Preface

This book deals with the long-time behavior of solutions of degenerate parabolic dissipative equations arising in the study of biological, ecological, and physical problems. It is well known that the long-time behavior of many dissipative systems generated by evolution partial differential equations (PDEs) of mathematical physics can be described in terms of the so-called global attractors. By definition, a global attractor is a compact invariant set in the phase space which attracts the images of all bounded subsets under the temporal evolution.

In particular, in the case of dissipative PDEs in bounded domains, this attractor usually has finite Hausdorff and fractal dimension; see [9], [93], [32], and the references therein. Hence, if the global attractor exists, its defining property guarantees that the dynamical system (DS) reduced to the attractor $\mathcal{A}$ contains all of the nontrivial dynamics of the original system and the reduced phase space $\mathcal{A}$ is really "thinner" than the initial phase space X. (We recall that in infinite-dimensional spaces, a compact set cannot contain, for instance, balls and as a result should be nowhere dense.) One of the important questions in this theory is, *In what sense is the dynamics reduced to the attractor finite or infinite dimensional?*

Usually for regular (nondegenerate) dissipative autonomous PDEs in a bounded spatial domain Ω, the Kolmogorov ε-entropy of their attractors has asymptotics such as

$$C_1|\Omega|\log_2\frac{1}{\varepsilon} \le H_\varepsilon(\mathcal{A}, X) \le C_2|\Omega|\log_2\frac{1}{\varepsilon}.$$

Consequently, in spite of the infinite dimensionality of the initial phase space, the reduced dynamics on the attractor is (in a sense) finite dimensional and can be studied by the methods of the classical (finite-dimensional) theory of dynamical systems.

In contrast, infinite-dimensional global/uniform attractors are typical for dissipative PDEs in *unbounded* domains and/or for *nonautonomous* equations. In order to study such attractors, one usually uses the concept of Kolmogorov ε-entropy and its asymptotics in various functional spaces; see the recently published book [32] for the systematic study and appropriate details.

However, we note that the above results have been obtained mainly for evolution PDEs with more or less regular structure (e.g., uniform parabolic). In contrast to this, very little is known about the long-time dynamics of degenerate parabolic equations, such as porous media equations, p-Laplacian and doubly nonlinear equations, as well as degenerate diffusion with chemotaxis and ODE-PDE coupling systems (and their degenerate extensions), etc., which also play a significant role in modern mathematical physics, biology, and ecology. In this book we aim to fill this gap. Therefore the main goal of the present book is to give a detailed and systematic study of the well-posedness and the dynamics of the associated semigroup generated by the degenerate parabolic equations mentioned above in terms of their global and exponential attractors (e.g., existence, convergence of the dynamics, and the rate of convergence) as well as studying fractal dimension and Kolmogorov entropy of corresponding attractors. Our analysis and results in this book show that there are new effects related to the attractor of such degenerate equations which cannot be observed in the case of nondegenerate equations in bounded domains.

The book consists of eleven chapters. In Chapter 1 for the convenience of the reader we give some details of several well-known facts which are used in the sequel. In particular, we recall asymptotics of the ε-Kolmogorov entropy in various functional spaces, L^q regularity and interior regularity of solutions for nondegenerate parabolic equations, classical embedding theorems as well as embedding theorems in weighted Sobolev spaces with degenerate weights. Moreover, Chapter 1 contains properties of Nemytskii (superposition) operators in Sobolev spaces and Hölder spaces which we use in the analysis of the next chapters.

Chapter 2 is concerned with the long-time behavior of solutions of evolution equations in terms of the global and regular attractors, including existence of attractors and properties of the attractor. Moreover we derive an estimate of time derivatives for nonautonomous perturbations of regular attractors. This is a cornerstone for developing a new method of proving stabilization to equilibria for solutions of an ODE-PDE coupling problem studied in Chapter 11, because, as we will see in Chapter 11, when the ODE part of the coupling is nonmonotone, the equilibria set of ODE-PDE coupling is not compact in any reasonable topology, and, as a result, the standard Lojasiewicz technique fails for stabilization of trajectories to the equilibria.

Chapter 3 is devoted to the systematic study of exponential attractors both for autonomous and for nonautonomous dynamical systems. We deal with existence theorems as well as perturbation theory of the exponential attractors and give some recent development on pull-back exponential attractors.

In Chapters 4–7 we are concerned with the well-posedness (global in time solutions) as well as long-time dynamics (finite and infinite dimensional) of porous medium and p-Laplacian equations both in homogeneous and heterogeneous media. In these chapters we present some new features related to the attractors of such equations that one cannot observe in nondegenerate cases, namely,

(a) the infinite dimensionality of the attractor,
(b) the polynomial asymptotics of its ε-Kolmogorov entropy,
(c) the difference in the asymptotics of the ε-Kolmogorov entropy depending on the choice of the underlying phase spaces.

These are the first examples in the mathematical literature of infinite-dimensional attractors admitting polynomial asymptotics of their ε-Kolmogorov entropy. It is worth noting that although infinite-dimensional global attractors are typical for nondegenerate equations in unbounded domains, even in that case the asymptotics of their Kolmogorov ε-entropy were always logarithmic in nature (such as $\left(\log_2 \frac{1}{\varepsilon}\right)^{n+1}$; see the book [32] for a systematic study of this issue).

We emphasize that, in our analysis in Chapters 4–7, to obtain properties (a)–(c) we cannot rely on the techniques that apply to nondegenerate parabolic equations. Indeed the usual method for obtaining lower bounds of the Kolmogorov entropy of attractors (as a result of its dimension) is based on the instability index of hyperbolic equilibria (see [9], [93], [32], and the references therein), which in turn requires differentiability of the associated semigroup with respect to the initial data. However, this method is not applicable for degenerate parabolic equations, since the associated semigroups (in contrast to nondegenerate parabolic equations) are usually not differentiable. That is why we were forced in Chapters 4–7 to develop alternative methods for proving properties (a)–(c) based on the existence of a localized solution and a scaling technique, which is closely related to the degenerate nature of the problem considered.

In Chapter 8 we give a detailed study of some classes of doubly nonlinear degenerate equations (we allow polynomial degeneration with respect to $\partial_t u$). We emphasize that the structure of a doubly degenerate equation considered in Chapter 8 does not fit the assumptions of the general fully nonlinear theory (see, e.g., [68] and [64]), so the highly developed classical theory in these books is not formally applicable. Thus we are forced to develop a new method, which in turn shows that a class of equations considered in Chapter 8 (under some assumptions on the data of the problem under consideration) possesses very good regularity properties and in particular has classical solutions. We believe that this phenomenon has a general nature and we clarify the difficulties related to finding stronger solutions of more general doubly nonlinear equations. Moreover, we obtain the uniqueness of solutions, which in fact was also a very delicate problem, because the simplest ODE example with polynomial degeneracy in $\partial_t u$ shows that the uniqueness of solutions fails. We also study the long-time behavior of solutions of doubly nonlinear degenerate problems in terms of the associated global and exponential attractors.

In Chapters 9 and 10 we consider both autonomous and nonautonomous chemotaxis systems with degenerate diffusion. Such classes of equations arise in the study of the role of chemotaxis for biofilm formation. We prove both global existence in time and uniqueness of solutions when the underlying domain is three dimensional. Such a well-posedness is done under certain "balance conditions" on the order of the porous medium degeneracy and the growth of the chemotactic functions.

The main aim of Chapter 11 is to study the long-time behavior of solutions of a class of degenerate parabolic systems describing the development of a forest ecosystem. From the mathematical point of view, the problem considered is a coupled system of second-order ODEs with a linear PDE (heat-like equation). Heuristically, it is clear that the dynamics of such coupled dissipative systems should depend drastically on the monotonicity properties of the ODE component. In this chapter we justify this in a mathematically rigorous way in the example of the ODE-PDE coupled system.

We finally note that the methods developed in Chapters 4–11 in order to study the long-time dynamics of certain classes of degenerate parabolic equations of different kinds seem to have a general nature and can be applied to other classes of degenerate equations, both autonomous and nonautonomous.

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