4.7) \qquad \lim_{j \to \infty} \langle T(b_1 \phi_{R_j}), g \rangle = \langle \beta, g \rangle$$

for all $g \in H^1$. In particular, (4.7) holds for all $g \in \{b_0 C_0^{\eta}\}_0$. Then, from (4.2) and (4.7) with the uniqueness of a limit, we can identify $T(b_1)$ (or rather $M_{b_0} T(b_1)$) with $\beta \in BMO$. See Remark 2.8.

Therefore, it remains to prove (4.6). Let $M \in \mathbb{N} \cup \{0\}$ be as in Theorem 1. Then, by imposing that $\|\partial^{\alpha} \phi\|_{L^{\infty}} \leq 1$ for all multi-indices α with $|\alpha| \leq M$, the function ϕ defined above is a normalized bump function of order M.

Fix a cube Q of side length $\ell > 0$ with center $x_0 \in \mathbb{R}^d$. Set $\phi_Q := \phi^{x_0, r}$, where $r := 6 \operatorname{diam}(Q) = 6\sqrt{d}\,\ell$. By writing $T(b_1 \phi_R)$ as

$$(4.8) \qquad T(b_1 \phi_R) = T(b_1 \phi_Q \phi_R) + T\big(b_1 (1 - \phi_Q) \phi_R\big),$$

we consider the first and second terms separately.

On the one hand, when $R \leq r$, write $\phi_Q \phi_R$ as

$$(4.9) \qquad \phi_Q(x) \phi_R(x) = \phi\big(\tfrac{R}{r} \tfrac{x}{R} - \tfrac{x_0}{r}\big) \phi\big(\tfrac{x}{R}\big) = [\psi_1 \phi]^{0,R}(x)$$

with $\psi_1(x) := \phi\big(\tfrac{R}{r} x - \tfrac{x_0}{r}\big)$. Note that $\psi_1 \phi$ is a normalized bump function. Then, by the Cauchy-Schwarz inequality and (2.9), we have

$$(4.10) \qquad \int_Q \big|T(b_1 \phi_Q \phi_R)\big| dx \leq |Q|^{\frac{1}{2}} \big\|T(b_1 [\psi_1 \phi]^{0,R})\big\|_{L^2} \lesssim R^{\frac{d}{2}} |Q|^{\frac{1}{2}} \lesssim |Q|.$$

On the other hand, when $R > r$, write $\phi_Q \phi_R$ as

$$(4.11) \qquad \phi_Q(x) \phi_R(x) = \phi\big(\tfrac{x - x_0}{r}\big) \phi\big(\tfrac{r}{R} \tfrac{x - x_0}{r} + \tfrac{x_0}{R}\big) = [\phi \psi_2]^{x_0, r}(x)$$

with $\psi_2(x) := \phi\big(\tfrac{r}{R} x + \tfrac{x_0}{R}\big)$. Then, noting that $\phi \psi_2$ is a normalized bump function, it follows from the Cauchy-Schwarz inequality and (2.9) that

$$(4.12) \qquad \int_Q \big|T(b_1 \phi_Q \phi_R)\big| dx \leq |Q|^{\frac{1}{2}} \big\|T(b_1 [\phi \psi_2]^{x_0, r})\big\|_{L^2} \lesssim R^{\frac{d}{2}} |Q|^{\frac{1}{2}} \lesssim |Q|.$$

Next, we estimate the second term in (4.8). From the support condition

$$(4.13) \qquad \operatorname{supp}(1 - \phi_Q) \subset \mathbb{R}^d \setminus B_{x_0}(3 \operatorname{diam}(Q)) \subset \mathbb{R}^d \setminus Q,$$

we have

$$T\big(b_1 (1 - \phi_Q) \phi_R\big)(x) = \int_{\mathbb{R}^d} K(x, y) b_1(y) \big(1 - \phi_Q(y)\big) \phi_R(y) dy,$$

for all $x \in Q$. Define $c_{Q,R}$ by

$$c_{Q,R} := \int_{\mathbb{R}^d} K(x_0, y) b_1(y) \big(1 - \phi_Q(y)\big) \phi_R(y) dy,$$

where x_0 is the center of the cube Q. Then, it follows from (2.2) with (4.13) that, for $x \in Q$, we have

$$\left|T\big(b_1(1 - \phi_Q)\phi_R\big)(x) - c_{Q,R}\right| \leq \int_{|x-x_0| \leq \text{diam}(Q) \leq \frac{1}{2}|x-y|} |K(x,y) - K(x_0,y)| dy$$

(4.14) $$\lesssim 1$$

uniformly in $R > 0$.

Hence, putting (4.8), (4.10), (4.12), and (4.14) together, we conclude that there exists $A > 0$ such that for each cube Q and $R > 0$, there exists a constant $c_{Q,R}$ such that

(4.15) $$\frac{1}{|Q|} \int_Q |T(b_1\phi_R)(x) - c_{Q,R}| dx \leq A.$$

Therefore, it follows from Proposition 7.1.2 in [7] that

$$\sup_{R>0} \|T(b_1\phi_R)\|_{BMO} \leq 2A.$$

This proves (4.7) and thus completes the proof of Theorem 1.

5. Proof of Theorem 2

Suppose that T is bounded on $L^4 \times L^4 \to L^2$. Then, given a normalized bump function ϕ, we have

$$\|T(b_1\phi^{x_1,R}, b_2\phi^{x_2,R})\|_{L^2} \lesssim \|b_1\|_{L^\infty} \|\phi^{x_1,R}\|_{L^4} \|b_2\|_{L^\infty} \|\phi^{x_2,R}\|_{L^4} \lesssim R^{\frac{d}{2}}$$

for any $x_1, x_2 \in \mathbb{R}^d$ and $R > 0$. This proves (3.5). A similar computation yields (3.6) and (3.7).

Next, we assume that conditions (3.5), (3.6), and (3.7) hold. It suffices to show that conditions (3.5), (3.6), and (3.7) imply conditions (i) and (ii) in Theorem E.

We first prove condition (i) in Theorem E. Let ϕ_j, $j = 0, 1, 2$, be normalized bump functions of order M. Then, it follows from the Cauchy-Schwarz inequality, (3.5), and (2.3) that

$$\left|\langle M_{b_0} T(b_1\phi_1^{x_1,R}, b_2\phi_2^{x_2,R}), \phi_0^{x_0,R}\rangle\right| \lesssim \|b_0\|_{L^\infty} \|T(b_1\phi_1^{x_1,R}, b_2\phi_2^{x_2,R})\|_{L^2} \|\phi_0^{x_0,R}\|_{L^2} \lesssim R^d$$

for all $x_0, x_1, x_2 \in \mathbb{R}^d$ and $R > 0$. This proves condition (i) in Theorem E.

Next, we prove condition (ii) in Theorem E. As in the proof of Theorem 1, we only show $M_{b_0}T(b_1, b_2) \in BMO$, assuming (3.5). The proof of the other two conditions follows in a similar manner in view of the symmetric condition in Definition 3.1.

Since T is a priori defined only on $b_1 C_0^\eta \times b_2 C_0^\eta$, we first extend T to $b_1 C_b^\eta \times b_2 C_b^\eta$. Fix $f_j \in b_j C_b^\eta$, $j = 1, 2$. Given $g \in \{b_0 C_0^\eta\}_0$, let $\psi \in C_0^\eta$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in a neighborhood of supp g. Then, we define the action of $T(f_1, f_2)$ on g by

$$\langle T(f_1, f_2), g\rangle := \langle T(f_1\psi, f_2\psi), g\rangle + \langle T(f_1(1-\psi), f_2\psi), g\rangle$$

(5.1) $$+ \langle T(f_1\psi, f_2(1-\psi)), g\rangle + \langle T(f_1(1-\psi), f_2(1-\psi)), g\rangle.$$

Note that the last three terms can be written as triple integrals of the form (3.4). From this, we see that this definition is independent of the choice of ψ.

Let $\phi \in C_0^\infty$ with $0 \le \phi \le 1$ such that $\phi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\operatorname{supp} \phi \subset B_0(1)$. Let $\phi_R(x) = \phi(R^{-1}x)$. Then, $T(b_1\phi_R, b_2\phi_R)$ converges to $T(b_1, b_2)$ in the weak-$*$ topology of $(\{b_0 C_0^\eta\}_0)'$. Namely, we have

$$(5.2) \qquad \langle T(b_1, b_2), g \rangle = \lim_{R \to \infty} \langle T(b_1\phi_R, b_2\phi_R), g \rangle$$

for all $g \in \{b_0 C_0^\eta\}_0$. See [12] for the proof of (5.2).

Suppose that we have

$$(5.3) \qquad \|T(b_1\phi_R, b_2\phi_R)\|_{BMO} \lesssim 1,$$

uniformly in $R > 0$. Then, as in the proof of Theorem 1, it follows from the Banach-Alaoglu theorem that there exists a sequence $\{R_j\}_{j=1}^\infty$ and $\beta \in BMO$ such that

$$(5.4) \qquad \lim_{j \to \infty} \langle T(b_1\phi_{R_j}, b_2\phi_{R_j}), g \rangle = \langle \beta, g \rangle$$

for all $g \in H^1$, in particular for all $g \in \{b_0 C_0^\eta\}_0$. Hence, from (5.2) and (5.4), we conclude that $M_{b_0} T(b_1, b_2) \in BMO$.

Therefore, it remains to prove (5.3). By imposing that $\|\partial^\alpha \phi\|_{L^\infty} \le 1$ for all multi-indices α with $|\alpha| \le M$, the function ϕ defined above is a normalized bump function of order M. As in the proof of Theorem 1, let Q be the cube of side length $\ell > 0$ with center $x_0 \in \mathbb{R}^d$. Set $\phi_Q = \phi^{x_0, r}$, where $r = 6 \operatorname{diam}(Q)$. Then, write $T(b_1\phi_R, b_2\phi_R)$ as

$$T(b_1\phi_R, b_2\phi_R) = T(b_1\phi_Q\phi_R, b_2\phi_Q\phi_R) + T\big(b_1(1 - \phi_Q)\phi_R, b_2\phi_Q\phi_R\big)$$
$$+ T\big(b_1\phi_Q\phi_R, b_2(1 - \phi_Q)\phi_R\big) + T\big(b_1(1 - \phi_Q)\phi_R, b_2(1 - \phi_Q)\phi_R\big)$$
$$(5.5) \qquad := \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

It follows from the Cauchy-Schwarz inequality and (3.5) with (4.9) and (4.11) that

$$(5.6) \qquad \int_Q |\mathrm{I}| \, dx \le \begin{cases} |Q|^{\frac{1}{2}} \big\| T\big(b_1[\psi_1\phi]^{0,R}, b_2[\psi_1\phi]^{0,R}\big) \big\|_{L^2} \lesssim |Q|, & \text{when } R \le r, \\ |Q|^{\frac{1}{2}} \big\| T\big(b_1[\phi\psi_2]^{x_0,r}, b_2[\phi\psi_2]^{x_0,r}\big) \big\|_{L^2} \lesssim |Q|, & \text{when } R > r. \end{cases}$$

Next, we consider the terms $\mathrm{II}, \mathrm{III}$, and IV. Let $\phi_Q^c := 1 - \phi_Q$. Then, from the support condition (4.13), we have

$$\mathrm{II}(x) = \int_{\mathbb{R}^{2d}} K(x, y, z) b_1(y) \phi_Q^c(y) \phi_R(y) b_2(z) \phi_Q(z) \phi_R(z) \, dy \, dz$$

for $x \in Q$. Define $c_{Q,R}^{(2)}$ by

$$c_{Q,R}^{(2)} := \int_{\mathbb{R}^{2d}} K(x_0, y, z) b_1(y) \phi_Q^c(y) \phi_R(y) b_2(z) \phi_Q(z) \phi_R(z) \, dy \, dz,$$

where x_0 is the center of the cube Q. Then, it follows from (3.2) with (4.13) that, for $x \in Q$, we have

$$|\mathrm{II}(x) - c_{Q,R}^{(2)}| \le \int_{\operatorname{supp} \phi_Q} \int_{|x - x_0| \le \operatorname{diam}(Q) \le \frac{1}{2}|x-y|} |K(x, y, z) - K(x_0, y, z)| \, dy \, dz$$
$$(5.7) \qquad\qquad \lesssim 1$$

uniformly in $R > 0$. By symmetry, the same estimate holds for Ⅲ. As for Ⅳ, by letting

$$c_{Q,R}^{(4)} := \int_{\mathbb{R}^{2d}} K(x_0, y, z) b_1(y) \phi_Q^c(y) \phi_R(y) b_2(z) \phi_Q^c(z) \phi_R(z) dy dz,$$

we have, for $x \in Q$,

$$|\text{Ⅳ}(x) - c_{Q,R}^{(4)}| \leq \int_{|x-x_0| \leq \text{diam}(Q) \leq \frac{1}{2} \min(|x-y|,|x-z|)} |K(x, y, z) - K(x_0, y, z)| dy dz$$

$$(5.8) \qquad\qquad \lesssim 1$$

uniformly in $R > 0$.

Hence, putting (5.5), (5.6), (5.7), and (5.8) together, we conclude that there exists $A > 0$ such that for each cube Q and $R > 0$, there exists a constant $\widetilde{c}_{Q,R}$ such that

$$(5.9) \qquad\qquad \frac{1}{|Q|} \int_Q |T(b_1 \phi_R, b_2 \phi_R)(x) - \widetilde{c}_{Q,R}| dx \leq A,$$

thus yielding (5.3). This completes the proof of Theorem 2.

Appendix A. On para-accretive functions

Para-accretive functions play an important role in the $T(b)$ theorems. In this paper, we used Definition 2.5 for para-accretivity. In [5], however, David, Journé, and Semmes used a different definition (see Definition A.1 below) and gave several equivalent characterizations for para-accretive functions (Proposition A.2 below). In this appendix, we show that these two definitions (Definition 2.5 and Definition A.1) are equivalent.

Definition A.1. A function $b \in L^\infty$ is para-accretive if $b^{-1} \in L^\infty$ and there exists a sequence $\{s_k\}_{k \in \mathbb{Z}}$ of functions $s_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ for which the following conditions hold: there exist $C > 0$ and $\alpha > 0$ such that for all $k \in \mathbb{Z}$,

(i) $|s_k(x,y)| \leq C2^{kd}$,   for all $x, y \in \mathbb{R}^d$,

(ii) $s_k(x,y) = 0$,   if $|x - y| \geq C2^{-k}$,

(iii) $s_k(x,y) = s_k(y,x)$,   for all $x, y \in \mathbb{R}^d$,

(iv) $|s_k(x,y) - s_k(x',y)| \leq C2^{k(d+\alpha)}|x - x'|$,   for all $x, x', y \in \mathbb{R}^d$,

(v) $\int s_k(x,y)b(y)dy = 1$,   for all $x \in \mathbb{R}^d$.

The following proposition states different characterizations for para-accretive functions according to Definition A.1.

Proposition A.2 (Proposition 2 in [5]). *Let $b \in L^\infty$ such that $b^{-1} \in L^\infty$. Then, the following statements are equivalent.*
(A) *A function b is para-accretive according to Definition* A.1.
(B) *There exists $\varepsilon > 0$ and $N > 0$ such that for all $k \in \mathbb{Z}$ and for any dyadic cube Q of side length $\ell(Q) = 2^{-k}$, there exists another dyadic cube $\widetilde{Q}$ of the same side length such that the distance between Q and $\widetilde{Q}$ is at most $N2^{-k}$ and*

$$\frac{1}{|\widetilde{Q}|} \left| \int_{\widetilde{Q}} b(x)dx \right| \geq \varepsilon.$$

(C) *There exist $C > 0$, $\delta > 0$, and $u_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that for all $k \in \mathbb{Z}$,*

  (i) $\quad |u_k(x, y)| \leq C2^{kd}, \quad$ *for all $x, y \in \mathbb{R}^d$,*

  (ii) $\quad u_k(x, y) = 0, \quad$ *if $|x - y| \geq C2^{-k}$,*

  (iii) $\quad |u_k(x, y) - u_k(x, y')| \leq C2^{k(d+\delta)}|y - y'|, \quad$ *for all $x, y, y' \in \mathbb{R}^d$,*

  (iv) *for all $x \in \mathbb{R}^d$,*

$$\frac{1}{C} \leq \left| \int u_k(x, y)b(y)dy \right| \leq C.$$

(D) *There exist $C > 0$, $\delta > 0$, and $v_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that for all $k \in \mathbb{Z}$, the conditions (i)-(iv) in (C) are satisfied. Moreover, the following extra conditions are satisfied:*

  (v) $\quad \displaystyle\int v_k(x, y)dy = 1, \quad$ *for all $x \in \mathbb{R}^d$,*

  (vi) $\quad \displaystyle\int v_k(x, y)dx = 1, \quad$ *for all $y \in \mathbb{R}^d$,*

and

  (vii) *For all $y \in \mathbb{R}^d$, the function $v_k(\,\cdot\,, y)$ is constant for each dyadic cube of side length 2^{-k}.*

A.1. Definition A.1 implies Definition 2.5.

Let $b \in L^\infty$ be para-accretive according to Definition A.1. In the following, we show that (2.5) in Definition 2.5 follows from Proposition A.2 (B).

Let $\varepsilon > 0$ and $N > 0$ be as in Proposition A.2 (B). Without loss of generality, we assume that $N \geq 10$. Given a dyadic cube Q centered at x_0, choose $k \in \mathbb{Z}$ such that

(A.1) $$10 \cdot N2^{-k} \leq \ell(Q) \leq 20 \cdot N2^{-k}.$$

Fix a dyadic cube $Q_1 \subset Q$ of side length 2^{-k} containing the center x_0 of the cube Q. Then, by Proposition A.2 (B), there exists another dyadic cube Q_2 of side length 2^{-k} within distance $N2^{-k}$ from Q_1 such that

(A.2) $$\frac{1}{|Q_2|} \left| \int_{Q_2} b(x)dx \right| \geq \varepsilon.$$

Note that $Q_2 \subset Q$. Moreover, from (A.1) and (A.2), we have

$$\frac{1}{|Q|} \left| \int_{Q_2} b(x)dx \right| \geq \frac{\varepsilon}{(20N)^d}.$$

Since the choice of Q was arbitrary, this shows that b is indeed para-accretive in the sense of Definition 2.5.

A.2. Definition 2.5 implies Definition A.1.

Let $b \in L^\infty$ be para-accretive according to Definition 2.5. It suffices to construct a sequence $\{u_k\}_{k \in \mathbb{Z}}$ of functions u_k on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying conditions (i)-(iv) in Proposition A.2 (C).

Let $\phi \in C_0^\infty$ be a normalized bump function of order 1 such that $\int_{\mathbb{R}^d} \phi(x)dx = \alpha^{-1} > 0$. Then, let $\phi_\varepsilon(x) = \varepsilon^{-d}\alpha\phi(\varepsilon^{-1}x)$, that is, $\{\phi_\varepsilon\}_{\varepsilon > 0}$ is an approximation to the identity.

Given $k \in \mathbb{Z}$, let Q_k be the cube of side length 2^{-k} centered at the origin and $Q_k^x := x + Q_k$ be the cube of side length 2^{-k} centered at $x \in \mathbb{R}^d$. Then, by Definition 2.5, there exists a subcube $\widetilde{Q}_k^x \subset Q_k^x$ such that

$$(A.3) \qquad 2^{kd} \left| \int \mathbf{1}_{\widetilde{Q}_k^x}(y) b(y) dy \right| \geq c_0.$$

Here, c_0 is uniform in all cubes $Q_k^x \supset \widetilde{Q}_k^x$. From (2.6), we also have

$$(A.4) \quad \ell_{x,k} := \ell(\widetilde{Q}_k^x) \geq c_1 \ell(Q_k^x) = c_1 2^{-k}, \qquad \text{where } c_1 = c_1(b) := \left(\frac{c_0}{\|b\|_{L^\infty}} \right)^{\frac{1}{d}}.$$

Note that

$$(A.5) \qquad \mathbf{1}_{\widetilde{Q}_k^x}(y) = \mathbf{1}_{Q_0}(\ell_{x,k}^{-1}(y - \widetilde{x})),$$

where $\widetilde{x}$ is the center of the subcube $\widetilde{Q}_k^x$. Then, by setting $\varepsilon = h \ell_{x,k}$ for $h > 0$, we have

$$\mathbf{1}_{\widetilde{Q}_k^x} * \phi_\varepsilon(y) = \int \mathbf{1}_{\widetilde{Q}_k^x}(y - z) \phi_\varepsilon(z) dz = \ell_{x,k}^{-d} \int \mathbf{1}_{Q_0}(\ell_{x,k}^{-1}(y - \widetilde{x} - z)) \phi_h(\ell_{x,k}^{-1} z) dz$$

$$(A.6) \qquad = \mathbf{1}_{Q_0} * \phi_h(\ell_{x,k}^{-1}(y - \widetilde{x})).$$

Then, it follows from from (A.5) and (A.6) that we can choose sufficiently small $h \ll 1$ such that

$$2^{kd} \left| \int \mathbf{1}_{\widetilde{Q}_k^x} * \phi_\varepsilon(y) - \mathbf{1}_{\widetilde{Q}_k^x}(y) dy \right| = \frac{|\widetilde{Q}_k^x|}{|Q_k^x|} \left| \int \mathbf{1}_{\widetilde{Q}_0} * \phi_h(y) - \mathbf{1}_{\widetilde{Q}_0}(y) dy \right|$$

$$(A.7) \qquad \leq \left| \int \mathbf{1}_{\widetilde{Q}_0} * \phi_h(y) - \mathbf{1}_{\widetilde{Q}_0}(y) dy \right| \leq \frac{c_0}{2\|b\|_{L^\infty}},$$

uniformly in $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$. Hence, using (A.3), (A.7), and the triangle inequality, we obtain

$$(A.8) \qquad 2^{kd} \left| \int \mathbf{1}_{\widetilde{Q}_k^x} * \phi_{h\ell_{x,k}}(y) b(y) dy \right| \geq \frac{1}{2} c_0.$$

In the following, we fix $h \ll 1$ such that (A.7) holds.

Now, let us define u_k by

$$(A.9) \qquad u_k(x, y) := |Q_k|^{-1} \mathbf{1}_{\widetilde{Q}_k^x} * \phi_{h\ell_{x,k}}(y) = 2^{kd} \mathbf{1}_{\widetilde{Q}_k^x} * \phi_{h\ell_{x,k}}(y).$$

Then, from (A.8) and Young's inequality, we have

$$\frac{1}{2} c_0 \leq \left| \int u_k(x, y) b(y) dy \right| \leq \frac{|\widetilde{Q}_k^x|}{|Q_k^x|} \|\phi_{h\ell_{x,k}}\|_{L^1} \|b\|_{L^\infty} \leq \|b\|_{L^\infty}$$

for all $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}$. Hence, (iv) holds.

By the mean value theorem and Young's inequality with (A.4), we have

$$|u_k(x, y) - u_k(x, y')| \leq 2^{kd} \|\mathbf{1}_{\widetilde{Q}_k^x} * \partial(\phi_{h\ell_{x,k}})\|_{L^\infty} |y - y'|$$

$$\leq \alpha 2^{kd} |\widetilde{Q}_k^x| (h\ell_{x,k})^{-d-1} |y - y'|$$

$$\leq \alpha c_1^{-1} h^{-d-1} 2^{k(d+1)} |y - y'|$$

for all $x, y, y' \in \mathbb{R}^d$. This proves (iii). By Young's inequality, we have

$$\|u_k(x, y)\|_{L^\infty} \leq 2^{kd} \|\mathbf{1}_{\widetilde{Q}_k^x}\|_{L^1} \|\phi_{h\ell_{x,k}}\|_{L^\infty} \leq \alpha 2^{kd} |\widetilde{Q}_k^x| (h\ell_{x,k})^{-d} = \alpha h^{-d} 2^{kd}$$

for all $x, y \in \mathbb{R}^d$. This proves (i). Lastly, from (A.6) and (A.9), we have

$$(A.10) \qquad u_k(x, y) = 2^{kd} \mathbf{1}_{Q_0} * \phi_h(\ell_{x,k}^{-1}(y - \widetilde{x})) = 0$$

for $|\widetilde{x} - y| \geq \left(1 + \frac{\sqrt{d}}{2}\right)\ell_{x,k}$ since $h \leq 1$. Note that (i) $\ell_{x,k} = \ell(\widetilde{Q}_k^x) \leq \ell(Q_k^x) = 2^{-k}$ and (ii) $|x - \widetilde{x}| \leq \frac{\sqrt{d}}{2}2^{-k}$, since x and $\widetilde{x}$ are the centers of the cubes Q_k^x and $\widetilde{Q}_k^x$, respectively. This in particular implies that (A.10) holds for $|x - y| \geq \left(1 + \sqrt{d}\right)2^{-k}$. This proves condition (ii). By Proposition A.2, we conclude that b is para-accretive in the sense of Definition A.1.

Therefore, Definitions 2.5 and A.1 are equivalent.

Acknowledgments

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