Introduction

The prototype for the Fekete-Szegö theorem with local rationality is Raphael Robinson's theorem on totally real algebraic integers in an interval:

Theorem (Robinson [**48**], 1964). *Let $[a, b] \subset \mathbb{R}$. If $b - a > 4$, then there are infinitely many totally real algebraic integers whose conjugates all belong to $[a, b]$. If $b - a < 4$, there are only finitely many.*

Robinson also gave a criterion for the existence of totally real units in $[a, b]$:

Theorem (Robinson [**49**], 1968). *Suppose $0 < a < b \in \mathbb{R}$ satisfy the conditions*

$$(0.1) \qquad\qquad \log(\frac{b-a}{4}) \; > \; 0 \; ,$$

$$(0.2) \qquad \log(\frac{b-a}{4}) \cdot \log(\frac{b-a}{4ab}) - \Big(\log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) \Big)^2 \; > \; 0 \; .$$

Then there are infinitely many totally real units α whose conjugates all belong to $[a, b]$. If either inequality is reversed, there are only finitely many.

David Cantor's "Fekete-Szegö theorem with splitting conditions" on $\mathbb{P}^1$ ([**14**], Theorem 5.1.1, 1980) generalized Robinson's theorems, reformulated them adelically, and set them in a potential-theoretic framework.

In this work we prove a strong form of Cantor's result, valid for algebraic curves of arbitrary genus over global fields of any characteristic.

Let K be a global field, a number field or a finite extension of $\mathbb{F}_p(T)$ for some prime p. Let $\widetilde{K}$ be a fixed algebraic closure of K, and let $\widetilde{K}^{\mathrm{sep}} \subseteq \widetilde{K}$ be the separable closure of K. We will write $\mathrm{Aut}(\widetilde{K}/K)$ for the group of automorphisms $\mathrm{Aut}(\widetilde{K}/K) \cong \mathrm{Gal}(\widetilde{K}^{\mathrm{sep}}/K)$. Let $\mathcal{M}_K$ be the set of all places of K. For each $v \in \mathcal{M}_K$, let K_v be the completion of K at v, let $\widetilde{K}_v$ be an algebraic closure of K_v, and let $\mathbb{C}_v$ be the completion of $\widetilde{K}_v$. We will write $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ for the group of continuous automorphisms of $\mathbb{C}_v/K_v$; thus $\mathrm{Aut}_c(\mathbb{C}_v/K_v) \cong \mathrm{Aut}(\widetilde{K}_v/K_v) \cong \mathrm{Gal}(\widetilde{K}_v^{\mathrm{sep}}/K_v)$.

Let $\mathcal{C}/K$ be a smooth, geometrically integral, projective curve. If F is a field containing K, put $\mathcal{C}_F = \mathcal{C} \times_K \mathrm{Spec}(F)$ and let $\mathcal{C}(F) = \mathrm{Hom}_F(\mathrm{Spec}(F), \mathcal{C}_F)$ be the set of F-rational points; let $F(\mathcal{C})$ be the function field of $\mathcal{C}_F$. When $F = K_v$, we write $\mathcal{C}_v$ for $\mathcal{C}_{K_v}$. Let $\mathfrak{X} = \{x_1, \dots, x_m\}$ be a finite, Galois-stable set points of $\mathcal{C}(\widetilde{K})$, and let $\mathbb{E} = \mathbb{E}_K = \prod_{v \in \mathcal{M}_K} E_v$ be a K-rational adelic set for $\mathcal{C}$, that is, a product of sets $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ such that each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$. For each v, fix an embedding $\widetilde{K} \hookrightarrow \mathbb{C}_v$ over K, inducing an embedding $\mathcal{C}(\widetilde{K}) \hookrightarrow \mathcal{C}_v(\mathbb{C}_v)$. In this way $\mathfrak{X}$ can be regarded as a subset of $\mathcal{C}_v(\mathbb{C}_v)$: since $\mathfrak{X}$ is Galois-stable, its image is independent of the choice of embedding. The same is true for any Galois-stable set of points in $\mathcal{C}(\widetilde{K})$, such as the set of $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of a point $\alpha \in \mathcal{C}(\widetilde{K})$.

We will call a set $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ an RL-domain ("Rational Lemniscate Domain") if there is a nonconstant rational function $f_v(z) \in \mathbb{C}_v(\mathcal{C}_v)$ such that $E_v = \{z \in \mathcal{C}_v(\mathbb{C}_v) : |f_v(z)|_v \leq 1\}$. This terminology is due to Cantor. By combining ([**26**], Satz 2.2) with ([**51**], Corollary 4.2.14), one sees that a set is an RL-domain if and only if it is a strict affinoid subdomain of $\mathcal{C}_v(\mathbb{C}_v)$, in the sense of rigid analysis.

Fix an embedding $\mathcal{C} \hookrightarrow \mathbb{P}_K^N = \mathbb{P}^N / \mathrm{Spec}(K)$ for an appropriate N, and equip $\mathbb{P}_K^N$ with a (K-rational) system of homogeneous coordinates. For each nonarchimedean v, this data determines a model $\mathfrak{C}_v / \mathrm{Spec}(\mathcal{O}_v)$. There is a natural metric $\|x, y\|_v$ on $\mathbb{P}_v^N(\mathbb{C}_v)$: the chordal distance associated to the Fubini-Study metric, if v is archimedean; the v-adic spherical metric, if v is nonarchimedean (see §3.4 below). The metric $\|x, y\|_v$ induces the v-topology on $\mathcal{C}_v(\mathbb{C}_v)$. Given $a \in \mathcal{C}_v(\mathbb{C}_v)$ and $r > 0$, we write $B(a, r)^- = \{z \in \mathcal{C}_v(\mathbb{C}_v) : \|z, a\|_v < r\}$ and $B(a, r) = \{z \in \mathcal{C}_v(\mathbb{C}_v) : \|z, a\|_v \leq r\}$ for the corresponding "open" and "closed" balls.

Sets in $\mathcal{C}_v(\mathbb{C}_v)$ that are well-behaved for capacity theory are called *algebraically capacitable* (see Definition 3.18 below). Finite unions of RL-domains and compact sets are algebraically capacitable (in the nonarchimedean case, this is follows from [**51**], Corollary 4.2.14 and Theorem 4.3.11). For the Fekete-Szegö theorem with local rationality, we need to restrict to a smaller class of sets:

DEFINITION 0.1. If $v \in \mathcal{M}_K$, and $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ is nonempty and stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$, we will say that E_v has a *finite K_v-primitive cover* if it can be written as a finite union $E_v = \bigcup_{\ell=1}^M E_{v,\ell}$, where
  (A) If v is archimedean and $K_v \cong \mathbb{C}$, then each $E_{v,\ell}$ is
      compact, connected, and bounded by finitely many Jordan curves.
  (B) If v is archimedean and $K_v \cong \mathbb{R}$, then each $E_{v,\ell}$ is either
      (1) compact, connected, and bounded by finitely many Jordan curves, or
      (2) is a closed subinterval of $\mathcal{C}_v(\mathbb{R})$ with nonempty interior.
  (C) If v is nonarchimedean, then each $E_{v,\ell}$ is either
      (1) an RL-domain,
      (2) a ball $B(a_\ell, r_\ell)$ with radius r_ℓ in the value group of $\mathbb{C}_v^\times$, or
      (3) is compact, and has the form $\mathcal{C}_v(F_{w,\ell}) \cap D_v$ for some ball or
          RL-domain D_v, and some finite separable extension $F_{w,\ell}/K_v$.

Note that the sets $E_{v,\ell}$ can overlap, sets $E_{v,\ell}$ of more than one type can occur for a given v, and the extensions $F_{w,\ell}/K_v$ need not be Galois.

DEFINITION 0.2. If v is a nonarchimedean place of K, a set $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ will be called $\mathfrak{X}$-*trivial* if $\mathfrak{C}_v$ has good reduction at v, if the points of $\mathfrak{X}$ specialize to distinct points $(\mathrm{mod}\ v)$, and if $E_v = \mathcal{C}_v(\mathbb{C}_v) \backslash \bigcup_{i=1}^m B(x_i, 1)^-$.

If E_v is $\mathfrak{X}$-trivial, it consists of all points of $\mathcal{C}_v(\mathbb{C}_v)$ which are $\mathfrak{X}$-integral at v for the model $\mathfrak{C}_v$, i.e., which specialize to points complementary to $\mathfrak{X}$ $(\mathrm{mod}\ v)$. In particular, it is an RL-domain and is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$.

DEFINITION 0.3. An adelic set $\mathbb{E} = \prod_{v \in \mathcal{M}_K} E_v \subset \prod_{v \in \mathcal{M}_K} \mathcal{C}_v(\mathbb{C}_v)$ will be called K-*rational* if each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$. It will be called *compatible with* $\mathfrak{X}$ if the following conditions hold:
  (1) Each E_v is bounded away from $\mathfrak{X}$ in the v-topology.
  (2) For all but finitely many v, E_v is $\mathfrak{X}$-trivial.

The properties of K-rationality and compatibility with $\mathfrak{X}$ are independent of the choice of projective embedding of $\mathcal{C}$.

When each E_v is algebraically capacitable, there is a potential-theoretic measure of size for the adelic set $\mathbb{E}$ relative to the set of global points $\mathfrak{X}$: the Cantor capacity $\gamma(\mathbb{E}, \mathfrak{X})$, defined in formula (0.10) below. Our main result is:

THEOREM 0.4 (The Fekete-Szegö Theorem with Local Rationality on Curves). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, geometrically integral, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{K}/K)$, and let $\mathbb{E} = \prod_v E_v \subset \prod_v \mathcal{C}_v(\mathbb{C}_v)$ be a K-rational adelic set compatible with $\mathfrak{X}$. (Thus, each E_v is bounded away from $\mathfrak{X}$ and stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$, and E_v is $\mathfrak{X}$-trivial for all but finitely many v.) Let $S \subset \mathcal{M}_K$ be a finite set of places containing all archimedean v and all nonarchimedean v such that E_v is not $\mathfrak{X}$-trivial. Assume that:*

    (A) For each $v \in S$, E_v has a finite K_v-primitive cover.
    (B) $\gamma(\mathbb{E}, \mathfrak{X}) > 1$.
Then there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to E_v.

The primary content of the theorem is the local rationality assertion (the fact that the conjugates belong to E_v, for each v); the Fekete-Szegö theorem without local rationality, which constructs points α whose conjugates belong to arbitrarily small $\mathcal{C}_v(\mathbb{C}_v)$-neighborhoods of E_v, was proved in ([**51**], Theorem 6.3.2). In §2.4 we provide examples due to Daeshik Park, showing the need for the hypothesis of separability for the extensions $F_{w,\ell}/K_v$ in part (C) of the definition of a finite K_v-primitive cover.

Suppose that in the theorem, for each $v \in S$ we have $E_v \subset \mathcal{C}_v(K_v)$. Then for each $v \in S$, the conjugates of α belong to $\mathcal{C}_v(K_v)$, which means that v splits completely in $K(\alpha)$. In this case, following ([**52**]) and ([**53**]), we speak of "the Fekete-Szegö theorem with splitting conditions".

Sometimes it is the corollaries of a theorem, which are weaker but easier to apply, that are most useful. The following corollary of Theorem 0.4 strengthens Moret-Bailly's theorem for "Incomplete Skolem Problems on Affine Curves" ([**39**], Théorème 1.3, p.182), but does not require evaluating capacities.

Suppose $\mathcal{A}/K$ is an affine curve, embedded in $\mathbb{A}^N$ for some N. Let $z_1, \ldots, z_N$ be the coordinates on $\mathbb{A}^N$; given $v \in \mathcal{M}_K$ and a point $P \in \mathcal{A}(\mathbb{C}_v)$, write $\|P\|_{\mathcal{A},v} = \max(|z_1(P)|_v, \ldots, |z_N(P)|_v)$. We will say that a set $E_v \subset \mathcal{A}(\mathbb{C}_v)$ has a finite K_v-primitive cover relative to $\mathcal{A}$ if it is bounded under $\|\cdot\|_{\mathcal{A},v}$ and satisfies the conditions of Definition 0.1 with $\mathcal{C}$ replaced by $\mathcal{A}$, using balls for the metric $\|x - y\|_{\mathcal{A},v}$.

COROLLARY 0.5 (Fekete-Szegö for Skolem Problems on Affine Curves). *Let K be a global field, and let $\mathcal{A}/K$ be a geometrically integral (possibly singular) affine curve, embedded in $\mathbb{A}^N$. Fix a place v_0 of K, and let $S \subset \mathcal{M}_K \backslash \{v_0\}$ be a finite set of places containing all archimedean $v \neq v_0$. For each $v \in S$, let $E_v \subset \mathcal{A}_v(\mathbb{C}_v)$ be nonempty and stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$, with a finite K_v-primitive cover relative to $\mathcal{A}$. Assume that for each $v \in \mathcal{M}_K \backslash (S \cup \{v_0\})$ there is a point $P \in \mathcal{A}(\mathbb{C}_v)$ with $\|P\|_{\mathcal{A},v} \leq 1$. Then there is a bound $R = R(\mathcal{A}, \{E_v\}_{v \in S}, v_0) < \infty$ such that there are infinitely many points $\alpha \in \mathcal{A}(\widetilde{K}^{\mathrm{sep}})$ for which*

    (1) for each $v \in S$, all the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates $\sigma(\alpha)$ belong to E_v;
    (2) for each $v \in \mathcal{M}_K \backslash (S \cup \{v_0\})$, all the conjugates satisfy $\|\sigma(\alpha)\|_{\mathcal{A},v} \leq 1$;
    (3) for $v = v_0$, all the conjugates satisfy $\|\sigma(\alpha)\|_{\mathcal{A},v_0} \leq R$.

In Chapter 1 below, we will give several variants of Theorem 0.4, including one involving "quasi-neighborhoods" analogous to the classical theorem of Fekete and Szegö, one for more general sets $\mathbb{E}$ using the inner Cantor capacity $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$, and two for sets on Berkovich curves. Theorem 0.4, Corollary 0.5, and the variants in Chapter 1 will be proved in Chapter 4.

Some History

The original theorem of Fekete and Szegö ([**25**], 1955) said that if $E \subset \mathbb{C}$ was compact and stable under complex conjugation, with logarithmic capacity $\gamma_\infty(E) > 1$, then every neighborhood U of E contained infinitely many conjugates sets of algebraic integers. (The neighborhood U was needed to 'fatten' sets like a circle $E = C(0, r)$ with transcendental radius r, which contain no algebraic numbers.)

A decade later Raphael Robinson gave the generalizations of the Fekete-Szegö theorem for totally real algebraic integers and totally real units stated above. Independently, Bertrandias gave an adelic generalization of the Fekete-Szegö theorem concerning algebraic integers with conjugates near sets E_p at a finite number of p-adic places as well as the archimedean place (see Amice [**3**], 1975).

In the 1970s David Cantor carried out an investigation of capacities on $\mathbb{P}^1$ dealing with all three themes: incorporating local rationality conditions, requiring integrality with respect to multiple poles, and formulating the theory adelically. In a series of papers culminating with ([**16**], 1980), he introduced the Cantor capacity $\gamma(\mathbb{E}, \mathfrak{X})$, which he called the extended transfinite diameter.

Cantor's capacity $\gamma(\mathbb{E}, \mathfrak{X})$ is defined by means of a minimax property which encodes a finite collection of linear inequalities; its definition is given in (0.10) below. The points in $\mathfrak{X}$ will be called the *poles* for the capacity. In the special case where $\mathcal{C} = \mathbb{P}^1$ and $\mathfrak{X} = \{0, \infty\}$, Cantor's conditions are equivalent to those in Robinson's unit theorem. Among the applications Cantor gave in ([**16**]) were generalizations of the Pólya-Carlson theorem and Fekete's theorem, and the Fekete-Szegö theorem with splitting conditions. Unfortunately, as noted in ([**53**]), the part of the proof concerning the satisfiability of the splitting conditions had errors. However, many of Cantor's ideas are used in this work.

In the 1980's the author ([**51**]) extended Cantor's theory to curves of arbitrary genus, and proved the Fekete-Szegö theorem on curves, *without* splitting conditions. As an application he obtained a local-global principle for the existence of algebraic integer points on absolutely irreducible affine algebraic varieties ([**55**]), which had been conjectured by Cantor and Roquette ([**17**]).

Laurent Moret-Bailly and Lucien Szpiro recognized that the theory of capacities (which imposes conditions at all places) was stronger than was needed for the existence of integral points. They reformulated the local-global principle in scheme-theoretic language as an "Existence Theorem" for algebraic integer points, and gave a much simpler proof. Moret-Bailly subsequently gave far-reaching generalizations of the Existence Theorem ([**38**], [**39**], [**40**]), which allowed imposition of F_w-rationality conditions at a finite number of places, for a finite Galois extension F_w/K_v, and applied to algebraic stacks as well as schemes. However, the method required that there be at least one place v_0 where no conditions are imposed. Roquette, Green, and Pop ([**50**]) independently proved the Existence Theorem with F_w-rationality conditions, and Green, Matignon, and Pop ([**30**]) have given very general conditions on the base field K for such theorems to hold. The author ([**55**]),

van den Dries ([**66**]), Prestel and Schmid ([**47**]), and others have given applications of these results to decision procedures in mathematical logic.

Recently Akio Tamagawa ([**63**]) proved an extension of the Existence Theorem in characteristic p, which produces points that are unramified outside v_0 and the places where the F_w-rationality conditions are imposed.

The Fekete-Szegö theorem with local rationality conditions constructs algebraic numbers satisfying conditions at *all* places. At its core it is analytic in character, while the Existence Theorem is algebraic. The proof of the Fekete-Szegö theorem involves a process called "patching", which takes an initial collection of local functions $f_v(z) \in K_v(\mathcal{C})$ with poles supported on $\mathfrak{X}$ and roots in E_v for each v, and constructs a global function $G(z) \in K(\mathcal{C})$ (of much higher degree) with poles supported on $\mathfrak{X}$, whose roots belong to E_v for all v. In his doctoral thesis, Pascal Autissier ([**6**]) gave a reformulation of the patching process in the context of Arakelov theory.

In ([**52**], [**53**]) the author proved the Fekete-Szegö theorem with splitting conditions for sets $\mathbb{E}$ in $\mathbb{P}^1$, when $\mathfrak{X} = \{\infty\}$. Those papers developed a method for carrying out the patching process in the p-adic compact case, and introduced a technique for patching together archimedean and nonarchimedean polynomials over number fields.

When $\mathcal{C} = \mathbb{P}^1/K$, with K a finite extension of $\mathbb{F}_p(T)$, the Fekete-Szegö theorem with splitting conditions was established in the doctoral thesis of Daeshik Park ([**45**]).

A Sketch of the Proof of the Fekete-Szegö Theorem

In outline, the proof of the classical Fekete-Szegö theorem ([**25**], 1955) is as follows. Let a compact set $E \subset \mathbb{C}$ and a complex neighborhood U of E be given. Assume E is stable under complex conjugation, and has logarithmic capacity $\gamma_\infty(E) > 1$. For simplicity, assume also that the boundary of E is piecewise smooth and the complement of E is connected.

Under these assumptions, there is a real-valued function $G(z, \infty; E)$, called the Green's function of E with respect to ∞, which is continuous on $\mathbb{C}$, 0 on E, harmonic and positive in $\mathbb{C} \backslash E$, and has the property that $G(z, \infty; E) - \log(|z|)$ is bounded as $z \to \infty$. (We write $\log(x)$ for $\ln(x)$.) The theorem on removable singularities for harmonic functions shows that the *Robin constant*, defined by

$$V_\infty(E) \;=\; \lim_{z \to \infty} G(z, \infty; E) - \log(|z|) \;,$$

exists. By definition $\gamma_\infty(E) = e^{-V_\infty(E)}$; our assumption that $\gamma_\infty(E) > 1$ means $V_\infty(E) < 0$. It can be shown that $V_\infty(E)$ is the minimum possible value of the "energy integral"

$$I_\infty(\nu) \;=\; \iint_{E \times E} -\log(|z - w|) \, d\nu(z) d\nu(w)$$

as ν ranges over all probability measures supported on E. There is a unique probability measure μ_∞ on E, called the equilibrium distribution of E with respect to ∞, for which

$$V_\infty(E) \;=\; \iint_{E \times E} -\log(|z - w|) \, d\mu_\infty(z) d\mu_\infty(w) \;.$$

The Green's function is related to the equilibrium distribution by

$$G(z, \infty; E) - V_\infty(E) \;=\; \int_E \log(|z - w|) \, d\mu_\infty(w) \ .$$

Because of its uniqueness, the measure μ_∞ is stable under complex conjugation. Taking a suitable discrete approximation $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(z)$ to μ_∞, stable under complex conjugation, one obtains a monic polynomial $f(z) = \prod_{i=1}^{N}(z - x_i) \in \mathbb{R}[z]$ such that $\frac{1}{N} \log(|f(z)|)$ approximates $G(z, \infty, E) - V_\infty(E)$ very well outside U. If the approximation is good enough, then since $V_\infty(E) < 0$, there will be an $\varepsilon > 0$ such that $\log(|f(z)|) > \varepsilon$ outside U.

One then uses the polynomial $f(z) \in \mathbb{R}[z]$ to construct a monic polynomial $G(z) \in \mathbb{Z}[z]$ of much higher degree, which has properties similar to those of $f(z)$. The construction is as follows. By adjusting the coefficients of $f(z)$ to be rational numbers and using continuity, one first obtains a polynomial $\phi(z) \in \mathbb{Q}[z]$ and an $R > 1$ such that $|\phi(z)| \geq R$ outside U. For suitably chosen n, the multinomial theorem implies that $\phi(z)^n$ will have a pre-designated number of high-order coefficients in $\mathbb{Z}$. By successively modifying the remaining coefficients of $G^{(0)}(z) := \phi(z)^n$ from highest to lowest order, writing $k = mN + r$ and adding $\delta_k \cdot z^r \phi(z)^m$ to change $a_k z^k$ with $a_k \in \mathbb{R}$ to $(a_k + \delta_k) z^k$ with $a_k + \delta_k \in \mathbb{Z}$ (the "patching" process), one obtains the desired polynomial $G(z) = G^{(n)}(z) \in \mathbb{Z}[z]$. One uses the polynomials $\delta_k z^r \phi(z)^m$ in patching, rather than simply the monomials $\delta_k z^k$, in order to control the sup-norms $\|z^r \phi(z)^m\|_E$. Each adjustment changes all the coefficients of order k and lower, but leaves the higher coefficients unchanged. Using a geometric series estimate to show that $|G(z)| > 1$ outside U, one concludes that $G(z)$ has all its roots in U. The algebraic integers produced by the classical Fekete-Szegö theorem are the roots of $G(z)^\ell - 1$ for $\ell = 1, 2, 3, \ldots$.

The proof of the Fekete-Szegö theorem with local rationality conditions on curves follows the same pattern, but with many complications. These arise from working on curves of arbitrary genus, from arranging that the zeros avoid the finite set $\mathfrak{X} = \{x_1, \ldots, x_m\}$ instead of a single point, from working adelically, and from imposing the local rationality conditions.

We will now sketch the proof in the situation where $E_v \subset \mathcal{C}_v(K_v)$ for each $v \in S$. The proof begins reducing the theorem to a setting where one is given a $\mathcal{C}_v(\mathbb{C}_v)$-neighborhood U_v of E_v for each v, with $U_v = E_v$ if $v \notin S$. One must then construct points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ whose conjugates belong to $U_v \cap \mathcal{C}_v(K_v)$ for each $v \in S$, and to U_v for each $v \notin S$. The strategy is to construct rational functions $G(z) \in K(\mathcal{C})$ with poles supported on $\mathfrak{X}$, whose zeros have the property above.

One first constructs an "initial approximating function" $f_v(z) \in K_v(\mathcal{C})$ for each $v \in S$. Each $f_v(z)$ has poles supported on $\mathfrak{X}$ and zeros in U_v, with the zeros in $\mathcal{C}_v(K_v)$ if $v \in S$. All the $f_v(z)$ have the same degree N, and they have the property that outside U_v the logarithms $\log_v(|f(z)|_v$ closely approximate a weighted sum of Green's functions $G(z, x_i; E_v)$. The weights are determined by $\mathbb{E}$ and $\mathfrak{X}$, through the definition of the Cantor capacity.

The construction of the initial approximating functions is one of the hardest parts of the proof. When working on curves of positive genus, one cannot simply take a discrete approximation to the equilibrium distribution, but must arrange that the divisor whose zeros come from that approximation and whose poles have the prespecified orders on the points in $\mathfrak{X}$, is principal. For places $v \in S$ there

are additional constraints. When $K_v \cong \mathbb{R}$ and $E_v \subset \mathcal{C}_v(\mathbb{R})$, one must assure that $f_v(z)$ is real-valued and oscillates between large positive and negative values on E_v (a property like that of Chebyshev polynomials, first exploited by Robinson). In this work, we give a general potential-theoretic construction of oscillating functions. When K_v is nonarchimedean and $E_v \subset \mathcal{C}_v(K_v)$, one must arrange that the zeros of $f_v(z)$ belong to $U_v \cap \mathcal{C}_v(K_v)$ and are uniformly distributed with respect to a certain generalized equilibrium measure. Both cases are treated by constructing a nonprincipal divisor with the necessary properties, and then carefully moving some of its zeros to obtain a principal divisor. In this construction, the "canonical distance function" $[x, y]_\zeta$, introduced in ([**51**], §2.1), plays an essential role: given a divisor D of degree 0, the canonical distance tells what the v-adic absolute of a function with divisor D "would be", if such a function were to exist.

A further complication is that for archimedean v, one must arrange that the leading coefficients of the Laurent expansions of $f_v(z)$ at the points $x_i \in \mathfrak{X}$ have a property of "independent variability". When $K_v \cong \mathbb{C}$, this was established in ([**51**]) by using a convexity property of harmonic functions. When $K_v \cong \mathbb{R}$, we prove it by a continuity argument using the Brouwer Fixed Point theorem.

Once the initial approximating functions $f_v(z)$ have been constructed, we modify them to obtain "coherent approximating functions" $\phi_v(z)$ with specified leading coefficients, using global considerations. We then use the $\phi_v(z)$ to construct "initial patching functions" $G_v^{(0)}(z) \in K_v(\mathcal{C})$ of much higher degree which still have their zeros in U_v (and in $\mathcal{C}_v(K_v)$, for $v \in S$). The $G_v^{(0)}(z)$ are obtained by raising the $\phi_v(z)$ to high powers, or by composing them with Chebyshev polynomials or generalized Stirling polynomials if $v \in S$. (This idea goes back to Cantor [**16**].)

We next "patch" the functions $G_v^{(0)}(z)$, inductively constructing K_v-rational functions $(G_v^{(k)}(z))_{v \in S}$, $k = 1, 2, \ldots, n$, for which more and more of the high order Laurent coefficients (relative to the points in $\mathfrak{X}$) are K-rational and independent of v. In the patching process, we take care that the roots of $G_v^{(k)}(z)$ belong to U_v for all v, and belong to $\mathcal{C}_v(K_v)$ for each $v \in S$. In then end we obtain a global K-rational function $G^{(n)}(z) = G_v^{(n)}(z)$ independent of v, which "looks like" $G_v^{(0)}(z)$ at each $v \in S$.

The patching process has two aspects, global and local.

The global aspect concerns achieving K-rationality for $G(z)$, while assuring that its roots remain outside the balls $B_v(x_i, 1)^-$ for the infinitely many v where E_v is $\mathfrak{X}$-trivial. It is necessary to carry out the patching process in a Galois-invariant way. For this, we construct an $\mathrm{Aut}(\widetilde{K}/K)$-equivariant basis for the space of functions in $K(\mathbb{C})$ with poles supported on $\mathfrak{X}$, and arrange that when the functions $G_v^{(k)}(z)$ are expanded relative to this basis, their coefficients are equivariant under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$.

The most delicate step involves patching the leading coefficients: one must arrange that they be S-units (the analogue of monicity in the classical case). The argument can succeed only if the orders of the poles of the $f_v(z)$ at the x_i lie in a prescribed ratio to each other. The existence of such a ratio is intimately related to the fact that $\gamma(\mathbb{E}, \mathfrak{X}) > 1$, and is at the heart of the definition of the Cantor capacity, as will be explained below.

The remaining coefficients must be patched to be S-integers. As in the classical case, patching the high-order coefficients presents special difficulties. In general there are both archimedean and nonarchimedean places in S. It is no longer possible to use continuity and the multinomial theorem as in the classical case; instead, we use a phenomenon of "magnification" at the archimedean places, first applied in ([**53**]), together with a phenomenon of "contraction" at the nonarchimedean places. In the function field case, additional complications arise from inseparability issues. A different method is used to patch the high order coefficients than in the number field case: in the construction of initial patching functions, we arrange that the high order coefficients are all 0, and that the patching process for the leading coefficients preserves this property.

The local aspect of the patching process consists of giving "confinement arguments" showing how to keep the roots of the $G_v^{(k)}(z)$ in the sets E_v, while modifying the Laurent coefficients. Four confinement arguments are required, corresponding to the cases $K_v \cong \mathbb{C}$, $K_v \cong \mathbb{R}$ with $E_v \subset \mathcal{C}_v(\mathbb{R})$, K_v nonarchimedean with E_v being an RL-domain, and K_v nonarchimedean with $E_v \subset \mathcal{C}_v(K_v)$. The confinement arguments in the first and third cases are adapted from ([**51**]), and those in the second and fourth cases are generalizations of those in ([**53**]). The fourth case involves locally expanding the functions $G_v^{(k)}(z)$ as v-adic power series, and extending the Newton polygon construction in ([**53**]) from polynomials to power series. A crucial step involves moving apart roots which have come close to each other. This requires the theory of the universal function developed in Appendix C, and the local action of the Jacobian developed Appendix D.

The Definition of the Cantor Capacity

We next discuss the Cantor capacity $\gamma(\mathbb{E}, \mathfrak{X})$, which is treated more fully in ([**51**], §5.1). Our purpose here is to explain its meaning and its role in the proof of the Fekete-Szegö theorem. First, we will need some notation.

If v is archimedean, write $\log_v(x) = \ln(x)$. If v is nonarchimedean, let q_v be the order of the residue field of K_v, and write $\log_v(x)$ for the logarithm to the base q_v. Put $q_v = e$ if $K_v \cong \mathbb{R}$ and $q_v = e^2$ if $K_v \cong \mathbb{C}$.

Define normalized absolute values on the K_v by letting $|x|_v = |x|$ if v is archimedean, and taking $|x|_v$ to be the the modulus of additive Haar measure if v is nonarchimedean. For $0 \neq \kappa \in K$, the product formula reads

$$\sum_v \log_v(|\kappa|_v) \log(q_v) = 0 .$$

Each absolute value has a unique extension to $\mathbb{C}_v$, which we still denote by $|x|_v$.

For each $\zeta \in \mathcal{C}_v(\mathbb{C}_v)$, the *canonical distance* $[z, w]_\zeta$ on $\mathcal{C}_v(\mathbb{C}_v) \backslash \{\zeta\}$ (constructed in §2.1 of [**51**]) plays a role in the definition of $\gamma(\mathbb{E}, \mathfrak{X})$ similar to the role of the usual absolute value $|z - w|$ on $\mathbb{P}^1(\mathbb{C}) \backslash \{\infty\}$ for the classical logarithmic capacity $\gamma(E)$. The canonical distance is a symmetric, real-valued, nonnegative function of $z, w \in \mathcal{C}_v(\mathbb{C}_v)$, with $[z, w]_\zeta = 0$ if and only if $z = w$. For each w, it has a "simple pole" as $z \to \zeta$. It is uniquely determined up to scaling by a constant. The constant can be specified by choosing a uniformizing parameter $g_\zeta(z) \in \mathbb{C}_v(\mathcal{C})$ at $z = \zeta$, and requiring that

(0.3)
$$\lim_{z \to \zeta} [z, w]_\zeta \cdot |g_\zeta(z)|_v = 1$$

for each w. One definition of the canonical distance is that for each w,

$$[z, w]_\zeta \;=\; \lim_{n \to \infty} |f_n(z)|_v^{1/\deg(f_n)}$$

where the limit is taken over any sequence of functions $f_n(z) \in \mathbb{C}_v(\mathcal{C})$ having poles only at ζ whose zeros approach w, normalized so that

$$\lim_{z \to \zeta} |f_n(z) g_\zeta(z)^{\deg(f_n)}|_v \;=\; 1 \;.$$

A key property of $[z, w]_\zeta$ is that it can be used to factor the absolute value of a rational function in terms of its divisor: for each $f(z) \in \mathbb{C}_v(\mathcal{C})$, there is a constant $C(f)$ such that

$$|f(z)|_v \;=\; C(f) \cdot \prod_{x \neq \zeta} [z, x]_\zeta^{\operatorname{ord}_x(f)}$$

for all $z \neq \zeta$. For this reason, it is "right" kernel for use in arithmetic potential theory.

The Cantor capacity is defined in terms of Green's functions $G(z, x_i; E_v)$. We first introduce the Green's function for compact sets $H_v \subset \mathcal{C}_v(\mathbb{C}_v)$, where there is a potential-theoretic construction like the one in the classical case. Suppose $\zeta \notin H_v$. For each probability measure ν supported on H_v, consider the *energy integral*

$$I_\zeta(\nu) \;=\; \iint_{H_v \times H_v} -\log_v([z, w]_\zeta) \, d\nu(z) d\nu(w) \;.$$

Define the *Robin constant*

(0.4) $$V_\zeta(H_v) \;=\; \inf_\nu I_\zeta(\nu) \;.$$

It can be shown that either $V_\zeta(H_v) < \infty$ for all $\zeta \notin E_v$, or $V_\zeta(H_v) = \infty$ for all $\zeta \notin E_v$ (see Lemma 3.15). In the first case we say that H_v has *positive inner capacity*, and the second case that it has *inner capacity* 0.

If H_v has positive inner capacity, there is a unique probability measure μ_ζ on H_v which achieves the infimum in (0.4). It is called the equilibrium distribution of H_v with respect to ζ. We define the Green's function by

(0.5) $$G(z, \zeta; H_v) \;=\; V_\zeta(H_v) + \int_{H_v} \log_v([z, w]_\zeta) \, d\mu_\zeta(w) \;.$$

It is nonnegative and has a logarithmic pole as $z \to \zeta$. If H_v has inner capacity 0, we put $G(z, \zeta; H_v) = \infty$ for all z, ζ.

The Green's function is symmetric for $z, \zeta \notin H_v$, and is monotone decreasing in the set H_v: for compact sets $H_v \subset H_v'$ and $z, \zeta \notin E_v'$,

(0.6) $$G(z, \zeta; H_v) \;\geq\; G(z, \zeta; H_v') \;.$$

If H_v has positive inner capacity, then for each neighborhood $U \supset H_v$, and each $\varepsilon > 0$, by taking a suitable discrete approximation to μ_ζ, one sees that there are an $N > 0$ and a function $f_v(z) \in \mathbb{C}_v(\mathcal{C})$ of degree N, with zeros in U and a pole of order N at ζ, such that

$$|G(z, \zeta; H_v) - \frac{1}{N} \log_v(|f_v(z)|_v)| \;<\; \varepsilon$$

for all $z \in \mathcal{C}_v(\mathbb{C}_v) \backslash (U \cup \{\zeta\})$.

In [**51**], Green's functions $G(z, \zeta; E_v)$ are defined for compact sets E_v in the archimedean case, and by a process of taking limits, for "algebraically capacitable"

sets in the nonarchimedean case. Algebraically capacitable sets include all sets that are finite unions of compact sets and affinoid sets; see ([**51**], Theorem 4.3.11). In particular, the sets E_v in Theorem 0.4 are algebraically capacitable.

We next define local and global "Green's matrices". Let L/K be a finite normal extension containing $K(\mathfrak{X})$. For each place v of K and each w of L with $w|v$, after fixing an isomorphism $\mathbb{C}_w \cong \mathbb{C}_v$, we can pull back E_v to a set $E_w \subset \mathcal{C}_v(\mathbb{C}_w)$. The set E_w is independent of the isomorphism chosen, since E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$. If we identify $\mathcal{C}_v(\mathbb{C}_v)$ and $\mathcal{C}_w(\mathbb{C}_w)$, then for $z, \zeta \notin E_v$,

$$(0.7) \qquad G(z, \zeta; E_w) \log(q_w) \ = \ [L_w : K_v] \cdot G(z, \zeta; E_v) \log(q_v) \ .$$

For each $x_i \in \mathfrak{X}$, fix a global uniformizing parameter $g_{x_i}(x) \in L(\mathcal{C})$ and use it to define the upper Robin constants $V_{x_i}(E_w)$ for all w. For each w, let the "local upper Green's matrix" be

$$(0.8) \quad \Gamma(E_w, \mathfrak{X}) \ = \ \begin{pmatrix} V_{x_1}(E_w) & G(x_1, x_2; E_w) & \cdots & G(x_1, x_m; E_w) \\ G(x_2, x_1; E_w) & V_{x_2}(E_w) & \cdots & G(x_2, x_m; E_w) \\ \vdots & \vdots & \ddots & \vdots \\ G(x_m, x_1; E_w) & G(x_m, x_2; E_w) & \cdots & V_{x_m}(E_w) \end{pmatrix} \ .$$

Symmetrizing over the places of L, we define the "global Green's matrix" by

$$(0.9) \qquad \Gamma(\mathbb{E}, \mathfrak{X}) \ = \ \frac{1}{[L : K]} \sum_{w \in \mathcal{M}_L} \Gamma(E_w, \mathfrak{X}) \log(q_w) \ .$$

If $\mathbb{E}$ is compatible with $\mathfrak{X}$, the sum defining $\Gamma(\mathbb{E}, \mathfrak{X})$ is finite. By the product formula, $\Gamma(\mathbb{E}, \mathfrak{X})$ is independent of the choice of the $g_{x_i}(z)$. By (0.7) it is independent of the choice of L.

The global Green's matrix is symmetric and nonnegative off the diagonal. Its entries are finite if and only if each E_v has positive inner capacity.

Finally, for each K-rational $\mathbb{E}$ compatible with $\mathfrak{X}$, we define the *Cantor capacity* to be

$$(0.10) \qquad \gamma(\mathbb{E}, \mathfrak{X}) \ = \ e^{-V(\mathbb{E}, \mathfrak{X})} \ ,$$

where $V(\mathbb{E}, \mathfrak{X}) = \mathrm{val}(\Gamma(\mathbb{E}, \mathfrak{X}))$ is the value of $\Gamma(\mathbb{E}, \mathfrak{X})$ as a matrix game. Here, for any $m \times m$ real-valued matrix Γ,

$$(0.11) \qquad \mathrm{val}(\Gamma) \ = \ \max_{\vec{s} \in \mathcal{P}^m} \min_{\vec{r} \in \mathcal{P}^m} {}^t\vec{s}\,\Gamma\vec{r} \ ,$$

where $\mathcal{P}^m = \{{}^t(s_1, \ldots, s_m) \in \mathbb{R}^m : s_1, \ldots, s_m \geq 0, \sum s_i = 1\}$ is the set of m-dimensional "probability vectors". Clearly $\gamma(\mathbb{E}, \mathfrak{X}) > 0$ if and only if each E_v has positive inner capacity.

The hidden fact behind the definition is that $\mathrm{val}(\Gamma)$ is a function of matrices which, for symmetric real matrices Γ which are nonnegative off the diagonal, is negative if and only if Γ is negative definite; this is a consequence of Frobenius' Theorem (see ([**51**], p.328 and p.331) and ([**28**], p.53). Thus, $\gamma(\mathbb{E}, \mathfrak{X}) > 1$ if and only if $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite.

If the matrix $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite, there is a unique probability vector $\widehat{s} = {}^t(\widehat{s}_1, \ldots, \widehat{s}_m)$ such that

$$(0.12) \qquad \Gamma(\mathbb{E}, \mathfrak{X})\,\widehat{s} \;=\; \begin{pmatrix} \widehat{V} \\ \vdots \\ \widehat{V} \end{pmatrix}$$

has all its coordinates equal. From the definition of val(Γ), it follows that $\widehat{V} = V(\mathbb{E}, \mathfrak{X}) < 0$. For simplicity, assume in what follows that $\widehat{s}$ has rational coordinates (in general, this fails; overcoming the failure is a major technical difficulty).

The probability vector $\widehat{s}$ determines the relative orders of the poles of the function $G(z)$ constructed in the Fekete-Szegö theorem. The idea is that the initial local approximating functions $f_v(z)$ should have polar divisor $\sum_{i=1}^{m} N\widehat{s}_i(x_i)$ for some N, and be such that for each v, outside the given neighborhood U_v of E_v,

$$\frac{1}{N}\log_v(|f_v(z)|_v) \;=\; \sum_{j=1}^{m} G(z, x_j; E_v)\widehat{s}_j \;.$$

(At archimedean places, this will only hold asymptotically as $z \to x_i$, for each x_i.) The fact that the coordinates of $\overline{\Gamma}(\mathbb{E}, \mathfrak{X})\widehat{s}$ are equal means it is possible to scale the $f_v(z)$ so that in their Laurent expansions at x_i, the leading coefficients $c_{v,i}$ satisfy

$$\sum_v \log_v(|c_{v,i}|_v)\log(q_v) \;=\; 0,$$

compatible with the product formula, allowing the patching process to begin. Reversing this chain of ideas lead Cantor to his definition of the capacity.

For readers familiar with intersection theory, we remark that an Arakelov-like adelic intersection theory for curves was constructed in ([**56**]). The arithmetic divisors in that theory include all pairs $\mathcal{D} = (D, \{G(z, D; E_v)\}_{v \in \mathcal{M}_K})$ where $D = \sum_{i=1}^{m} s_i(x_i)$ is a K-rational divisor on $\mathcal{C}$ with real coefficients and $G(z, D; E_v) = \sum_{i=1}^{m} s_i G(z, x_i; E_v)$. If $\vec{s} = \widehat{s}$ is the probability vector constructed in (0.12), then relative to that intersection theory

$$V(\mathbb{E}, \mathfrak{X}) \;=\; {}^t\vec{s}\,\Gamma(\mathbb{E}, \mathfrak{X})\vec{s} \;=\; \mathcal{D} \cdot \mathcal{D} \;<\; 0 \;.$$

As noted by Moret-Bailly, this says that the Fekete-Szegö theorem with local rationality conditions can be viewed as a kind of arithmetic contractibility theorem.

Outline of the Book

In this section we outline the content and main ideas of this book.

The Introduction and Chapters 1 and 2 are expository, intended to give perspective on the Fekete-Szegö theorem. In Chapter 1 we state six variants of the theorem, which extend it in different directions. These include a version producing points in "quasi-neighborhoods" of $\mathbb{E}$, generalizing the classical Fekete-Szegö theorem; a version producing points in $\mathbb{E}$ under weaker conditions than those of Theorem 0.4; a version which imposes ramification conditions at finitely many primes outside S; a version for algebraically capacitable sets which expresses the Fekete/Fekete-Szegö dichotomy in terms of the global Green's matrix $\Gamma(\mathbb{E}, \mathfrak{X})$; and two versions for Berkovich curves.

In Chapter 2 we give numerical examples illustrating the theorem on $\mathbb{P}^1$, elliptic curves, Fermat curves, and modular curves. We begin by proving several formulas for capacities and Green's functions of archimedean and nonarchimedean sets, aiming to collect formulas useful for applications and going beyond those tabulated in ([**51**], Chapter 5). In the archimedean case, we give formulas for capacities and Green's functions of one, two, and arbitrarily many intervals in $\mathbb{R}$. The formulas for two intervals involve classical theta-functions, and those for multiple intervals (due to Harold Widom) involve hyperelliptic integrals. In the nonarchimedean case we give a general algorithm for computing capacities of compact sets. We determine the capacities and Green's functions of rings of integers, groups of units, and bounded tori in local fields. We also give the first known computation of a capacity of a nonarchimedean set where the Robin constant is not a rational number.

In the global case, we give numerical criteria for the existence/nonexistence of infinitely many algebraic integers and units satisfying various geometric conditions. The existence of such criteria, for which the prototypes are Robinson's theorems for totally real algebraic integers and units, is one of the attractive features of the subject. In applying a general theorem like the Fekete-Szegö theorem with local rationality conditions, it is often necessary to make clever reductions in order to obtain interesting results, and we have tried to give examples illustrating some of the reduction methods that can be used.

Our results for elliptic curves include a complete determination of the capacities (relative to the origin) of the integral points on Weierstrass models and Néron models. Our results for Fermat curves are based on McCallum's description of the special fiber for a regular model of the Fermat curve $\mathcal{F}_p$ over $\mathbb{Q}_p(\zeta_p)$. They show how the geometry of the model (in particular the number of "tame curves" in the special fibre) is reflected in the arithmetic of the curve. Our results for the modular curves $X_0(p)$ use the Deligne-Rapoport model. In combination, they illustrate a general principle that it is usually possible to compute nonarchimedean local capacities on a curve of higher genus, if a regular model of the curve is known.

Beginning with Chapter 3, we develop the theory systematically.

Chapter 3 covers notation, conventions, and foundational material about capacities and Green's functions used throughout the work. An important notion is the $(\mathfrak{X}, \vec{s})$-canonical distance $[z, w]_{\mathfrak{X}, \vec{s}}$. Given a curve $\mathcal{C}/K$ and a place v of K, we will be interested in constructing rational functions $f \in \mathcal{C}_v(\mathbb{C}_v)$ whose poles are supported on a finite set $\mathfrak{X} = \{x_1, \ldots, x_m\}$ and whose polar divisor is proportional to $\sum_{i=1}^m s_i(x_i)$, where $\vec{s} = (s_1, \ldots, s_m)$ is a fixed probability vector. The $(\mathfrak{X}, \vec{s})$-canonical distance enables to treat $|f(z)|_v$ like the absolute value of a polynomial, factoring it in terms of the zero divisor of f as

$$|f(z)|_v \;=\; C(f) \cdot \prod_{\text{zeros } \alpha_i \text{ of } f} [z, \alpha_i]_{\mathfrak{X}, \vec{s}} \;.$$

Furthermore, the product on the right, which we call an $(\mathfrak{X}, \vec{s})$-*pseudopolynomial*, is defined and continuous even for divisors which are not principal. This enables us to separate analytic and algebraic issues in the construction of f.

Put $L = K(\mathfrak{X}) = K(x_1, \ldots, x_m)$, and let L^{sep} be the separable closure of K in L. Other important technical tools from Chapter 3 are the L-rational and L^{sep}-rational bases. These are multiplicatively finitely generated sets of functions which can be used to expand rational functions with poles supported on $\mathfrak{X}$, much

like the monomials $1, z, z^2, \ldots$ can be used to expand polynomials. As their names indicate, the functions in the L-rational basis are defined over L, and those in the L^{sep}-rational basis are defined over L^{sep}. The construction arranges that the transition matrix between the two bases is block diagonal, and has bounded norm at each place w of L.

In Chapter 4 we state a version of the Fekete-Szegö theorem with local rationality conditions for "K_v-simple sets" (Theorem 4.2), and we reduce Theorem 0.4, Corollary 0.5, and the variants stated in Chapter 1 to it. The rest of the book (Chapters 5–11 and Appendices A–D) is devoted to the proof of Theorem 4.2.

Chapters 5 and 6 contain the constructions of the initial approximating functions needed for Theorem 4.2. Four constructions are needed: for archimedean sets $E_v \subset \mathcal{C}_v(\mathbb{C})$ when the ground field is $\mathbb{C}$ and $\mathbb{R}$, and for nonarchimedean sets $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ which are RL-domains or are compact. The first and third were done in ([**51**]); the second and fourth are done here.

The probability vector $\vec{s}$ ultimately used in the construction is determined by $\mathbb{E}$ and $\mathfrak{X}$, through the global Green's matrix $\Gamma(\mathbb{E}, \mathfrak{X})$. This means that for each E_v, the local constructions must be carried out in a uniform way for all $\vec{s}$. In Appendix A we develop potential theory with respect to the kernel $[z, w]_{\mathfrak{X}, \vec{s}}$. There are $(\mathfrak{X}, \vec{s})$-capacities, $(\mathfrak{X}, \vec{s})$-Green's functions, and $(\mathfrak{X}, \vec{s})$-equilibrium distributions with properties analogous to the corresponding objects in classical potential theory. The initial approximating functions are $(\mathfrak{X}, \vec{s})$-functions whose normalized logarithms $\deg(f)^{-1} \log_v(|f(z)|_v)$ closely approximate the $(\mathfrak{X}, \vec{s})$-Green's function outside a neighborhood of E_v, and whose zeros are roughly equidistributed like the $(\mathfrak{X}, \vec{s})$-equilibrium distribution.

Chapter 5 deals with the construction of initial approximating functions $f(z) \in \mathbb{R}(\mathcal{C}_v)$ when the ground field K_v is $\mathbb{R}$, for Galois-stable sets $E_v \subset \mathcal{C}_v(\mathbb{C})$ which are finite unions of intervals in $\mathcal{C}_v(\mathbb{R})$ and closed sets in $\mathcal{C}_v(\mathbb{C})$ with piecewise smooth boundaries. The desired functions must oscillate with large magnitude on the real intervals. The construction has two parts: a potential-theoretic part carried out in Appendix B, which constructs "$(\mathfrak{X}, \vec{s})$-pseudopolynomials" whose absolute value behaves like that of a Chebyshev polynomial, and an algebraic part which involves adjusting the divisor of the pseudopolynomial to make it principal. The first part of the argument requires subdividing the real intervals into "short" segments, where the notion of shortness depends only on the deviation of the canonical distance $[z, w]_{\mathfrak{X}, \vec{s}}$ from $|z - w|$ in local coordinates, and is uniform over compact sets. The second part of the argument uses a variant of the Brouwer Fixed Point theorem. An added difficulty involves assuring that the "logarithmic leading coefficients" of f are independently variable over a range independent of $\vec{s}$, which is needed as an input to the global patching process in Chapter 7.

Chapter 6 deals with the construction of initial approximating functions $f \in K_v(\mathcal{C}_v)$ when the ground field K_v is a nonarchimedean local field, and the sets E_v are Galois-stable finite unions of balls in $\mathcal{C}_v(F_{w,\ell})$, for fields $F_{w,\ell}$ are which are finite separable extensions of K_v. Again the construction has two parts: an analytic part, which constructs an $(\mathfrak{X}, \vec{s})$-pseudopolynomial by transporting Stirling polynomials for the rings of integers of the $F_{w,\ell}$ to the balls, and an algebraic part, which involves moving some of the roots of the pseudopolynomial to make its divisor principal. When $\mathcal{C}_v$ has positive genus g, this uses an action of a neighborhood of the origin in $\text{Jac}(\mathcal{C})(\mathbb{C}_v)$ on $\mathcal{C}_v(\mathbb{C}_v)^g$ constructed in Appendix D.

Chapter 7 contains the global patching argument for Theorem 4.2, which breaks into two cases: when $\text{char}(K) = 0$, and when $\text{char}(K) = p > 0$. The two cases involve different difficulties. When $\text{char}(K) = 0$, the need to patch archimedean and nonarchimedean initial approximating functions together is the main constraint, and the most serious bottleneck involves patching the leading coefficients. The ability to independently adjust the logarithmic leading coefficients for the archimedean initial approximating functions allows us to accomplish this. When $\text{char}(K) = p > 0$, the leading coefficients are not a problem, but separability/inseparability issues drive the argument. These are dealt with by simultaneously monitoring the patching process relative to the L-rational and L^{sep}-rational bases from Chapter 3.

Chapters 8–11 contain the local patching arguments needed for Theorem 4.2. Chapter 8 concerns the case when $K_v \cong \mathbb{C}$, Chapter 9 concerns the case when $K_v \cong \mathbb{R}$, Chapter 10 concerns the nonarchimedean case for RL-domains, and Chapter 11 concerns the nonarchimedean case for compact sets. Each provides geometrically increasing bounds for the amount the coefficients can be varied, while simultaneously confining the movement of the roots, as the patching proceeds from high order to low order coefficients.

Chapter 8 gives the local patching argument when $K_v \cong \mathbb{C}$. The aim of the construction is to confine the roots of the function to a prespecified neighborhood U_v of E_v, while providing the global patching construction with increasing freedom to modify the coefficents relative to the L-rational basis, as the degree of the basis functions goes down. For the purposes of the patching argument, the coefficients are grouped into "high-order", "middle" and "low-order". The construction begins by raising the initial approximating function to a high power n. A "magnification argument", similar to the ones in ([**52**]) and ([**53**]), is used to gain the freedom needed to patch the high-order coefficients.

Chapter 9 gives the local patching argument when $K_v \cong \mathbb{R}$. Here the construction must simultaneously confine the roots to a set U_v which is the union of $\mathbb{R}$-neighborhoods of the components of E_v in $\mathcal{C}_v(\mathbb{R})$, and $\mathbb{C}$-neighborhoods of the other components. We call such a set a "quasi-neighborhood" of E_v. The construction is similar to the one over $\mathbb{C}$, except that it begins by composing the initial approximating function with a Chebyshev polynomial of degree n. Chebyshev polynomials have the property that they oscillate with large magnitude on a real interval, and take a family of confocal ellipses in the complex plane to ellipses. Both properties are used in the confinement argument.

Chapter 10 gives the local patching construction when K_v is nonarchimedean and E_v is an RL-domain. The construction again begins by raising the initial approximating function to a power n. To facilitate patching the high-order coefficients, we require that n be divisible by a high power of the residue characteristic p. If K_v has characteristic 0, this makes the high order coefficients be p-adically small; if K_v has characteristic p, it makes them vanish (apart from the leading coefficients), so they do not need to be patched at all.

Chapter 11 gives the local patching construction when K_v is nonarchimedean and E_v is compact. This case is by far the most intricate, and begins by composing the initial approximating function with a Stirling polynomial. If K_v has characteristic 0, this makes the high order coefficients be p-adically small; if K_v has characteristic p, it makes them vanish. The confinement argument generalizes

those in ([**52**], [**53**]), and the roots are controlled by tracking their positions within "ψ_v-regular sequences".

A ψ_v-regular sequence is a finite sequence of roots which are v-adically spaced like an initial segment of the integers, viewed as embedded in $\mathbb{Z}_p$ (see Definition 11.3). The local rationality of each root is preserved by an argument involving Newton polygons for power series. In the initial stages, confinement of the roots depends on the fact that the Stirling polynomial factors completely over K_v. Some roots may move quite close to others in early steps of the patching process, and the the middle part of argument involves an extra step of separating roots, first used in ([**52**]). This is accomplished by multiplying the partially patched function with a carefully chosen rational function whose zeros and poles are very close in pairs. This function is obtained by specializing the "universal function" constructed in Appendix C, which parametrizes all functions of given degree by means of their roots and poles and value at a normalizing point.

Appendix A develops potential theory with respect the kernel $[z, w]_{\mathfrak{X}, \vec{s}}$, paralleling the classical development of potential theory over $\mathbb{C}$ given in ([**65**]). There are $(\mathfrak{X}, \vec{s})$-equilibrium distributions, potential functions, transfinite diameters, Chebyshev constants, and capacities with the same properties as in the classical theory. A key result is Proposition A.5, which asserts that "$(\mathfrak{X}, \vec{s})$-Green's functions", obtained by subtracting an "$(\mathfrak{X}, \vec{s})$-potential function" from an "$(\mathfrak{X}, \vec{s})$-Robin constant", are given by linear combinations of the Green's functions constructed in ([**51**]). Other important results are Lemmas A.6 and A.7, which provide uniform upper and lower bounds for the mass the $(\mathfrak{X}, \vec{s})$-equilibrium distribution can place on a subset, independent of $\vec{s}$; and Theorem A.13, which shows that nonarchimedean $(\mathfrak{X}, \vec{s})$-Green's functions and equilibrium distributions can be computed using linear algebra.

Appendix B constructs archimedean local oscillating functions for short intervals, and gives the potential-theoretic input for the construction of the initial approximating functions over $\mathbb{R}$ in Chapter 5. In classical potential theory, the equality of the transfinite diameter, Chebyshev constant, and logarithmic capacity of a compact set $E \subset \mathbb{C}$ is shown by means of a "rock-paper-scissors" argument proving in a cyclic fashion that each of the three quantities is greater than or equal to the next. Here, a rock-paper-scissors argument is used to prove Theorem B.13, which says that the probability measures associated to the roots of weighted Chebyshev polynomials for a set E_v converge to the $(\mathfrak{X}, \vec{s})$-equilibrium measure of E_v.

Appendix C studies the "universal function" of degree d on a curve, used in Chapter 11. We give two constructions for it, one by Robert Varley using Grauert's theorem, the other by the author using the theory of the Picard scheme. We then use local power series parametrizations, together with a compactness argument, to obtain uniform bounds for the change in the norm of a function outside a union of balls containing its divisor, if its zeros and poles are moved a distance at most δ (Theorem C.2).

Appendix D shows that in the nonarchimedean case, if the genus g of $\mathcal{C}$ is positive, then at generic points of $\mathcal{C}_v(\mathbb{C}_v)^g$ there is an action of a neighborhood of the origin of the Jacobian on $\mathcal{C}_v(\mathbb{C}_v)^g$, which makes $\mathcal{C}_v(\mathbb{C}_v)^g$ into a local principal homogeneous space. This is used in Chapters 6 and 11 in adjusting nonprincipal

divisors to make them principal. The action is obtained by considering the canonical map $\mathcal{C}_v^g(\mathbb{C}_v) \to \mathrm{Jac}(\mathcal{C})(\mathbb{C}_v)$, which is locally an isomorphism outside a set of codimension 1, pulling back the formal group of the Jacobian, and using properties of power series in several variables. Theorem D.2 gives the most general form of the action.

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