CHAPTER 2

# Global attractors for autonomous evolution equations

## 2.1. Existence theorem for the global attractor

As already mentioned in the introduction, one of the main concepts of the modern theory of DS in infinite dimensions is that of the *global attractor*. We give below its definition for an abstract semigroup $S(t)$ acting on a metric space $\Phi$, although, without loss of generality, the reader may think that $(S(t), \Phi)$ is just a DS associated with one of the PDEs described in the introduction.

To this end, we first recall that a subset $K$ of the phase space $\Phi$ is an attracting set of the semigroup $S(t)$ if it attracts the images of all the *bounded* subsets of $\Phi$, i.e., for every bounded set $B$ and every $\varepsilon > 0$, there exists a time $T$ (depending in general on $B$ and $\varepsilon$) such that the image $S(t)B$ belongs to the $\varepsilon$-neighborhood of $K$ if $t \geq T$. This property can be rewritten in the equivalent form

$$\lim_{t\to\infty} \operatorname{dist}_H(S(t)B, K) = 0, \tag{2.1}$$

where $\operatorname{dist}_H(X, Y) := \sup_{x\in\mathbb{X}} \inf_{y\in\mathbb{Y}} d(x, y)$ is the nonsymmetric Hausdorff distance between subsets of $\Phi$.

We now give the definition of a global attractor, following Babin-Vishik [9].

Definition 2.1. A set $\mathcal{A} \subset \Phi$ is a global attractor for the semigroup $S(t)$ if
(1) $\mathcal{A}$ is *compact* in $\Phi$;
(2) $\mathcal{A}$ is *strictly invariant*: $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
(3) $\mathcal{A}$ is an *attracting* set for the semigroup $S(t)$.

Thus, the second and third properties guarantee that a global attractor, if it exists, is unique and that the DS reduced to the attractor contains all the nontrivial dynamics of the initial system. Furthermore, the first property indicates that the reduced phase space $\mathcal{A}$ is indeed "thinner" than the initial phase space $\Phi$ (we recall that, in infinite dimensions, a compact set cannot contain, e.g., balls and should thus be nowhere dense).

In most applications, one can use the following attractor existence theorem.

Theorem 2.1. *Let a DS $(S(t), \Phi)$ possess a compact attracting set and let the operators $S(t) : \Phi \to \Phi$ be continuous for every fixed $t$. Then, this system possesses the global attractor $\mathcal{A}$ which is generated by all the trajectories of $S(t)$ which are defined for all $t \in \mathbb{R}$ and are globally bounded.*

The strategy for applying this theorem to concrete equations of mathematical physics is the following. In the first step, one verifies the so-called *dissipative* estimate which usually has the form

$$\|S(t)u_0\|_\Phi \leq Q(\|u_0\|_\Phi)e^{-\alpha t} + C_*, \quad u_0 \in \Phi, \tag{2.2}$$

where $\|\cdot\|_\Phi$ is a norm in the function space $\Phi$ and the positive constants $\alpha$ and $C_*$ and the monotonic function $Q$ are independent of $t$ and $u_0 \in \Phi$ (usually, this estimate follows from energy estimates and is sometimes even used in order to "define" a dissipative system). This estimate obviously gives the existence of an attracting set for $S(t)$ (e.g., the ball of radius $2C_*$ in $\Phi$), which is, however, *noncompact* in $\Phi$. In order to overcome this problem, one usually derives, in the second step, a *smoothing* property for the solutions, which can be formulated as follows:

$$\|S(1)u_0\|_{\Phi_1} \le Q_1(\|u_0\|_\Phi), \quad u_0 \in \Phi, \tag{2.3}$$

where $\Phi_1$ is another function space which is *compactly* embedded into $\Phi$. In applications, $\Phi$ is usually the space $L^2(\Omega)$ of square integrable functions, $\Phi_1$ is the Sobolev space $H^1(\Omega)$ of the functions $u$ such that $u$ and $\nabla_x u$ belong to $L^2(\Omega)$, and estimate (2.2) is a classical smoothing property for solutions of parabolic equations (for parabolic equations in unbounded domains and for hyperbolic equations, a slightly more complicated *asymptotic* smoothing property should be used instead of (2.2); see [32]).

Since the continuity of the operators $S(t)$ usually poses no difficulty (if the uniqueness is proven), then the above scheme indeed gives the existence of the global attractor for most of the PDE of mathematical physics in bounded domains.

Remark 2.1. As was shown in [9] the assumption that $S(t) : \Phi \to \Phi$ be continuous for every fixed $t$ can be replaced by the closedness of the graph $\{(u_0, S(t)u_0), u_0 \in \Phi\}$.

Remark 2.2. Although the global attractor usually has a very complicated geometric structure, there exists one exceptional class of DS for which the global attractor has a relatively simple structure which is completely understood, namely the DS having a global Lyapunov function. We recall that a continuous function $\mathcal{L} : \Phi \to \mathbb{R}$ is a global Lyapunov function if

(1) $\mathcal{L}$ is nonincreasing along the trajectories, i.e., $\mathcal{L}(S(t)u_0) \le \mathcal{L}(u_0)$, for all $t \ge 0$;

(2) $\mathcal{L}$ is *strictly* decreasing along all nonequilibrium solutions, i.e., $\mathcal{L}(S(t)u_0) = \mathcal{L}(u_0)$ for some $t > 0$ and $u_0$ implies that $u_0$ is an equilibrium of $S(t)$.

It is well known that if a DS possesses a global Lyapunov function, then, at least under the generic assumption that the set $\mathcal{R}$ of equilibria is finite, every trajectory $u(t)$ *stabilizes* to one of these equilibria as $t \to +\infty$. Moreover, every complete bounded trajectory $u(t)$, $t \in \mathbb{R}$, belonging to the attractor is a heteroclinic orbit joining two equilibria. Thus, the global attractor $\mathcal{A}$ can be described as follows [9], [32], [53], [66], [93]:

$$\mathcal{A} = \bigcup_{u_0 \in \mathcal{R}} \mathcal{M}^+(u_0),$$

where $\mathcal{M}^+(u_0)$ is the so-called unstable set of the equilibrium $u_0$ (which is generated by all heteroclinic orbits of the DS which start from the given equilibrium $u_0 \in \mathcal{A}$). It is also known that if the equilibrium $u_0$ is *hyperbolic* (generic assumption [9]), then the set $\mathcal{M}^+(u_0)$ is a $\kappa$-dimensional submanifold of $\Phi$, where $\kappa$ is the instability index of $u_0$. Thus, under the generic hyperbolicity assumption on the equilibria, the attractor $\mathcal{A}$ of a DS having a global Lyapunov function is a finite union of smooth finite-dimensional submanifolds of the phase space $\Phi$. These attractors are called *regular* (following Babin-Vishik (see [9])).

It is also worth emphasizing that, in contrast to general global attractors, regular attractors are robust under perturbations. Moreover, in some cases, it is also possible to verify the so-called *transversality* conditions (for the intersection of stable and unstable manifolds of the equilibria) and, thus, verify that the DS considered is a Morse-Smale system. In particular, this means that the dynamics restricted to the regular attractor $\mathcal{A}$ are also preserved (up to homeomorphisms) under perturbations.

In the sequel we will apply Theorem 2.1 or Remark 2.1 (whenever necessary) to a class of PDEs arising in mathematical physics. We especially emphasize that one of the challenging questions in the theory of attractors is, in which sense are the dynamics on the global attractor finite dimensional? As already mentioned, the global attractor is usually not a manifold but has a rather complicated geometric structure. So, it is natural to use the definitions of dimensions adopted for the study of fractal sets here. We restrict ourselves to the so-called fractal (or box-counting, entropy) dimension, although other dimensions (e.g., Hausdorff, Lyapunov, etc.) are also used in attractor theory. Here the so-called Mañé theorem (which can be considered as a generalization of the classical Whitney embedding theorem for fractal sets) plays an important role in finite-dimensional reduction theory [93].

THEOREM 2.2. *Let $\Phi$ be a Banach space and let $\mathcal{A}$ be a compact set such that $d_f(\mathcal{A}) < N$ for some $N \in \mathbb{N}$. Then, for "almost all" $(2N+1)$-dimensional planes $L$ in $\Phi$, the corresponding projector $\Pi_L : \Phi \to L$ restricted to the set $\mathcal{A}$ is a Hölder continuous homeomorphism.*

Thus, if the finite fractal dimensionality of the attractor is established, then, fixing a hyperplane $L$ satisfying the assumptions of the Mañé theorem and projecting the attractor $\mathcal{A}$ and the DS $S(t)$ restricted to $\mathcal{A}$ onto this hyperplane ($\bar{\mathcal{A}} := \Pi_L \mathcal{A}$ and $\bar{S}(t) := \Pi_L \circ S(t) \circ \Pi_L^{-1}$), we indeed obtain a reduced DS $(\bar{S}(t), \bar{\mathcal{A}})$ which is defined on a finite-dimensional set $\bar{\mathcal{A}} \subset L \sim \mathbb{R}^{2N+1}$. Moreover, this DS will be Hölder continuous with respect to the initial data.

REMARK 2.3. Note that good estimates on the dimension of the attractors in terms of the physical parameters are crucial for the finite-dimensional reduction described above and (consequently) there exists a highly developed machinery for obtaining such estimates. The best known upper estimates are usually obtained by the so-called volume contraction method which is based on the study of the evolution of infinitesimal $k$-dimensional volumes in the neighborhood of the attractor (and, if the DS considered contracts the $k$-dimensional volumes, then the fractal dimension of the attractor is less than $k$; see, e.g., [9], [93]).

REMARK 2.4. Lower bounds on the dimension are usually based on the observation that the global attractor always contains the unstable manifolds of the (hyperbolic) equilibria. Thus, the instability index of a properly constructed equilibrium gives a lower bound on the dimension of the attractor; see, e.g., [9], [93].

Theorem 2.3 below plays a decisive role in the study of the dimension of attractors, which in turn does not require differentiability of the associated semigroup in contrast to [9], [32], [93]. We especially emphasize that for a quite large class of degenerate parabolic systems arising in the modelling of life science problems (see [32]) the associated semigroup is not differentiable. We denote it by $S := S(1)$.

THEOREM 2.3. *Let $H_1$ and $H$ be Banach spaces, let $H_1$ be compactly embedded in $H$, and let $K \subset\subset H$. Assume that there exists a map $S : K \to K$, such that $S(K) = K$ and the following "smoothing" property is valid:*

$$\|S(k_1) - S(k_2)\|_{H_1} \le C\|k_1 - k_2\|_H \tag{2.4}$$

*for every $k_1, k_2 \in K$. Then the fractal dimension of $K$ in $H$ is finite and can be estimated in the following way:*

$$d_F(K, H) \le \mathbb{H}_{\frac{1}{4}C}(B(1, 0, H_1), H) \tag{2.5}$$

*where $C$ is the same as in (2.4) and $B(1, 0, H_1)$ denotes the unit ball in the space $H_1$.*

PROOF. Let $\{B(\varepsilon, k_i, H)\}_{i=1}^{N_\varepsilon}$, $k_i \in K$, be some *$\varepsilon$-covering* of the set $K$ (here and below we denote by $B(\varepsilon, k, V)$ the *$\varepsilon$-ball* in the space $V$, centered in $k$). Then according to (2.4), the system $\{B(C\varepsilon, S(k_i), H_i)\}_{i=1}^{N_\varepsilon}$ of $C\varepsilon$-balls in $H_1$ covers the set $S(K)$ and consequently (since $S(K) = K$) the same system covers the set $K$.

Now cover every $H_1$-ball with radius $C\varepsilon$ by a finite number of $\frac{\varepsilon}{4}$-balls in $H$. By definition, the minimal number of such balls equals

$$N_{\varepsilon/4}(B(C\varepsilon, S(k_i), H_1), H) = N_{\varepsilon/4}(B(C\varepsilon, 0, H_1), H) = N_{\frac{1}{4}C}(B(1, 0, H_1), H) \equiv \mathcal{N}.$$

Note that the centers of the $\varepsilon/4$-covering thus obtained do not necessarily belong to $K$ but we evidently can construct the $\varepsilon/2$-covering with centers in $K$ and with the same number of balls.

Thus, having the initial $\varepsilon$-covering of $K$ in $H$ with the number of balls $N_\varepsilon$, we have constructed the $\varepsilon/2$-covering with the number of balls $N_{\varepsilon/2} = \mathcal{N}N_\varepsilon$.

Consequently, the $\varepsilon$-entropy of the set $K$ possesses the following estimate. In fact the assertion of the theorem is a corollary of this recurrent estimate. Indeed, since $K \subset\subset H$, then there exists $\varepsilon_0$ such that $K \subset B(\varepsilon_0, k_0, H)$ and consequently

$$\mathbb{H}_{\varepsilon_0}(K, N) = 0. \tag{2.6}$$

Iterating the estimate (2.6) $n$ times, we obtain that

$$\mathbb{H}_{\varepsilon_0/2^n}(k, H) \le n \log_2 \mathcal{N}. \tag{2.7}$$

Now fix an arbitrary $\varepsilon > 0$ and choose $n = n(\varepsilon)$ in such a way that

$$\frac{\varepsilon_0}{2^n} \le \varepsilon \le \frac{\varepsilon_0}{2^n - 1}.$$

Then

$$\mathbb{H}_\varepsilon(K) \le \mathbb{H}_{\varepsilon/2^n}(K) \le n \log_2 \mathcal{N} \le \log_2\left(\frac{2\varepsilon_0}{\varepsilon}\right) \log_2 \mathcal{N}. \tag{2.8}$$

Thus (2.5) is an immediate consequence of (2.8). Theorem 2.3 is proved. □

## 2.2. Estimation of time derivatives for nonautonomous perturbations of regular attractors

The aim of this section is to verify the auxiliary estimate for nonautonomous perturbations of regular attractors. To be more precise, consider an ODE in $\mathbb{R}^n$:

$$u'(t) = F(u(t)), \quad u(0) = u_0, \tag{2.9}$$

for some $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. We assume that, for every $u_0 \in \mathbb{R}^n$, this equation is globally (for $t \geq 0$) solvable and the associated semigroup $S(t)u_0 := u(t)$ is dissipative, i.e.,

$$\|S(t)u_0\| \leq Q(\|u_0\|)e^{-\beta t} + C_* \tag{2.10}$$

for some positive $\beta$ and $C_*$ and monotone $Q$. Therefore, equation (2.9) possesses a global attractor $\mathcal{A}$ in $\mathbb{R}^n$. Our main assumption is that this attractor is *regular* in the sense of Remark 2.2, i.e., all of the equilibria $u_0 \in \mathcal{R}$ are hyperbolic, every trajectory, belonging to the attractor $\mathcal{A}$ is a heteroclinic orbit connecting two *different* equilibria, and the attractor $\mathcal{A}$ is a finite union of the unstable manifolds $\mathcal{M}^+_{u_0}$ associated with the equilibria $u_0 \in \mathcal{R}$:

$$\mathcal{A} = \bigcup_{u_0 \in \mathcal{R}} \mathcal{M}^+_{u_0}. \tag{2.11}$$

Finally, we assume that the so-called no-cycle condition is satisfied, i.e., the attractor $\mathcal{A}$ does not contain any heteroclinic cycles. As we know, that is always true in the case when (2.9) possesses a global Lyapunov function.

Consider now the following small nonautonomous perturbation of equation (2.9):

$$u' = F(u) + h(t), \quad u(0) = u_0, \quad t \geq 0, \tag{2.12}$$

where the nonautonomous external force is uniformly small:

$$\|h\|_{W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)} \leq \varepsilon \ll 1. \tag{2.13}$$

The main result of this section is the following estimate.

Proposition 2.1. *Let the above assumptions hold and let the external force $h \in W^{1,\infty}(\mathbb{R}_+)$ satisfy estimate* (2.13) *for sufficiently small $\varepsilon > 0$. Then, any solution $u(t)$ of the perturbed problem* (2.12) *satisfies the following estimate:*

$$\int_0^T \|u'(t)\|\, dt \leq C_1 + C_2 \int_0^T \|h'(t)\|\, dt, \tag{2.14}$$

*where the positive constants $C_1$ and $C_2$ depend only on the norm of $u(0)$ and are independent of $T$ and the concrete choice of $u(0)$ and $h(t)$.*

Proof. Indeed, using the standard regular attractor perturbation arguments, one can check that for every bounded set $B$ of $\mathbb{R}^n$ and every $\delta > 0$, there exist $T = T(B, \delta)$ and $\varepsilon_0 = \varepsilon_0(B, \delta)$ such that, for every $\varepsilon \leq \varepsilon_0$ and every trajectory $u(t)$ starting from $B$, we can find a sequence $u_1, \dots, u_N$ of *different* equilibria $u_i \in \mathcal{R}$ (of problem (2.9)!) and a sequence of times

$$0 = T_0^+ \leq T_1^- \leq T_1^+ < T_2^- \leq T_2^+ < \dots < T_N^- < T_N^+ = \infty$$

such that

$$u(t) \in \mathcal{O}_\delta(u_i), \quad t \in (T_i^-, T_i^+), \quad T_i^- - T_{i-1}^+ \leq T, \ i = 1, \dots, N. \tag{2.15}$$

In other words, the sequence of equilibria $u_i$ and the values of $T_l^\pm$ depend on the concrete choice of $u(0)$ and $h$, but the number $N$ of equilibria is bounded by the whole number $\#\mathcal{R} < \infty$ of possible equilibria (since the equilibria must be different) and the differences $T_i^+ - T_{i-1}^-$ are also uniformly bounded by $T$; see [9], [42] for the details.

Thus, any trajectory starting from $B$ spends at most time $T_{\text{out}} := \#\mathcal{R} \cdot T$ outside of the $\delta$-neighborhood $\mathcal{O}_\delta(\mathbb{R})$ of the equilibria set $\mathcal{R}$ and this time depends only on $B$ and $\delta$. For this reason, the part of the trajectory lying outside of $\mathcal{O}_\delta(\mathbb{R})$ gives only a *finite and uniformly bounded* impact on the integral (2.14) (which can be included in the constant $C_1$). So, we only need to estimate the left-hand side of (2.14) for the case where $u(t)$ belongs to a small neighborhood of a single fixed equilibrium $u_0 \in \mathcal{R}$ only.

To this end, we will use the hyperbolicity assumption on $u_0$. Indeed, the implicit function theorem implies the existence of $\varepsilon_0 > 0$ and $\delta > 0$ such that, for every $\varepsilon \le \varepsilon_0$, there exists a unique solution $U_{u_0,h}(t)$ of (2.12) belonging to the $\delta$-neighborhood of $u_0$ for all $t$. Moreover, this solution, in fact, belongs to the $C\varepsilon$-neighborhood of $u_0$ and the following estimate holds:

$$|U'_{u_0,h}(t)| \le C \int_{\mathbb{R}} e^{-\kappa|t-s|} |h'(s)|\, ds, \tag{2.16}$$

where the constant $C$ and the hyperbolicity exponent $\kappa$ are independent of the concrete choice of $u_0 \in \mathcal{R}$ and the external force $h$ satisfying (2.13); see [50].

Furthermore, since $u_0$ is hyperbolic, the trajectory $U_{u_0,h}(t)$ will also be hyperbolic and we will have an exponential dichotomy in a small $\delta$-neighborhood of $U_{u_0,h}$. In particular, every trajectory $u(t)$ belonging to $\mathcal{O}_\delta(u_0)$ for $t \in [0, S]$, $S \gg 1$, will tend exponentially to $U_{u_0,h}(t)$ inside of the interval

$$|u(t) - U_{u_0,h}(t)| + |u'(t) - U'_{u_0,h}(t)| \le C(e^{-\kappa|t|} + e^{-\kappa|S-t|}) \tag{2.17}$$

and $C$ and $\kappa$ are independent of the concrete choice of $u$ and $h$; see [50] for the details. Therefore,

$$\int_0^t |u'(s)|\, ds \le C + \int_0^t |U'_{u_0,h}(s)|\, ds \tag{2.18}$$

for $t \in [0, S]$ and $u(t) \in \mathcal{O}_\delta(u_0)$.

Thus, we have proved that

$$\int_0^T |u'(s)|\, ds \le C_1 + C_2 \sum_{u_0 \in \mathcal{R}} \int_0^T |U'_{u_0,h}(s)|\, ds, \tag{2.19}$$

where the constants $C_i$ depend only on the radius of $B$.

In order to deduce (2.14) from (2.19), we will use estimate (2.16). Indeed, integrating it over $t \in [0, T]$ and using that $h$ can be extended for $t \le 0$ by zero, we have

$$\begin{aligned} \int_0^T |U'_{u_0,h}(t)|\, dt \le & C \int_0^T |h'(t)|\, dt + C \int_T^\infty e^{-\kappa|t-T|} |h'(s)|\, ds \\ \le & C_1 + C \int_0^T |h'(t)|\, dt. \end{aligned} \tag{2.20}$$

Inserting this estimate into the right-hand side of (2.19), we obtain (2.14) and finish the proof of the proposition. □

Remark 2.5. As we can see from the proof, estimate (2.14) has a general nature which is not restricted by the class of ordinary differential equations. However, since we intend to use it only for the ODE case (see Chapter 11), in order to avoid the technicalities related to the formulation for a "general" PDE, we restrict our consideration to the case of an ODE (see also [44]).