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# The Backward Shift on the Hardy Space

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## Preface

Shift operators on Hilbert spaces of analytic functions play an important role in the study of bounded linear operators on Hilbert spaces since they often serve as “models” for various classes of linear operators. For example, “parts” of direct sums of the backward shift operator on the classical Hardy space  $H^2$  model certain types of contraction operators and potentially have connections to understanding the invariant subspaces of a general linear operator.

In this book, we do not want to give a general treatment of the backward shift on  $H^2$  and its connections to problems in operator theory. This has been done quite thoroughly by Nikolskiĭ in his book [65]. Instead, we wish to work in the Banach (and  $F$ -space) setting of  $H^p$  ( $0 < p < \infty$ ) where we will focus primarily on characterizing the backward shift invariant subspaces of  $H^p$ . When  $p \in (1, \infty)$ , this characterization problem was solved by R. Douglas, H. S. Shapiro, and A. Shields in a well known paper [29] which employed the concept of a ‘pseudocontinuation’ developed earlier by Shapiro [84]. When  $p \in (0, 1)$ , the characterization problem is more difficult, due to some topological differences between the two settings  $p \in [1, \infty)$  and  $p \in (0, 1)$ , and was solved in a paper of A. B. Aleksandrov [3] which was never translated from its original Russian and hence is not readily available in the West. The Aleksandrov paper is also quite complicated and makes use of the distribution theory and Coifman’s atomic decomposition for the Hardy spaces of the upper half plane, a topic we feel is not always at the fingertips of those schooled, as we were, in classical function theory and operator theory. It is for these reasons that we gather up these results, along with the necessary background material, and put them all under one roof.

In developing the necessary background results, we do not wish to reproduce the material in the books of Duren [31] or Garnett [39] (for a general treatment of Hardy spaces) or Stein [95] (for a detailed treatment of harmonic analysis and real variable  $H^p$  theory). Instead, we will only review this material and refer the interested reader to the appropriate places in these texts for the proofs. The reader is expected to have a reasonable background in functional analysis and function theory (including the basics of  $H^p$  theory), but might want to have Rudin’s functional analysis book [78], Duren’s  $H^p$  book [31], and Stein’s harmonic analysis book [95] at the ready while reading this book. We will try to develop the more specialized topics as we need them.

The authors wish to thank several people who helped us along the way. First, we thank A. B. Aleksandrov, who, through many e-mails, helped us understand the more difficult parts of his papers. Secondly, we thank Alec Matheson and Don Sarason, who read a draft of this book and provided us with useful suggestions and corrections. Thirdly, we thank Olga Troyanskaya, who translated the Aleksandrov paper [3] from the original Russian. Finally, the second author wishes to thank

the mathematics department of the University of North Carolina, Chapel Hill, for the comfortable setting for the semester in which he finally got to work with the first author face to face (and not over the Internet) where they assembled the final version of this book.

JAC AND WTR

## CHAPTER 1

### Overview

In this monograph, we will discuss the invariant subspaces of the backward shift operator on the Hardy space. Here, for  $p \in (0, \infty)$ , the Hardy space  $H^p$  is the set of analytic functions  $f$  on the open unit disk  $\mathbb{D} = \{|z| < 1\}$  for which the quantity

$$(1.0.1) \quad \|f\|_p := \sup_{0 < r < 1} \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right\}^{1/p}$$

is finite. The backward shift operator  $B$  on  $H^p$  is the continuous linear operator defined for a function  $f = \sum_{n=0}^{\infty} a_n z^n \in H^p$  by

$$Bf := \frac{f - f(0)}{z} = a_1 + a_2 z + a_3 z^2 + \dots.$$

By “invariant subspace” for the operator  $B$ , we mean a closed linear manifold (i.e., a subspace)  $\mathcal{M}$  of  $H^p$  for which  $B\mathcal{M} \subset \mathcal{M}$ . Although there are various aspects of the backward shift on the Hardy spaces that are certainly worthy of attention, this book will focus on only one of these topics:

*the characterization of the backward shift invariant subspaces.*

For  $p \in (1, \infty)$  the  $B$ -invariant subspaces of  $H^p$  were described in a well known paper of R. Douglas, H. S. Shapiro, and A. Shields [29]; while for  $p \in (0, 1]$ , they were described by A. B. Aleksandrov [3] in a remarkable paper which was never translated into English and as a result, is not readily available in the West. The main purpose of this book is to give a thorough treatment of all this material, replete with the appropriate background material and full details of the proofs.

Although the backward shift operator is interesting to study in its own right, it has important connections to the general study of bounded linear operators on Hilbert spaces. Work of Rota [76], de Branges and Rovnyak [26], and Foiaş [36] show that the restriction of a direct sum of backward shifts on  $H^2$  to an invariant subspace is often the “model” for important classes of bounded linear operators. For example, Rota showed that every strict contraction (a bounded operator  $T$  from a Hilbert space to itself with operator norm  $\|T\|$  strictly less than one) is similar to a direct sum of backward shift operators on  $H^2$  restricted to an invariant subspace (of the direct sum). One often uses the phrase  $T$  is similar to “part of a direct sum of backward shifts”. de Branges and Rovnyak, and Foiaş brought this model theory to fruition and proved that if  $T$  satisfies the two conditions

$$\|T\| \leq 1 \text{ and } \|T^n x\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad \forall x,$$

then  $T$  is unitarily equivalent to part of a direct sum of backward shifts. We refer the reader to [65] [75] [98] for a more detailed treatment of model theory and its connections to backward shifts.

Though the invariant subspaces of  $B$  on  $H^2$  are certainly not as complicated as the invariant subspaces of a direct sum of backward shifts, they do form interesting classes of functions whose description involves the boundary behavior of analytic functions near (and across) the unit circle  $\mathbb{T} = \partial\mathbb{D}$ . To describe these results, we recall some basic facts about  $H^p$  functions and refer the reader to Chapter 3 where they will be discussed in greater detail.

It can be shown from eq.(1.0.1) that for  $p \in [1, \infty)$ , the quantity  $\|f\|_p$  defines a norm on  $H^p$  which makes it a Banach space; while for  $p \in (0, 1)$ , the metric

$$\rho(f, g) := \|f - g\|_p^p$$

makes  $H^p$  an  $F$ -space. It is also known that functions belonging to  $H^p$  are reasonably well-behaved near  $\mathbb{T}$  and in fact, for any  $f \in H^p$ , the radial limit

$$f(\zeta) := \lim_{r \rightarrow 1^-} f(r\zeta)$$

exists for almost every  $\zeta \in \mathbb{T}$ . One of the most important properties of this boundary function is that  $f(\zeta)$  belongs to  $L^p = L^p(\mathbb{T}, dm)$  (where  $dm := d\theta/2\pi$  is normalized Lebesgue measure on the unit circle  $\mathbb{T}$ ) and the quantity in eq.(1.0.1) is equal to the  $L^p$ -norm of the boundary function; that is,

$$\|f\|_p := \sup_{0 < r < 1} \left\{ \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) \right\}^{1/p} = \left\{ \int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta) \right\}^{1/p}.$$

The linear mapping which takes a function  $f \in H^p$  to its boundary function  $f(\zeta)$  in  $L^p$  is an isometric mapping from  $H^p$  onto a closed subspace of  $L^p$  which we shall call  $H^p(\mathbb{T})$ . For  $p \in [1, \infty)$ , one can identify  $H^p(\mathbb{T})$  by examining the Fourier coefficients

$$\hat{f}(n) := \int_{\mathbb{T}} f(\zeta) \bar{\zeta}^n dm(\zeta), \quad n \in \mathbb{Z},$$

of  $f$ . Indeed, a theorem of F. and M. Riesz says that for  $f \in L^p$ ,

$$f \in H^p(\mathbb{T}) \Leftrightarrow \hat{f}(n) = 0, \quad \forall n < 0.$$

Moreover, the power series coefficients of  $f \in H^p$  are the same as the Fourier series coefficients of the boundary function  $f(\zeta)$  in  $H^p(\mathbb{T})$ . The behavior of this boundary function is not only useful in describing the topology of  $H^p$ , as seen above, but as we shall see shortly, is also instrumental in describing the backward shift invariant subspaces of  $H^p$ .

The primary tool used in describing the  $B$ -invariant subspaces of  $H^p$ , at least for  $p \in [1, \infty)$ , is the duality theory of  $H^p$  spaces. For  $p \in (1, \infty)$ , the norm dual,  $(H^p)^*$ , of  $H^p$  can be identified with  $H^q$ , where  $1/p + 1/q = 1$ , via the pairing

$$(1.0.2) \quad \langle f, g \rangle := \int_{\mathbb{T}} f(\zeta) \bar{g}(\zeta) dm(\zeta), \quad f \in H^p, \quad g \in H^q.$$

Furthermore, if  $S : H^q \rightarrow H^q$  is the forward shift operator  $(Sg)(z) := zg(z)$ , then it is easy to check that

$$\langle Bf, g \rangle = \langle f, Sg \rangle, \quad \forall f \in H^p, \quad g \in H^q$$

and so for a subspace  $\mathcal{M} \subset H^p$ ,

$$B\mathcal{M} \subset \mathcal{M} \Leftrightarrow S\mathcal{M}^\perp \subset \mathcal{M}^\perp,$$

where

$$\mathcal{M}^\perp := \{ g \in (H^p)^* : \langle f, g \rangle = 0 \quad \forall f \in \mathcal{M} \}$$

is the annihilator of  $\mathcal{M}$ . By a celebrated result of Beurling [14], every (non-trivial) forward shift invariant subspace of  $H^q$  is of the form  $IH^q$ , where  $I$  is a bounded analytic function on  $\mathbb{D}$  whose boundary values are unimodular almost everywhere on  $\mathbb{T}$  ( $|I(\zeta)| = 1$  almost everywhere on  $\mathbb{T}$ ). Such a function is called an “inner function”. By the Hahn-Banach separation theorem,

$$\mathcal{M} = (\mathcal{M}^\perp)^\perp = (IH^q)^\perp = \{ f \in H^p : \langle f, Ih \rangle = 0 \ \forall h \in H^q \}.$$

Using the F. and M. Riesz theorem and equating functions in  $\mathcal{M}$  with their boundary functions on  $\mathbb{T}$ , one can show that

$$\mathcal{M} = H^p(\mathbb{T}) \cap \overline{IH_0^p(\mathbb{T})},$$

where  $H_0^p = \{f \in H^p : f(0) = 0\}$  and  $H_0^p(\mathbb{T})$  are the boundary functions for  $H_0^p$ . For notational convenience, we write  $\mathcal{M} = H^p \cap \overline{IH_0^p}$  and understand this as a space of functions on  $\mathbb{T}$ .

Douglas, Shapiro, and Shields, were able to further describe the functions in  $(IH^q)^\perp = H^p \cap \overline{IH_0^p}$  by means of a “continuation” across  $\mathbb{T}$ . To motivate this idea, notice that

$$f \in (IH^q)^\perp \Rightarrow \langle f, Iz^n \rangle = 0 \ \forall n \in \mathbb{N} \cup \{0\}.$$

However,

$$\langle f, Iz^n \rangle = \int_{\mathbb{T}} f(\zeta) \overline{I(\zeta)} \zeta^{-n} dm(\zeta) = \widehat{f\overline{I}}(n)$$

which is the non-negative Fourier coefficient of the  $L^p$  function  $f\overline{I}$ . This means that  $f\overline{I}$  has Fourier expansion

$$(1.0.3) \quad f\overline{I} \sim b_1 \overline{\zeta} + b_2 \overline{\zeta}^2 + \dots$$

If one defines the function  $g$  on the extended exterior disk  $\mathbb{D}_e := \mathbb{C}_\infty \setminus \mathbb{D}^-$  by

$$g(z) := \frac{b_1}{z} + \frac{b_2}{z^2} + \dots, \quad z \in \mathbb{D}_e,$$

one shows (see Chapter 5) that

$$(1.0.4) \quad \sup_{r>1} \int_{\mathbb{T}} |g(r\zeta)|^p dm(\zeta)/r$$

is finite. Analytic functions  $g$  on  $\mathbb{D}_e$  for which eq.(1.0.4) is finite are said to belong to  $H^p(\mathbb{D}_e)$ , the Hardy space of the extended exterior disk. One can also show, as suggested by eq.(1.0.3), that the boundary values of  $g$  are almost always the same as the boundary values of the meromorphic function  $f/I$ , that is to say

$$\lim_{r \rightarrow 1^+} g(r\zeta) = \lim_{r \rightarrow 1^-} \frac{f(r\zeta)}{I(r\zeta)}$$

almost everywhere. Thus if  $f \in (IH^q)^\perp$ , then the boundary values of the meromorphic function  $f/I$  are the same, almost everywhere, as the boundary values of some  $H^p(\mathbb{D}_e)$  function  $g$  which vanishes at infinity. Using a term of H.S. Shapiro [84], we say that  $g$  is a pseudocontinuation of  $f/I$ . A more formal definition will be given in Chapter 2. Douglas, Shapiro, and Shields [29] were able to show the converse to the above statement which is the first main result of this monograph.

**THEOREM 1.0.5** (Douglas, Shapiro, Shields [29]). *Let  $p \in (1, \infty)$  and  $\mathcal{M}$  be a non-trivial  $B$ -invariant subspace of  $H^p$ . Then there is an inner function  $I$  such that  $\mathcal{M} = H^p \cap \overline{IH_0^p}$ . Moreover,  $f \in \mathcal{M}$  if and only if  $f/I$  has a “pseudocontinuation” to a function belonging to  $H^p(\mathbb{D}_e)$  which vanishes at infinity.*

In fact, even more can be said about functions belonging to  $(IH^q)^\perp$ . If

$$\sigma(I) := \{ \lambda \in \mathbb{D}^- : \liminf_{z \rightarrow \lambda} |I(z)| = 0 \}$$

denotes the “lim inf zero set” for  $I$  (also called the “spectrum” of  $I$ ), then every  $f \in (IH^q)^\perp$  has an analytic continuation to

$$\mathbb{C}_\infty \setminus \{ 1/\bar{z} : z \in \sigma(I) \}.$$

For  $p = 1$ , the description of the  $B$ -invariant subspaces is the same as before, namely  $\mathcal{M} = H^1 \cap \overline{IH_0^1}$ , except that the proof needs to be different due to some technicalities which arise from a slightly different dual pairing. For instance, the norm dual of  $H^1$  can be identified not with  $H^\infty$ , but with the somewhat larger class of functions  $BMOA$ , the analytic functions of bounded mean oscillation (see Chapter 3 for a definition) via the pairing

$$(1.0.6) \quad \lim_{r \rightarrow 1} \int_{\mathbb{T}} f(r\zeta) \bar{g}(\zeta) dm(\zeta), \quad f \in H^1, g \in BMOA.$$

The limit is needed here since  $f\bar{g}$  is not always integrable. Unfortunately, this identification makes the Fourier analysis used in the  $p > 1$  case unworkable. Despite these difficulties, Aleksandrov is able to identify the  $B$ -invariant subspaces of  $H^1$  and we will give a thorough treatment of his proof (which is hinted at in [3]) in Chapter 5 of this monograph. His technique is quite delicate and involves representing the duality in eq.(1.0.6) in a more usable form by using the truncation operator on functions of bounded mean oscillation.

Also included in our discussion of the backward shift on  $H^p$  ( $1 \leq p < \infty$ ) will be a closer look at some of the functional analysis properties of a typical invariant subspace  $H^p \cap \overline{IH_0^p}$  as well as the spectral properties of the operator  $B|_{H^p \cap \overline{IH_0^p}}$ . It is well known that

$$\sigma(B) = \mathbb{D}^-, \quad \sigma_p(B) = \mathbb{D}.$$

The description of  $\sigma(B|_{H^p \cap \overline{IH_0^p}})$ , due to Moeller [63], is the following.

**THEOREM 1.0.7** (Moeller [63]). *Let  $p \in [1, \infty)$ ,  $I$  be an inner function, and let  $\mathcal{M} := H^p \cap \overline{IH_0^p}$ . Then*

$$\sigma(B|\mathcal{M}) = \overline{\sigma(I)}.^1$$

Moreover

1.  $\sigma_p(B|\mathcal{M}) \cap \mathbb{D} = \overline{\sigma(I)} \cap \mathbb{D} = \{ \lambda \in \mathbb{D} : (1 - \lambda z)^{-1} \in \mathcal{M} \}$ .
2.  $\bar{\zeta} \notin \sigma(B|\mathcal{M}) \cap \mathbb{T}$  if and only if there is an open neighborhood  $U$  of the point  $\zeta$  such that every  $f \in \mathcal{M}$  has an analytic continuation to  $U$ .

For  $p \in (1, \infty)$ , the dual of  $H^p$  can be identified with  $H^q$  ( $1/p + 1/q = 1$ ) via the pairing eq.(1.0.2); while for  $p = 1$ , the dual of  $H^1$  can be identified with  $BMOA$  via the pairing eq.(1.0.6). The following theorem identifies the dual of  $H^p \cap \overline{IH_0^p}$ .

<sup>1</sup>As mentioned in the section on “numbering and notation”,  $\overline{\sigma(I)} = \{ \bar{\lambda} : \lambda \in \sigma(I) \}$  and not the closure of  $\sigma(I)$ . The closure of a set  $A$  is denoted by  $A^-$ .

THEOREM 1.0.8. *If  $p \in (1, \infty)$  and  $I$  is an inner function, then the dual of  $H^p \cap \overline{IH_0^p}$  can be identified with  $H^q \cap \overline{IH_0^q}$  via the pairing*

$$\int f \bar{g} \, dm, \quad f \in H^p \cap \overline{IH_0^p}, \quad g \in H^q \cap \overline{IH_0^q}.$$

For  $1 \leq p' < p < \infty$ , the inclusion operator  $i : H^p \cap \overline{IH_0^p} \rightarrow H^{p'} \cap \overline{IH_0^{p'}}$  is continuous. In fact, more is true.

THEOREM 1.0.9 (Aleksandrov [6]). *For  $1 \leq p' < p < \infty$ , the inclusion operator  $i : H^p \cap \overline{IH_0^p} \rightarrow H^{p'} \cap \overline{IH_0^{p'}}$  is compact.*

Due to significant differences in both the functional analysis and the function theory when moving from  $p \in [1, \infty)$  to  $p \in (0, 1)$ , the description of the  $B$ -invariant subspaces of  $H^p$  ( $0 < p < 1$ ) is intensely more complicated as is the technique used to prove it. Some of the classical tools of functional analysis, for example, the Hahn-Banach separation theorem, fail in the  $p \in (0, 1)$  setting. There are indeed proper  $B$ -invariant subspaces  $\mathcal{M}$  of  $H^p$  for which  $\mathcal{M}^\perp = (0)$ . Here the dual of  $H^p$  can be identified with a space of Lipschitz or Zygmund functions, see Chapter 3. In light of a general result of Kalton [46], the failure of the Hahn-Banach separation property should not be surprising due to the fact that the metric topology on  $H^p$  is not locally convex. Important results from function theory, for example, the Cauchy integral theorem, an important function theoretic tool which allows us to recapture a function in  $H^p$  ( $1 \leq p < \infty$ ) from its boundary values, also fail when  $p \in (0, 1)$ . In fact, some of the most important examples of functions in  $H^p$  ( $0 < p < 1$ ), for example,

$$\frac{1}{(1 - \bar{\zeta}z)^k}, \quad \zeta \in \mathbb{T}, \quad k \in \mathbb{N} \cap [1, 1/p),$$

cannot be represented as Cauchy integrals against their boundary values. Closed linear spans of certain subsets of these functions also serve as examples of proper  $B$ -invariant subspaces  $\mathcal{M}$  of  $H^p$  for which  $\mathcal{M}^\perp$  is zero.

Aleksandrov is able to overcome these obstacles and successfully describe the  $B$ -invariant subspaces of  $H^p$  ( $0 < p < 1$ ). His description is complicated and involves several parameters which we now take a moment to describe. The first parameter is the integer

$$n_p := \max\{n \in \mathbb{N} \cap [1, 1/p)\} = [1/p].$$

A routine computation shows that for all  $\zeta \in \mathbb{T}$ ,

$$\frac{1}{(1 - \bar{\zeta}z)^j} \in H^p \quad \forall j = 1, \dots, n_p.$$

Informally,  $n_p$  is the largest order pole an  $H^p$  function can have at a point  $\zeta \in \mathbb{T}$ . The second parameter will be an inner function  $I$ . The third parameter will be a closed set  $F \subset \mathbb{T}$  which we will assume (we will see why later) contains  $\sigma(I) \cap \mathbb{T}$ . The final parameter will be a function  $k : F \rightarrow \mathbb{N} \cap [1, n_p]$ . Consider the space  $\mathcal{E}^p(I, F, k)$ , which will turn out to be a typical  $B$ -invariant subspace, consisting of  $H^p$  functions  $f$  such that

- (i)  $f \in H^p \cap \overline{IH_0^p}$
- (ii)  $f$  has an analytic continuation to a neighborhood of  $\mathbb{T} \setminus F$
- (iii) for each  $\zeta \in F_0 \setminus \sigma(I)$ , where  $F_0$  is the set of isolated points of  $F$ ,  $f$  has a pole of order at most  $k(\zeta)$ .

It is not difficult to show, as was the case when  $p \in [1, \infty)$ , that  $f \in H^p \cap \overline{IH_0^p}$  ( $0 < p < 1$ ) if and only if  $f/I$  has a pseudocontinuation to a function belonging to  $H^p(\mathbb{D}_e)$  which vanishes at infinity. However, in contrast to the  $p \in [1, \infty)$  case, functions in  $H^p \cap \overline{IH_0^p}$  need not have analytic continuations to  $\mathbb{C}_\infty \setminus \{1/\bar{z} : z \in \sigma(I)\}$ . In fact for any inner function  $I$ ,

$$\frac{1}{(1 - \bar{\zeta}z)^j} \in H^p \cap \overline{IH_0^p}, \quad \forall j = 1, \dots, n_p, \quad \zeta \in \mathbb{T}.$$

This is why the second and third properties in the definition of  $\mathcal{E}^p(I, F, k)$  are important. Aleksandrov's result is the following.

**THEOREM 1.0.10** (A. B. Aleksandrov [3]). *For each  $p \in (0, 1)$  and triple  $I, F$ , and  $k$  as above, the linear manifold  $\mathcal{E}^p(I, F, k)$  is a (closed)  $B$ -invariant subspace of  $H^p$ . Moreover, every non-trivial  $B$ -invariant subspace of  $H^p$  is of this form.*

We will certainly take the time in Chapter 6 to explain the parameters in detail. For example, why does  $F$  need to contain  $\sigma(I) \cap \mathbb{T}$ ? Why do we only specify the order of the pole at  $F_0 \setminus \sigma(I)$ ? Why is  $\mathcal{E}^p(I, F, k)$  closed?

Given a  $B$ -invariant subspace  $\mathcal{M}$ , one might wonder how to obtain the parameters  $I, F$ , and  $k$ . This is done in the following way: Given  $\mathcal{M}$ , the subspace  $\mathcal{M} \cap H^2$  is a (possibly zero)  $B$ -invariant subspace of  $H^2$  which, by the Douglas-Shapiro-Shields result, is of the form  $H^2 \cap \overline{IH_0^2}$ .<sup>2</sup> With some work, one shows that  $\mathcal{M} \subset H^p \cap \overline{IH_0^p}$ . To get the closed set  $F$ , we define

$$F := \left\{ \zeta \in \mathbb{T} : \frac{1}{1 - \bar{\zeta}z} \in \mathcal{M} \right\}$$

and prove that  $F$  is indeed a closed subset of  $\mathbb{T}$ . It can be shown that if  $\zeta \in \sigma(I) \cap \mathbb{T}$ , then  $\zeta \in F$ , which is why we assumed from the onset that  $F$  contains  $\sigma(I) \cap \mathbb{T}$ . To obtain the function  $k$ , define  $k : F \rightarrow \mathbb{N} \cap [1, n_p]$  by

$$k(\zeta) := \max \left\{ j \in \mathbb{N} : \frac{1}{(1 - \bar{\zeta}z)^j} \in \mathcal{M} \right\}.$$

We will see that if  $\zeta$  belongs to either  $F \setminus F_0$  (the non-isolated points of  $F$ ) or  $\sigma(I) \cap \mathbb{T}$ , then  $k(\zeta) = n_p$ , which is why we only specify the poles at  $F_0 \setminus \sigma(I)$ . Given  $\mathcal{M}$ , we now have our three parameters  $I, F$ , and  $k$ .

The first important result here is that the  $B$ -invariant linear manifold  $\mathcal{E}^p(I, F, k)$  is actually closed in  $H^p$  and

$$\mathcal{M} \subset \mathcal{E}^p(I, F, k).$$

To obtain the reverse inclusion, Aleksandrov defines

$$e^p(I, F, k) := \left( H^2 \cap \overline{IH_0^2} \right) \bigvee \left\{ \frac{1}{(1 - \bar{\zeta}z)^j} : \zeta \in F, 1 \leq j \leq k(\zeta) \right\}.$$

By the definitions of our three parameters, we have

$$e^p(I, F, k) \subset \mathcal{M} \subset \mathcal{E}^p(I, F, k).$$

The equality  $e^p(I, F, k) = \mathcal{E}^p(I, F, k)$  is obtained by an ingenious rational approximation argument developed especially for this situation using the Coifman atomic decomposition [21] to control the order and location of the poles. A major portion of Chapter 6 will be dedicated to giving the full details and background material needed to understand this argument.

<sup>2</sup>If  $\mathcal{M} \cap H^2 = (0)$ , take the inner function  $I$  to be the constant function one.

As we did when  $p \in [1, \infty)$ , we will look at some of the functional analysis properties of the space  $\mathcal{E}^p(I, F, k)$  as well as the operator theory properties of  $B|\mathcal{E}^p(I, F, k)$ . For example, one can compute the spectrum of  $B|\mathcal{E}^p(I, F, k)$ .

**THEOREM 1.0.11.**  $\sigma(B|\mathcal{E}^p(I, F, k)) = \sigma_p(B|\mathcal{E}^p(I, F, k)) = \overline{\sigma(I) \cup F}$ .

When  $p \in [1, \infty)$ , one is able to identify the dual of a  $B$ -invariant subspace of  $H^p$  as well as the commutant of  $B$  when restricted to an invariant subspace. When  $p \in (0, 1)$ , these two problems become more complicated and we will make some comments on the difficulties that arise.

The layout of this book is as follows. Chapter 2 (Classical boundary value results): Since the description of the backward shift invariant subspaces of the Hardy space depends heavily on the notion of a non-tangential limit and a pseudocontinuation, we will review, without proof, the fundamentals of these topics. We start with the definitions of a radial limit and a non-tangential limit for analytic functions on the unit disk and recall the classical existence and uniqueness results of Fatou [33], F. and M. Riesz [72], Plessner [66], Lusin, and Privalov [59] [69] [68]. We then proceed to define the term pseudocontinuation, as developed by H. S. Shapiro [84], and give several examples.

Chapter 3 (The Hardy space of the disk): We will assume the reader is somewhat familiar with the results from the theory of Hardy spaces and we will not attempt to rewrite the material from the books of Duren [31], Garnett [39], Hoffman [44], Koosis [51], and Zygmund [106]. We quickly review, without proof, the basic definitions of the Hardy spaces and catalog the main properties of these spaces for their use in later chapters. The results include: definition of the Hardy spaces, boundary values, factorization, Hardy spaces and Fourier analysis, the F. and M. Riesz theorem, Riesz projections and harmonic conjugation, duality, bounded mean oscillation, Lipschitz classes, Cauchy transforms, Kolmogorov's theorem, and Fatou's jump theorem.

Chapter 4 (The Hardy space of the upper-half plane): It is well known that functions in  $H^p$  ( $1 \leq p < \infty$ ) can be recovered from their non-tangential boundary values by means of the Cauchy integral formula. Although this is no longer true for  $H^p$  ( $0 < p < 1$ ), there is a deep theory of Fefferman, Stein, and Coifman which says that functions in the Hardy space correspond to a certain class of distributions on the circle and that the functions can be recovered from their corresponding distributions by means of a distributional form of the Cauchy integral formula via the atomic decomposition. We will discuss the atomic decomposition of a distribution arising from a Hardy space function since it will be a key ingredient in the rational approximation arguments mentioned earlier. This entire theory was originally formulated for spaces of the upper half-plane, hence the title of this chapter. This slight detour to the upper half plane is beneficial to us since the atomic decomposition theory for the Hardy spaces of the upper half plane is developed, with complete proofs, in Stein's harmonic analysis book [95]. In this chapter, we will develop and state the needed results and will refer the reader to the appropriate places in Stein's book for the proofs. These results include: definition and basic properties of the Hardy spaces of the upper-half plane, Poisson and conjugate Poisson integrals, maximal functions, the Hilbert transform, the harmonic Hardy space, distributions and the harmonic Hardy space, and Coifman's atomic decomposition.

Chapter 5 (The backward shift on  $H^p$  for  $p \in [1, \infty)$ ): We will include three different proofs of the description of the  $B$ -invariant subspaces of  $H^p$ . We do this

not only to demonstrate some of the techniques in this field, but more importantly, to show the reader how the techniques in two of these three proofs can be used to examine the  $B$ -invariant subspaces of other spaces of analytic functions. As mentioned above, every  $B$ -invariant subspace is of the form  $(IH^q)^\perp$ . All three proofs describe this annihilator but do it in different ways. The first is the simplest and most direct and uses the F. and M. Riesz theorem. The second one is more complicated but allows the reader to see an actual formula for the pseudocontinuation of the functions in  $(IH^q)^\perp$ , a formula which is not readily transparent from the first proof. A variation of the second proof can be used to examine the backward shift invariant subspaces of the Bergman spaces as was done by Aleman, Richter, and Ross [8] [9]. The second proof can also be used to describe  $\sigma(B|H^p \cap \overline{IH_0^p})$ . The third proof is due to Aleksandrov [5] and gives a unified approach to characterizing the  $B$ -invariant subspaces of other spaces of functions such as  $VMOA$ ,  $BMOA$ , and  $L^1/\overline{H_0^1}$  (Chapter 3 will have the formal definitions of these spaces). We end the chapter with some remarks about some of the functional analysis properties of  $H^p \cap \overline{IH_0^p}$  as well as the operator theory properties of  $B|H^p \cap \overline{IH_0^p}$ .

Chapter 6 (The backward shift on  $H^p$  for  $p \in (0, 1)$ ): We finally arrive at the full proof of Aleksandrov's characterization of the  $B$ -invariant subspaces for  $H^p$ . After proving some basic facts about the subspaces  $\mathcal{E}^p(I, F, k)$ , we then proceed to his rational approximation scheme which has been laid out in several papers [2] [3] [4]. As with many of the results in  $H^p$  theory for  $p \in (0, 1)$  (see [32] for example) we need to consider two cases:  $1/p \in \mathbb{N}$  and  $1/p \notin \mathbb{N}$ , the former usually the most difficult case to handle. Finally, we make some remarks about dual of  $\mathcal{E}^p(I, F, k)$  as well as the commutants of  $B$  and  $B|\mathcal{E}^p(I, F, k)$ .