CHAPTER 1

The Water Waves Problem and Its Asymptotic Regimes

We derive here various equivalent mathematical formulations of the water waves problem (and some extensions to the two-fluids problem). We then propose a dimensionless version of these equations that is well adapted to the qualitative description of the solutions. The way we nondimensionalize the water waves equations relies on a rough analysis of their linearization around the rest state and shows the relevance of various dimensionless parameters, namely, the *amplitude parameter* ε, the *shallowness parameter* μ, the *topography parameter* β, and the *transversality parameter* γ. The linear analysis of the equations is also used to introduce the concept of wave packets and modulation equations.

With the relevant physical dimensionless parameters introduced, we then identify *asymptotic regimes* (the shallow water regime for instance) as conditions on these dimensionless parameters (e.g., $\mu \ll 1$ for the shallow water regime). Finally, we present two natural extensions of the problems addressed in this book: the case of moving bottoms and of rough topographies. A discussion of the main physical assumptions (e.g., homogeneity, inviscidity, incompressibility, etc.) and some comments on possible extensions (such as taking into account Coriolis effects, or a nonconstant external pressure) are then briefly addressed in the last section.

As everywhere throughout this book, $d = 1, 2$ denotes the spacial dimension of the surface of the fluid. The spatial variable is written $X \in \mathbb{R}^d$ and the vertical variable is denoted by z. We also write $X = (x, y)$ when $d = 2$ and $X = x$ when $d = 1$.

1.1. Mathematical formulation

1.1.1. Basic assumptions. The water waves problem consists in describing the motion, under the influence of gravity, of a fluid occupying a domain delimited below by a fixed bottom and above by a free surface that separates it from vacuum (that is, from a fluid whose density is considered negligible, such as for the air-water interface). The following assumptions are made on the fluid and the flow:

(H1) The fluid is homogeneous and inviscid.
(H2) The fluid is incompressible.
(H3) The flow is irrotational.
(H4) The surface and the bottom can be parametrized as graphs above the still water level.
(H5) The fluid particles do not cross the bottom.
(H6) The fluid particles do not cross the surface.

(H7) There is no surface tension[1] and the external pressure is constant.
(H8) The fluid is at rest at infinity.
(H9) The water depth is always bounded from below by a nonnegative constant.

These assumptions are discussed with more detail in §1.8.1 below. Assumptions (H1) and (H2) imply that the fluid motion is governed by the incompressible Euler equation inside the fluid domain.

The irrotationality assumption (H3) is useful but not necessary (see §1.8.1) but is commonly made in coastal oceanography since for most applications rotational effects are negligible up to the breaking point of the waves.

The assumption (H4) excludes overhanging waves; this is of course not a restriction for our present purpose of describing asymptotic dynamics of interest in coastal oceanography. Describing the free surface with a parametrized hypersurface (see §1.8), it is, however, possible to take overhanging waves into account.

Assumptions (H5) and (H6) provide boundary conditions to the Euler equations: (H5) implies that the normal component of the velocity must vanish at the bottom while (H6) provides a (nonlinear) kinematic boundary condition at the surface.

Neglecting the surface tension as in (H7) is completely reasonable in coastal oceanography since the typical scale for which surface tension occurs is 1.6 cm (see Chapter 9 for more details). Dealing with a nonconstant external pressure does not raise any particular difficulty (see §1.8.1).

Assumption (H8) is quite natural as long as one considers infinite domains that satisfy (H9). The latter condition excludes beaches (seen as vanishing shorelines); this is obviously a serious restriction for applications to coastal oceanography, but removing this assumption remains to this day an open mathematical problem.

1.1.2. The free surface Euler equations. The free surface Euler equations are just a mathematical restatement of assumptions (H1)–(H9). Let us first introduce the following notation:

- The domain occupied by the fluid at time t is denoted $\Omega_t \subset \mathbb{R}^{d+1}$.
- The velocity of the fluid particle located at $(X, z) \in \Omega_t$ at time t is written $\mathbf{U}(t, X, z) \in \mathbb{R}^{d+1}$. Its horizontal and vertical components are denoted by $V(t, X, z) \in \mathbb{R}^d$ and $w(t, X, z) \in \mathbb{R}$, respectively.
- We write $P(t, X, z) \in \mathbb{R}$ for the pressure at time t at the point $(X, z) \in \Omega_t$.
- The (constant) acceleration of gravity is denoted by $-g\mathbf{e}_z$, where $g > 0$ and $\mathbf{e}_z$ is the unit (upward) vector in the vertical direction.
- The (constant) density of the fluid is written ρ.

The motion of a homogeneous, inviscid, incompressible, and irrotational fluid (assumptions (H1)–(H3)) is governed by the Euler equation, with constraints on the divergence and curl of the velocity field:

$$\begin{array}{lll} (\mathrm{H1})' & \partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla_{X,z})\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}P - g\mathbf{e_z} & \text{in} \quad \Omega_t, \\ (\mathrm{H2})' & \operatorname{div} \mathbf{U} = 0 & \text{in} \quad \Omega_t, \\ (\mathrm{H3})' & \operatorname{curl} \mathbf{U} = 0 & \text{in} \quad \Omega_t. \end{array}$$

There also exist two functions $b : \mathbb{R}^d \to \mathbb{R}$ and $\zeta : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ $(T > 0)$ such that

$$(\mathrm{H4})' \qquad \forall t \in [0, T), \qquad \Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

[1] We remove this assumption in Chapter 9.

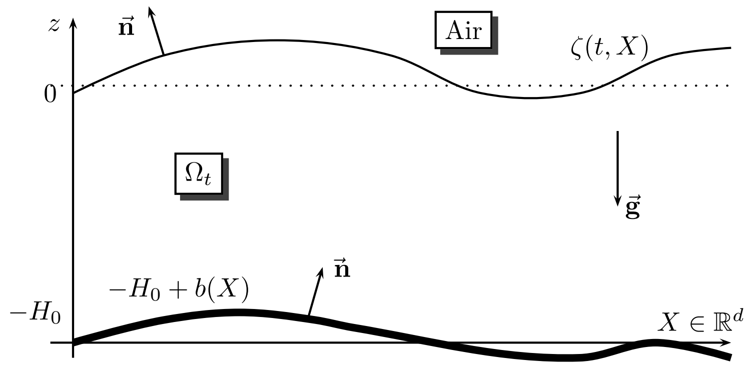

FIGURE 1.1. Main notation.

where $H_0 > 0$ is a constant reference depth introduced for later convenience; note that $z = 0$ corresponds to the still water level.

Denoting by $\mathbf{n}$ the unit normal vector to the fluid domain pointing upwards, we can reformulate[2] (H5) and (H6) as

$$\begin{array}{lll} (\mathrm{H5})' & \mathbf{U} \cdot \mathbf{n} = 0 & \text{on} \quad \{z = -H_0 + b(X)\}, \\ (\mathrm{H6})' & \partial_t \zeta - \sqrt{1 + |\nabla\zeta|^2}\mathbf{U} \cdot \mathbf{n} = 0 & \text{on} \quad \{z = \zeta(t, X)\}. \end{array}$$

Denoting by P_{atm} the (constant) atmospheric pressure, assumption (H7) can be restated as

$$(\mathrm{H7})' \qquad P = P_{atm} \quad \text{on} \quad \{z = \zeta(t, X)\},$$

while (H8) and the nonvanishing shoreline assumption (H9) can be written, respectively, as

$$(\mathrm{H8})' \qquad \forall t \in [0, T), \qquad \lim_{(X,z) \in \Omega_t, |(X,z)| \to \infty} |\zeta(t, X)| + |\mathbf{U}(t, X, z)| = 0$$

and

$$(\mathrm{H9})' \qquad \exists H_{min} > 0, \quad \forall (t, X) \in [0, T) \times \mathbb{R}^d, \qquad H_0 + \zeta(t, X) - b(X) \geq H_{min}.$$

Equations (H1)′–(H8)′ are called *free surface Euler equations.*

1.1.3. The free surface Bernoulli equations. The free surface Bernoulli equations are another formulation of the free surface Euler equations based on the representation of the velocity field in terms of avelocity potential. More precisely, there exists a mapping $\Phi : [0, T) \times \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\begin{array}{lll} (\mathrm{H3})'' & \mathbf{U} = \nabla_{X,z}\Phi & \text{in} \quad \Omega_t, \\ (\mathrm{H2})'' & \Delta_{X,z}\Phi = 0 & \text{in} \quad \Omega_t, \\ (\mathrm{H1})'' & \partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) & \text{in} \quad \Omega_t. \end{array}$$

[2]Let Γ_t be a hypersurface given implicitly by an equation $\gamma(t, X, z) = 0$, and denote by $M(t) = (X(t), z(t))$ the position of a fluid particle at time t. It is on the hypersurface Γ_t if and only if $\gamma(t, M(t)) = 0$ and stays on Γ_t for all times if $\frac{d}{dt}\gamma(t, M(t)) = 0$, or equivalently $\partial_t\gamma + \frac{d}{dt}M \cdot \nabla_{X,z}\gamma = 0$. Since by definition $\frac{d}{dt}M = \mathbf{U}$, we get $\partial_t\gamma + \mathbf{U} \cdot \nabla_{X,z}\gamma = 0$. The conditions (H5)′ and (H6)′ are therefore deduced from (H5) and (H6) by taking $\gamma(t, X, z) = z - H_0 + b(X)$ and $\gamma(t, X, z) = z - \zeta(t, X)$, respectively.

Note that (H1)$''$ is obtained upon integrating (H1)$'$ with respect to the spatial variables and is thus defined up to a time-dependent function. Changing Φ if necessary, it is possible to assume, as we did, that this function identically vanishes.

Recalling that $\partial_{\mathbf{n}}$ always stands for the upwards normal derivative, assumptions (H5)$'$ and (H6)$'$ can be recast in terms of the velocity potential Φ:

$$\begin{array}{lll} \text{(H5)}'' & \partial_{\mathbf{n}}\Phi = 0 & \text{on} \quad \{z = -H_0 + b(X)\}, \\ \text{(H6)}'' & \partial_t\zeta - \sqrt{1+|\nabla\zeta^2|}\partial_{\mathbf{n}}\Phi = 0 & \text{on} \quad \{z = \zeta(t,X)\}. \end{array}$$

Equations (H1)$''$–(H6)$''$, complemented with (H7)$'$, are called *free surface Bernoulli equations.*

1.1.4. The Zakharov/Craig-Sulem formulation. In his paper [**333**], Zakharov remarked that the knowledge of the free surface elevation ζ and the trace of the velocity potential at the surface $\psi = \Phi_{|z=\zeta}$ fully define the flow, and Craig, Sulem, and Sulem [**110, 111**] gave an elegant formulation of the equations involving the Dirichlet-Neumann operator. The latter is the mathematical formulation of the water waves problem that will be used throughout these notes because it is well adapted to the study of the asymptotic dynamics of the water waves.

Let us now proceed to the derivation of these equations. As mentioned above, the first step consists in noting that ζ and $\psi = \Phi_{|z=\zeta}$ fully determine the flow. Indeed, the velocity potential $\Phi(t,\cdot,\cdot)$ is recovered with (H2)$''$ and (H5)$''$ by solving

$$\left\{ \begin{array}{ll} \Delta_{X,z}\Phi = 0 & \text{in } \Omega_t, \\ \Phi_{|z=\zeta} = \psi, & \partial_{\mathbf{n}}\Phi_{|z=-H_0+b} = 0. \end{array} \right. \tag{1.1}$$

Note that the resolution of the Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions is possible under reasonable regularity assumptions on ζ and ψ if the nonvanishing shoreline assumption (H9)$'$ is satisfied and if the flow is at rest at infinity as in (H8)$'$; we refer to Chapter 2 for more details.

Now, knowing Φ, one gets the velocity field $\mathbf{U}$ through (H3)$''$ and the pressure P through (H1)". We are thus led to find a set of two equations that determines ζ and ψ (and thus all the physical quantities relevant to the water waves problem). To this end, it is convenient to introduce the Dirichlet-Neumann operator:

$$G[\zeta,b] : \psi \mapsto \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi_{|z=\zeta}, \tag{1.2}$$

where Φ solves (1.1). This operator is linear with respect to ψ but highly nonlinear with respect to the surface and bottom parametrizations ζ and b, which play a role through the fluid domain Ω_t on which (1.1) is solved. We refer to Chapter 3 for more details on the construction of the Dirichlet-Neumann operator (1.2). A straightforward application of the chain rule then yields the relations

$$\begin{array}{rcl} (\partial_t\Phi)_{|z=\zeta} & = & \partial_t\psi - (\partial_z\Phi)_{|z=\zeta}\partial_t\zeta, \\ (\nabla\Phi)_{|z=\zeta} & = & \nabla\psi - (\partial_z\Phi)_{|z=\zeta}\nabla\zeta, \\ (\partial_z\Phi)_{|z=\zeta} & = & \dfrac{G[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi}{1+|\nabla\zeta|^2}. \end{array}$$

Note now that, owing to (H7)$'$, the r.h.s. of (H1)$''$ vanishes at the free surface, so that using the above relations provides us with an evolution equation on ψ in terms of ζ and ψ (and b) only. Since (H6)$''$ provides an evolution equation on ζ in terms of the same quantities, the water waves problem can be reduced to the following system of two scalar evolution equations:

$$\left\{\begin{array}{l}\partial_t\zeta - G[\zeta,b]\psi = 0,\\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta,b]\psi + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0.\end{array}\right. \tag{1.3}$$

As noted by Zakharov [**333**], this system has a Hamiltonian structure in the canonical variables (ζ,ψ). Indeed, (1.3) takes the form

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\partial_\zeta H\\ \partial_\psi H\end{pmatrix},$$

with the Hamiltonian H given by

$$H = K + E, \tag{1.4}$$

where the kinetic and potential energies K and E are defined as

$$K = \frac{1}{2}\int_{\mathbb{R}^d}\int_{-H_0+b(X)}^{\zeta(X)} |\nabla_{X,z}\Phi(X,z)|^2 dzdX = \frac{1}{2}\int_{\mathbb{R}^d}\psi G[\zeta,b]\psi$$

(the second formula following from Green's identity) and

$$E = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2.$$

REMARK 1.1. It follows from the analysis above that the Hamiltonian H is a conserved quantity. This is not the only one. For instance, in the case of a flat bottom ($b=0$), the following quantities are conserved:

$$Q_x = \int_{\mathbb{R}^d}\zeta\partial_x\psi, \qquad Q_y = \int_{\mathbb{R}^d}\zeta\partial_y\psi, \quad m = \int_{\mathbb{R}^d}\zeta;$$

the first two are related to the horizontal momentum and the third one is the excess mass. We refer to [**24**] for more details on conserved quantities.

1.2. Other formulations of the water waves problem

Throughout these notes, we use the Zakharov-Craig-Sulem formulation of the water waves equations (1.3), which seems to be the one most adapted for the derivation of shallow water asymptotic models. However, many other formulations of the water waves problem exist and may be adapted in other contexts (singularity formation for instance). We briefly review some of these alternative formulations here. Since most of them have been derived in the context of infinite depth, we consider in this section that $H_0 = \infty$.

1.2.1. Lagrangian parametrizations of the free surface. The free surface Euler equations (H1)′–(H7)′ can be reformulated in Lagrangian variables.

Let $M(t,X)\in\mathbb{R}^{d+1}$ be the Lagrangian representation of the free surface defined as

$$\left\{\begin{array}{l}\partial_t M(t,\alpha,\beta) = \mathbf{U}\big(t,M(t,\alpha,\beta)\big),\\ M(0,\alpha,\beta) = (\alpha,\beta,\zeta^0(\alpha,\beta)),\end{array}\right. \tag{1.5}$$

for all $\alpha,\beta\in\mathbb{R}$ (with straightforward adaptation if $d=1$). We also denote by Γ_t the free surface at time t,

$$\Gamma_t = \{M(t,\alpha,\beta), \alpha,\beta\in\mathbb{R}\},$$

with here again straightforward adaptation if $d = 1$.

REMARK 1.2. The surface is here represented at time t as a parametrized hypersurface, which is not necessarily a graph. We could also have chosen a more general initial condition

$$M(0, \alpha, \beta) = \big((m_1^0(\alpha, \beta), m_2^0(\alpha, \beta), m_2^0(\alpha, \beta)\big)$$

allowing, for instance, overhanging waves.

Time differentiation of (1.5) yields

$$\begin{aligned} \partial_t^2 M &= \partial_t \mathbf{U} + \partial_t M \cdot \nabla_{X,z} \mathbf{U} \\ &= \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} \\ &= -g\mathbf{e}_z - \frac{1}{\rho} \nabla_{X,z} P(M), \end{aligned}$$

where we used the Euler equation (H1)$'$ to derive the last identity. We also deduce from (H7)$'$ that tangential derivatives of the pressure vanish at the surface, so that $\nabla_{X,z} P(M) = (\partial_{\mathbf{n}} P)\mathbf{n}$. We deduce therefore that

$$\partial_t^2 M + g\mathbf{e}_z = \frac{1}{\rho}(-\partial_{\mathbf{n}} P)\mathbf{n}, \tag{1.6}$$

where $\mathbf{n}$ is the unit normal pointing out of the fluid domain.

1.2.1.1. *Nalimov's formulation in dimension $d = 1$.* Let us identify canonically $\mathbb{R}^{d+1}$ ($d = 1$) with the complex plane $\mathbb{C}$: $(x, z) \in \mathbb{R}^d \rightsquigarrow x + iz \in \mathbb{C}$. This identification allows one to see the incompressibility and irrotationality assumptions (H2)$'$–(H3)$'$ as the Cauchy-Riemann relations for the conjugate of the velocity field (seen as a complex-valued vector field), denoted $\overline{\mathbf{U}} = u - iw$. It follows that $\overline{\mathbf{U}}$ is holomorphic in the fluid domain Ω_t, and there exists a singular integral operator on the surface recovering boundary values of w from boundary values of u — denoted, respectively, by $\underline{w}(t, \alpha) = w(t, M(t, \alpha))$ and $\underline{u}(t, \alpha) = u(t, M(t, \alpha))$,

$$\underline{w} = K[\Gamma_t]\underline{u} \quad \text{or, owing to (1.5),} \quad \partial_t M_2 = K[\Gamma_t]\partial_t M_1,$$

where we used the notation $M(t) = (M_1(t), M_2(t))$.

REMARK 1.3. The operator $K[\Gamma_t]$ is related to the Hilbert transform associated to the fluid domain. Actually, it is often convenient [**328, 315**] to replace the relation $\underline{w} = K[\Gamma_t]\underline{u}$ by

$$\underline{\overline{\mathbf{U}}} = \mathfrak{H}[\Gamma_t]\underline{\overline{\mathbf{U}}} \qquad (\text{with } \underline{\overline{\mathbf{U}}} = \underline{u} - i\underline{v}).$$

Here, $\mathfrak{H}[\Gamma_t]$ is the Hilbert transform on Γ_t associated with the parametrization $M(t, \cdot)$,

$$\mathfrak{H}[\Gamma_t] f(t, \alpha) = \frac{1}{\pi i} \text{p.v.} \int \frac{f(t, \alpha')\partial_\alpha M(t, \alpha')}{M(t, \alpha) - M(t, \alpha')} d\alpha' \tag{1.7}$$

where p.v. stands for "principal value" and $\mathbb{R}^2$ has been identified with $\mathbb{C}$ (we also recall that a function f is holomorphic in a domain Ω with boundary $\partial\Omega$ if and only if $\mathfrak{H}[\partial\Omega] f_{|\partial\Omega} = f_{|\partial\Omega}$).

Multiplying (1.6) by the tangent vector $(\partial_\alpha M_1, \partial_\alpha M_2)^T$, we get Nalimov's formulation of the one-dimensional water waves problem,

$$\begin{cases} \partial_\alpha M_1 \partial_t^2 M_1 + (g + \partial_t^2 M_2)\partial_\alpha M_2 = 0, \\ \partial_t M_2 = K[\Gamma_t]\partial_t M_1. \end{cases} \tag{1.8}$$

Using this formulation, Nalimov [**265**] was the first to prove a local well-posedness result for the water waves equation with Sobolev (small) initial data (other results deal with analytic data [**300, 198**]). Nalimov dealt with the case of infinite depth, but Yosihara extended his work to finite depth [**330**], while Craig used it [**101**] to provide the first justification of the KdV approximation (see Chapter 7).

1.2.1.2. *Wu's formulation.* The main limitation of Nalimov's result is that it requires a smallness condition on the initial data. In [**326**], S. Wu managed to remove this smallness condition using a modification of Nalimov's formulation. One of the key points of [**326**] is the observation that $(-\partial_{\mathbf{n}}P) > 0$ in the infinite depth case considered in [**326**] (see Proposition 4.32 in Chapter 4), so that one gets from (1.6) that $N := (-\partial_\alpha M_2, \partial_\alpha M_1) = |\partial_\alpha M|\mathbf{n}$ is given by

$$N = \frac{\partial_t^2 M + g\mathbf{e}_z}{|\partial_t^2 M + g\mathbf{e}_z|}|\partial_\alpha M|.$$

Denoting by $\mathfrak{a}$ the Rayleigh-Taylor coefficient,

$$\mathfrak{a} = \frac{1}{|\partial_\alpha M|}\frac{1}{\rho}\big(-\partial_{\mathbf{n}}P(M)\big), \tag{1.9}$$

and differentiating (1.6) with respect to time, we get

$$\partial_t^3 M - \mathfrak{a}\partial_t N = \partial_t \mathfrak{a} N;$$

using the expression derived above for N and recalling that $\partial_t M = \underline{\mathbf{U}}(t, M)$, we get, with the notation $\underline{\mathbf{U}}(t,\alpha,\beta) = \mathbf{U}(t, M(t,\alpha,\beta))$,

$$\partial_t^2\underline{\mathbf{U}} - \mathfrak{a}\partial_t N = \frac{\partial_t\underline{\mathbf{U}} + g\mathbf{e}_z}{|\partial_t\underline{\mathbf{U}} + g\mathbf{e}_z|}|\partial_\alpha M|\partial_t\mathfrak{a}.$$

Moreover, since $\partial_t N = (-\partial^2_{\alpha t}M_2, \partial^2_{\alpha t}M_1)$, we get after differentiating (1.5) with respect to α that

$$\partial_t N = \big(-\partial_\alpha M\cdot\nabla_{X,z}w(M), \partial_\alpha M\cdot\nabla_{X,z}u(M)\big)^T.$$

From the irrotationality and incompressibility conditions, we also have $\partial_x w = \partial_z u$ and $\partial_z w = -\partial_x u$, so that

$$\begin{aligned} -\partial_\alpha M\cdot\nabla_{X,z}w &= -\partial_\alpha M_1\partial_x w - \partial_\alpha M_2\partial_z w \\ &= -\partial_\alpha M_1\partial_z u + \partial_\alpha M_2\partial_x u \\ &= -|\partial_\alpha M|\partial_{\mathbf{n}}u, \end{aligned}$$

and similarly $\partial_\alpha M\cdot\nabla_{X,z}u = -|\partial_\alpha M|\partial_{\mathbf{n}}w$. It is consistent[3] with the definition (1.2) of the Dirichlet-Neumann operator to use the notation

$$G[\Gamma_t]\underline{\mathbf{U}} = \big(G[\Gamma_t]\underline{u}, G[\Gamma_t]\underline{w}\big)^T := |\partial_\alpha M|(\partial_{\mathbf{n}}u, \partial_{\mathbf{n}}v)^T.$$

In the one-dimensional case $d = 1$, Wu's formulation of the water waves equation can then be put under the form[4]

$$\begin{cases} \partial_t^2\underline{\mathbf{U}} + \mathfrak{a}G[\Gamma_t]\underline{\mathbf{U}} = \dfrac{\partial_t\underline{\mathbf{U}} + g\mathbf{e}_z}{|\partial_t\underline{\mathbf{U}} + g\mathbf{e}_z|}|\partial_\alpha M|\partial_t\mathfrak{a}, \\ \underline{\mathbf{U}} = \mathfrak{H}[\Gamma_t]\underline{\mathbf{U}}, \end{cases} \tag{1.10}$$

[3]If Γ_t is a graph parametrized by $(x, \zeta(x))$, then $G[\Gamma_t]\psi = G[\zeta]\psi$ where $G[\zeta]$ is defined as in (1.1)–(1.2) with the homogeneous Neumann condition at the bottom in (1.1) replaced by $\nabla_{X,z}\Phi \to 0$ as $z \to -\infty$.

[4]This formulation is actually not the one used by S. Wu in [**326**] but rather the adaptation to the one-dimensional case of the formulation Wu used in [**327**] for the case $d = 2$.

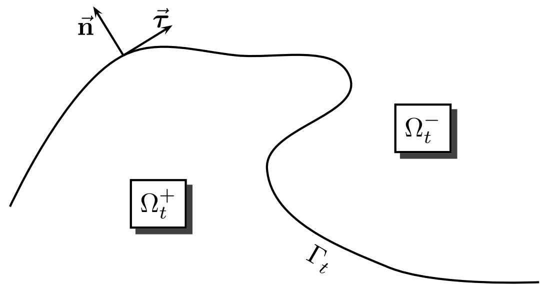

FIGURE 1.2. A generalization of the water waves problem: two-fluids interfaces.

where $\mathfrak{a} = \dfrac{1}{|\partial_\alpha M|}|\partial_t \underline{\mathbf{U}} + g\mathbf{e}_2|$ and $\mathfrak{H}(\Gamma_t)$ is the Hilbert transform introduced in Remark 1.3. S. Wu showed [**326**] that the right-hand side of the first equation consists of lower order terms in $\underline{\mathbf{U}}$, so that (1.10) is a quasilinear weakly hyperbolic problem, for which Wu constructed solutions by an iterative scheme without any smallness assumption on the initial data. This formulation is also the starting point for [**328**] where an almost global existence result is proved, for [**337**] where nonzero vorticity is allowed and for [**315**] where the nonlinear Schrödinger approximation is justified.

Wu's formulation (1.10) also holds in dimension $d = 2$. The derivation of the first equation of (1.10) is absolutely similar to the derivation sketched above in the case $d = 1$. The second equation of (1.10) also holds true when $d = 2$; this only requires an extension of the formula (1.7) for the Hilbert transform to the two-dimensional case. This extension is derived in [**327**] using Clifford analysis instead of complex analysis, but more standard tools could of course be used. In [**329**], global well-posedness in dimension $d = 2$ is proved using this formulation of the water waves equations.

1.2.2. Other interface parametrizations and extension to two-fluids interfaces. We have considered in §1.2.1 the equations obtained when using a Lagrangian parametrization of the surface Γ_t. However, other parametrizations are possible. In order to underline the importance of the choice of the parametrization of the interface, we consider here a generalization of the water waves problem, namely, the case of two-fluids interfaces. For the sake of simplicity, we consider the one dimensional case $d = 1$.

The interface Γ_t separates two fluids of densities $\rho_+ \geq \rho_-$, the heavier fluid being below the interface (see Figure 1.2). The water waves problem corresponds therefore to the endpoint case $\rho_+ = \rho$, $\rho_- = 0$. The discussion below is inspired by [**18**], devoted to a more general review on interface evolution.

Let $r(t, \cdot)$ be a parametrization of the interface,

$$\Gamma_t = \{r(t, \alpha), \alpha \in \mathbb{R}\};$$

we assume that the flow is incompressible on the whole space $\mathbb{R}^{d+1}$ and irrotational above and below the interface. We therefore have

$$\nabla_{X,z} \cdot \mathbf{U} = 0, \qquad \nabla_{X,z} \times \mathbf{U} = \tilde{\omega} \otimes \delta_{\Gamma_t}, \tag{1.11}$$

where δ_{Γ_t} is the Dirac measure associated to the interface, and $\tilde{\omega}$ is the *vorticity density* associated to the parametrization $r(t,\alpha)$,

$$\tilde{\omega}(t,\alpha) = |\partial_\alpha r|(\mathbf{U}_+^\tau - \mathbf{U}_-^\tau)_{|(t,r(t,\alpha))},$$

with $\mathbf{U}_\pm^\tau$ the tangential components of $\mathbf{U}_\pm$ evaluated at the interface. We therefore deduce from (1.11) the following jump relations on the normal and tangential components of the velocity at the interface,

$$[\mathbf{U}_\pm^n] = 0, \qquad [\mathbf{U}_\pm^\tau] = \frac{1}{|\partial_\alpha r|}\tilde{\omega}.$$

Let us now consider the conservation equations of mass and momentum,

$$\begin{aligned} \partial_t(\rho\mathbf{U}) + \nabla_{X,z}\cdot(\rho\mathbf{U}\otimes\mathbf{U}) + \nabla_{X,z}P = -\rho g\mathbf{e}_z,\\ \partial_t\rho + \nabla_{X,z}\cdot(\rho\mathbf{U}) = 0, \end{aligned}$$

written in the sense of distributions in the whole space $\mathbb{R}^{d+1}$, with $\rho = \rho_+$, $\mathbf{U} = \mathbf{U}_+$ below Γ_t and $\rho = \rho_-$, $\mathbf{U} = \mathbf{U}_-$ above. We deduce from the weak formulation of these equations the following two evolution equations (see [**306**], and [**97**] for a slightly different but equivalent formulation),

$$(\partial_t r - v)\cdot\vec{n} \quad = \quad 0 \tag{1.12}$$

$$\begin{aligned}\partial_t(\frac{1}{2}\widetilde{\omega} - a|\partial_\alpha r|v^\tau) \quad &+ \quad \partial_\alpha\Big(\frac{1}{|\partial_\alpha r|}(\frac{1}{2}\widetilde{\omega} - a|\partial_\alpha r|v^\tau)(v-\partial_t r)\cdot\vec{\tau}\Big)\\ &- \quad a\partial_\alpha\Big(\frac{1}{8}\frac{\widetilde{\omega}^2}{|\partial_\alpha r|^2} - \frac{1}{2}|v|^2\Big) = -a|\partial_\alpha r|g\mathbf{e}_z\cdot\vec{\tau},\end{aligned} \tag{1.13}$$

where v is the average velocity at the interface, and a the Atwood number,

$$v = \frac{1}{2}\big(\mathbf{U}_+ + \mathbf{U}_-\big)_{|\Gamma_t}, \qquad a = \frac{\rho_- - \rho_+}{\rho_- + \rho_+}.$$

Observe that the above system does not fully determine the evolution of the vector $r(\lambda,t)$. Equation (1.12) concerns only the normal component of its time derivative. This corresponds to the fact that it is only the global evolution of the interface and not its parametrization that matters. Depending on the situation, different choices can be made for the parametrization $\alpha\mapsto r(t,\alpha)$ of Γ_t:

- When $a = 0$ (this corresponds to $\rho_- = \rho_+$ i.e., to the so-called Kelvin-Helmholtz problem), it is convenient to choose a Lagrangian parametrization,

  $$r_t(t,\alpha) = v(t,r(t,\alpha)) = \frac{\mathbf{U}_+ + \mathbf{U}_-}{2}(t,r(t,\alpha));$$

  equations (1.12) and (1.13) then reduce to

  $$\partial_t\tilde{\omega} = 0 \quad \text{ and thus } \quad \tilde{\omega}(t,\alpha) = \tilde{\omega}(0,\alpha).$$

  Moreover, since we can recover v in terms of $\tilde{\omega}$ through Biot-Savart's law

  $$v(t,r(t,\alpha)) = \text{p.v.}\int_{\mathbb{R}}\nabla^\perp G(r(t,\alpha),r(t,\alpha'))\widetilde{\omega}(t,\alpha')d\alpha',$$

  where G is the Green function of the Laplacian on $\mathbb{R}^{1+1}$, we get

  $$\partial_t r(t,\alpha) = \frac{1}{2\pi}R_{\frac{\pi}{2}}\text{p.v.}\int\frac{r(t,\alpha)-r(t,\alpha')}{|r(t,\alpha)-r(t,\alpha')|^2}\tilde{\omega}(0,\alpha')d\alpha',$$

where $R_{\frac{\pi}{2}}$ is the rotation of angle $\pi/2$.
Therefore, it is natural to reparametrize the interface by the (time independent) arc length *of its initial position*, $\Gamma_t = \{r(t, s_0^{-1}(s))\}$, where $s_0(\alpha) = \int_0^\alpha \partial_\alpha r(0, \alpha')d\alpha'$. One can further identify the fluid domain with the complex plane, i.e. $z(t,s) \sim r(t, s_0^{-1}(s))$, to obtain:

$$\partial_t \overline{z}(t,s) = \frac{1}{2\pi i}\text{p.v.}\int \frac{\gamma(0,s')}{z(t,s)-z(t,s')}ds',$$

where $\gamma(0,s)$ is the *vortex strength*[5] at $t=0$.
Moreover, if $\gamma(s,0)$ has *a distinguished sign* (which is the case when the initial vorticity is a Radon measure), then we can use the mapping

$$\alpha_0(s) = \int_0^s \gamma(0,s')ds'$$

to perform another (time independent) reparametrization of the interface, namely, $\Gamma_t = \{z(t, \alpha_0^{-1}(\alpha))\}$. One thus obtains the *Birkhoff-Rott equation* for the variable $\mathbf{z}(t,\alpha) = z(t, \alpha_0^{-1}(\alpha))$,

$$\partial_t \overline{\mathbf{z}}(t,\alpha) = \frac{1}{2\pi i}\text{p.v.}\int \frac{d\alpha'}{\mathbf{z}(t,\alpha)-\mathbf{z}(t,\alpha')}. \tag{1.14}$$

The Birkhoff-Rott equation corresponds to a parametrization by the circulation; indeed, one readily checks that $|\partial_\alpha \mathbf{z}(t,\alpha_0(s))| = \frac{1}{\gamma(0,s)}$.

- When $a=-1$ (this corresponds to $\rho_- = 0$, and thus to the water waves problem), several choices are possible. One can, for instance, as in [**306**], choose a Lagrangian parametrization with the velocity given by the trace of the velocity field in Ω_t^+ at the surface; this situation has been described in §1.2.1.
  Another choice, adopted in [**14**] (and [**15**] for a generalization to the two-dimensional case $d=2$), consists of parametrizing the surface by its (renormalized) arc-length. It is also possible to use the particular case $a=-1$ of the following point when the surface is a graph.
- When the interface is a graph, we can also choose the canonical parametrization by x, $\Gamma_t = \{(x, \zeta(t,x)\}$. This is the choice made in the derivation of (1.3) and it will be used throughout these notes.
- The fact that one can add any tangential term to the velocity field at the interface without modifying the geometric evolution of the curve has been used by several authors for the water waves problem or two-fluids interfaces [**175, 97**] but also for interface evolutions in other contexts such as the Muskat problem [**66, 98**].

1.2.3. Variational formulations. Variational formulations for Euler's equations go back at least to [**171, 143**]. We briefly sketch here two different variational approaches. The first approach follows the geometric path of Arnold [**16**], who showed that Euler's equations can be seen as a geodesic flow on the group of volume-preserving homeomorphisms. The second approach (see Luke [**243**]), is more formal but has been historically used to derive several important asymptotic models.

[5]The vortex strength is defined as the vorticity density $\tilde{\omega}$ associated to the arc-length parametrization.

1.2.3.1. *The geometric approach.* It was observed in [**16**] that Euler equations for the motion of an inviscid incompressible fluid can be viewed as the geodesic flow on the infinite-dimensional manifold of volume-preserving diffeomorphisms. This approach has been used by several authors, including [**53, 142, 301**]. More recently, Beyer and Günther [**29, 30**] used this geometric approach to study the free surface for capillary water waves (surface tension but no gravity) in a bounded domain. In a series of papers [**297, 298, 299**], Shatah and Zeng dealt with the rotational case and addressed several related problems (such as two-fluid interfaces). Here we sketch their approach.

Let us denote by Σ the Lagrangian parametrization of the fluid domain associated to the velocity field $\mathbf{U}$,

$$\begin{cases} \partial_t \Sigma(t,X,z) = \mathbf{U}\big(t,\Sigma(t,X,z)\big), \\ \Sigma(0,X,z) = (X,z), \end{cases} \tag{1.15}$$

for all $(X,z) \in \Omega_0$ (the fluid domain at time $t=0$). Since $\mathbf{U}$ is divergence free, one has $\Sigma \in \mathcal{H}$, where $\mathcal{H}$ is the set of all volume-preserving homeomorphisms,

$$\begin{aligned} \mathcal{H} \quad = \quad & \{\Sigma : \Omega_0 \to \mathbb{R}^{d+1}, \\ & \Sigma \text{ is a volume-preserving homeomorphism from } \Omega_0 \text{ onto } \Sigma(\Omega_0)\}. \end{aligned}$$

Let us now write the conserved energy H introduced in (1.4) under the form

$$\begin{aligned} H \quad &= \quad \frac{1}{2}\int_{\Omega_t} |\mathbf{U}|^2 + \frac{g}{2}\int_{\mathbb{R}^d} \zeta^2 \\ &= \quad \frac{1}{2}\int_{\Omega_0} |\mathbf{U}\circ\Sigma|^2 + gG(\Sigma), \end{aligned}$$

where $G(\Sigma)$ is the volume delimited by the free surface and the asymptotic (rest) state. Owing to (1.15), we have

$$H = L(\Sigma, \partial_t\Sigma) := \frac{1}{2}\int_{\Omega_0} |\partial_t\Sigma|^2 + gG(\Sigma).$$

In order to derive the Euler-Lagrange equations associated with the Lagrangian action L, we can consider $\mathcal{H}$ as a manifold with tangent space

$$T_\Sigma\mathcal{H} = \{\Sigma' : \Omega_0 \to \mathbb{R}^{d+1}, \nabla_{X,z}\cdot(\Sigma'\circ\Sigma^{-1}) = 0\}.$$

The conservation of energy suggests that $T\mathcal{H}$ can be endowed with the L^2 metric, and one can show [**16, 297**] that the water waves equations coincide with the geodesic equation on $\mathcal{H}$, namely,

$$\partial_t^2\Sigma + gG'(\Sigma) = \text{Lagrange multiplier},$$

where $G'(\Sigma)$ denotes the tangential gradient of $G(\Sigma)$; the covariant derivative $\mathcal{D}_t\partial_t\Sigma$ (or equivalently the tangential component of $\partial_t^2\Sigma$) therefore satisfies

$$\mathcal{D}_t\partial_t\Sigma + gG'(\Sigma) = 0. \tag{1.16}$$

Shatah and Zeng studied the linearization of (1.16) to obtain local well-posedness results under the assumption that the Rayleigh-Taylor criterion is satisfied (i.e., that the Rayleigh-Taylor coefficient (1.9) is strictly positive).

1.2.3.2. *Luke's variational formulation.* The water waves equation can also be derived formally from Hamilton's principle using the Lagrangian derived by Luke [**243**]. Though we do not use this formulation in these notes, we briefly mention it for the sake of completeness. The Lagrangian $\widetilde{l}$ is given by the vertical integration of the pressure,

$$\widetilde{l} = \int_{-H_0+b(X)}^{\zeta(X)} (P - P_{atm})(t, X, z) dz.$$

Using the Bernoulli equation (H1)$''$, we can define a new Lagrangian $l = l(\zeta, \Phi)$ (deduced from $\widetilde{L}$ by dropping constant terms that obviously have no incidence on the minimization principle),

$$l = -\rho\big[g\frac{\zeta^2}{2} + \int_{-H_0+b}^{\zeta} \big(\partial_t\Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2\big)dz\big].$$

In [**243**] (which we refer to for the details of the computations) Luke observed that the water waves equations could be recovered from the variational principle,

$$\delta J(\zeta, \Phi) = 0 \quad \text{ where } \quad J(\zeta, \Phi) = \int\int l(\zeta, \Phi) dX dt.$$

As shown by Miles and Salmon [**261**], adding some constraints to this minimization principle allows one to recover some of the asymptotic models derived here (e.g., the Green-Naghdi equation over flat bottom). See also [**82**] for a similar approach.

1.2.4. Free surface Euler equations in Lagrangian formulation. Written in the Lagrangian coordinates (1.15), and denoting

$$\begin{aligned}
\tilde{\mathbf{U}}(t, X, z) &= \mathbf{U}(t, \Sigma(t, X, z)), \\
\tilde{P}(t, X, z) &= P(t, \Sigma(t, X, z)), \\
A(t, X, z) &= [\nabla_{X,z}\Sigma(t, X, z)]^{-1},
\end{aligned}$$

the free surface Euler equations become (without the irrotationality assumption (H3)$'$)

$$\begin{aligned}
\Sigma = \mathrm{Id} + \int_0^t \tilde{\mathbf{U}} \qquad & \text{in } \Omega_0,\ t \geq 0, \\
\partial_t\tilde{\mathbf{U}} + A\nabla_{X,z}\tilde{P} + g\mathbf{e}_z = 0 \qquad & \text{in } \Omega_0,\ t \geq 0, \\
\mathrm{Tr}(A\nabla_{X,z}\tilde{\mathbf{U}}) = 0 \qquad & \text{in } \Omega_0,\ t \geq 0, \\
\tilde{P} = P_{atm} \qquad & \text{on } \Gamma_0,\ t \geq 0.
\end{aligned}$$

The advantage of this formulation is that it is cast on a fixed domain Ω_0, which eases the construction of a solution by an iterative scheme. This has been done in [**78, 236, 237**] using refined geometric estimates and in [**99**] using a clever smoothing of the equations (see also [**75**] for an extension of this method to two-fluids interfaces).

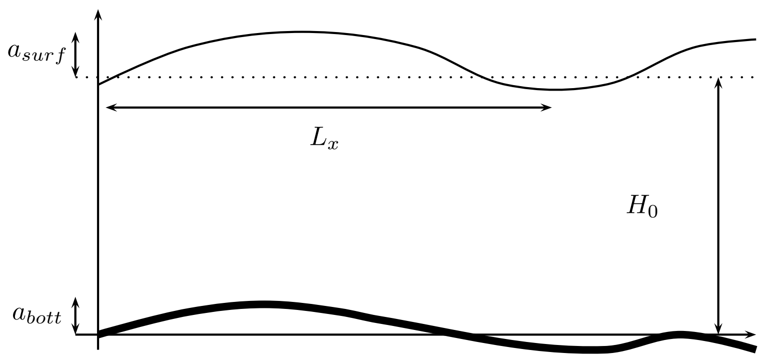

FIGURE 1.3. Length scales involved in the wave motion.

1.3. The nondimensionalized equations

The water waves equations (1.3) have a particularly rich structure and it is possible to exhibit solutions with dramatically different properties, depending on the physical characteristic of the flow. For instance, dispersive effects are more important in deep water than in shallow water, and nonlinear effects become more important when the amplitude of the waves grows larger, etc.

It is easier to comment on the qualitative properties of the solutions to (1.3) if we nondimensionalize these equations using the physical characteristics of the flow. This is the goal of this section.

1.3.1. Dimensionless parameters. We introduce here dimensionless parameters based on the characteristic scales of the wave motion. Four (when $d = 1$) or five (when $d = 2$) main length scales are involved in this problem:

(1) The characteristic water depth H_0
(2) The characteristic horizontal scale L_x in the longitudinal direction
(3) The characteristic horizontal scale L_y in the transverse direction (when $d = 2$)
(4) The order of the free surface amplitude a_{surf}
(5) The order of bottom topography variation a_{bott}

Four independent dimensionless parameters can be formed from these five scales. We choose:

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma, \tag{1.17}$$

where ε is often called the *nonlinearity* parameter, while μ is the *shallowness* parameter. We will also refer to β and γ as the *topography* and *transversality* parameters.

REMARK 1.4.

i- It is implicitly assumed here that the characteristic horizontal scales for the variations of the bottom are the same as for the free surface. This is of course not the case when a wave propagates over a "rough" bottom; in such configurations, the characteristic length is much smaller for the bottom than for the surface. See §1.7 for more comments on this point.

ii- Quite obviously, the set of dimensionless parameters (1.17) is not adapted to study the propagation of waves over infinite depth. So in this case, one uses

$$\epsilon = \frac{a_{surf}}{L_x},$$

where ϵ is the *steepness* of the wave. However, since $\epsilon = \varepsilon\sqrt{\mu}$ one can stick to (1.17) even in very deep water, provided that the dependence on ε and μ appears through $\varepsilon\sqrt{\mu}$ only.

EXAMPLE 1.5 (Tsunami). For the 2004 Indian ocean tsunami, the characteristic length L_x is given by the size of the fault slip, between 160 and 240 km. The depth H_0 of the ocean ranges between 1 km and 4 km, and satellite observations suggest an amplitude a_{surf} of the surface wave of 60 cm. The nonlinearity and shallowness parameters are thus estimated by

$$1.5 \times 10^{-4} < \varepsilon < 6 \times 10^{-4} \quad \text{and} \quad 1.7 \times 10^{-5} < \mu < 6.2 \times 10^{-4}.$$

A tsunami is therefore an example of wave propagating in a shallow-water regime (even though the depth H_0 is very large). We refer to [**115, 138**] and references therein for details on tsunami modeling and simulation.

EXAMPLE 1.6 (Coastal oceanography). For a swell of wavelength $L_x \sim 100m$ and amplitude $a_{surf} \sim 1m$ over a continental shelf of depth $H_0 \sim 10m$, one has

$$\varepsilon \sim 10^{-1} \quad \text{and} \quad \mu \sim 10^{-2}.$$

Though the shallow water assumption $\mu \ll 1$ is satisfied, this is far less obvious than for the tsunami mentioned in Example 1.5—even though the depth is 400 times smaller here! It is therefore important to keep in mind that *shallowness always concerns the ratio H_0/L_x and not H_0 itself.* Another important consideration is that since μ is not extremely small here, it is important to find asymptotic models with a wide range of validity, i.e., that remain correct for not so small values of μ and not only as $\mu \to 0$ (see, for instance, §5.2).

1.3.2. Linear wave theory. The easiest way to get some information on the order of magnitude of wave motion variables is to investigate the behavior of the water waves equations around the rest state. Linearizing (1.3) around $(\zeta, \psi) = (0, 0)$ (and considering flat bottoms, $b = 0$), one gets the following system

$$\begin{cases} \partial_t \zeta - G[0,0]\psi = 0, \\ \partial_t \psi + g\zeta = 0. \end{cases} \tag{1.18}$$

In particular, the surface elevation ζ solves a second-order evolution equation,

$$\partial_t^2 \zeta + gG[0,0]\zeta = 0. \tag{1.19}$$

Using the definition of the Dirichlet-Neumann operator (1.2), we have

$$G[0,0]\psi = \partial_z \Phi_{|z=0},$$

where

$$\begin{cases} \Delta_{X,z}\Phi = 0 \qquad \text{for } -H_0 < z < 0, \\ \Phi_{|z=0} = \psi, \quad \partial_z \Phi_{|z=-H_0} = 0. \end{cases}$$

Taking the Fourier transform of this equation with respect to the horizontal variables, one is led to solve a second-order ODE in z with boundary conditions at

$z = 0$ and $z = -H_0$; one readily checks that there exists a unique solution given by

$$\forall(\xi, z) \in \mathbb{R}^{d+1}, \qquad \widehat{\Phi}(\xi, z) = \frac{\cosh\big((z + H_0)|\xi|\big)}{\cosh(H_0|\xi|)}\widehat{\psi}(\xi),$$

where $\widehat{\cdot}$ denotes the Fourier transform. Therefore, we have

$$\widehat{\partial_z\Phi}(\xi, 0) = \partial_z\widehat{\Phi}(\xi, 0) = |\xi|\tanh(H_0|\xi|)\widehat{\psi}(\xi)$$

and the Dirichlet-Neumann operator admits the following explicit expression,

$$G[0, 0]\psi = |D|\tanh(H_0|D|)\psi,$$

where we used the Fourier multiplier notation:

NOTATION 1.7 (Fourier multipliers). Let $f \in L^\infty(\mathbb{R}^d)$; the Fourier multiplier $f(D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is then defined by

$$\forall u \in L^2(\mathbb{R}^d), \quad \forall\xi \in \mathbb{R}^d, \qquad \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

This definition can be generalized to other functional settings. (For instance, it is easy to check that $G[0, 0] : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is well defined and bounded.)

The second-order evolution equation (1.19) can therefore be recast as

$$\partial_t^2\zeta + g|D|\tanh(H_0|D|)\zeta = 0. \tag{1.20}$$

Taking initial conditions for (1.20) of the form

$$\zeta_{|t=0} = \zeta_0 \quad \text{and} \quad \partial_t\zeta_{|t=0} = |D|\tanh(H_0|D|)\psi_0$$

(which corresponds to initial condition $(\zeta, \psi)_{|t=0} = (\zeta_0, \psi_0)$ to (1.18)), the general form of the solutions of (1.20) is given by

$$\begin{aligned}\zeta(t, X) = &\frac{1}{2}\int_{\mathbb{R}^d} e^{i(\xi\cdot X - \omega(\xi)t)}\big(\widehat{\zeta_0}(\xi) + i\frac{\omega(\xi)}{g}\widehat{\psi_0}(\xi)\big)d\xi \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} e^{i(\xi\cdot X + \omega(\xi)t)}\big(\widehat{\zeta_0}(\xi) - i\frac{\omega(\xi)}{g}\widehat{\psi_0}(\xi)\big)d\xi,\end{aligned} \tag{1.21}$$

where

$$\omega(\xi) = \big(g|\xi|\tanh(H_0|\xi|)\big)^{1/2} \tag{1.22}$$

is the *dispersion relation* of the linearized water waves equations; the *wave celerity* at the wave number ξ is defined as

$$c(|\xi|) = \frac{\omega(\xi)}{|\xi|} = \big(g\frac{\tanh(H_0|\xi|)}{|\xi|}\big)^{1/2}. \tag{1.23}$$

These relations can be used to determine some typical order of magnitude for various characteristics of the flow in terms of the physical quantities introduced in the previous section. Let us focus on the one-dimensional case $d = 1$ for the sake of simplicity.

If the typical wavelength of the wave under consideration is L_x, as assumed in §1.3.1, then in Fourier space, the wave is concentrated around the wave number $\xi_0 = \frac{2\pi}{L_x}$. The typical wave celerity $c_0 = \omega(\xi_0)/|\xi_0|$ is then given by

$$c_0 = \big(gL_x\frac{1}{2\pi}\tanh(2\pi\frac{H_0}{L_x})\big)^{1/2} = \sqrt{gH_0\nu}, \tag{1.24}$$

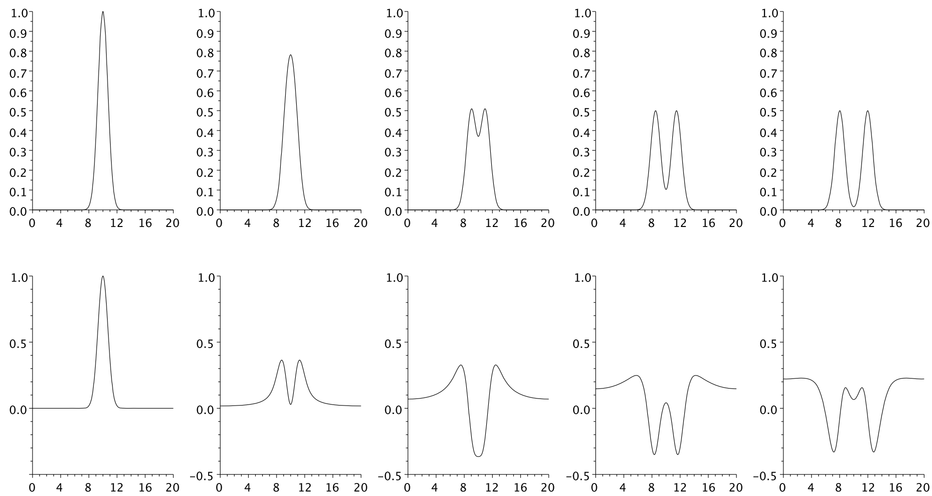

FIGURE 1.4. Time evolution of the same initial data in shallow (up) and deep (down) water for the linear equation (1.20).

where ν is a dimensionless parameter defined as

$$\nu = \frac{\tanh(2\pi\sqrt{\mu})}{2\pi\sqrt{\mu}}. \tag{1.25}$$

Similarly, it follows from (1.21) that if the typical order of magnitude of ζ (and ζ_0) is a_{surf}, as assumed in §1.3.1, then $\frac{\omega(\xi_0)}{g}\psi_0$ must also be roughly of size a_{surf}. A typical order of magnitude for the velocity potential ψ is thus given by

$$\Phi_0 = 2\pi g \frac{a_{surf}}{\omega(\xi_0)} = \frac{a_{surf}}{H_0} L_x \sqrt{\frac{gH_0}{\nu}} \tag{1.26}$$

(the factor 2π is used here to simplify computations to be done later).

REMARK 1.8. In shallow water ($\mu \ll 1$), one has $\nu \sim 1$ while in deep water ($\mu \gg 1$), one has $\nu \sim (2\pi\sqrt{\mu})^{-1}$. It follows that the typical wave celerity is $\sqrt{gH_0}$ in shallow water and $\sqrt{\frac{gL_x}{2\pi}}$ in deep water. In particular, water waves are *dispersive* in deep water (their celerity depends on the wavelength) but *nondispersive* in shallow water. In view of this fundamental qualitative difference, it is not surprising that different nondimensionalizations are used to describe shallow and deep water waves propagation (see §1.3.3 below). See also Figures 1.4 and 1.5 for examples of the very different behavior of shallow and deep water waves.

1.3.3. Nondimensionalization of the variables and unknowns. Some variables and unknowns can be nondimensionalized in a very simple way using the typical lengths introduced in §1.3.1. Namely, let us introduce

$$x' = \frac{x}{L_x}, \qquad y' = \frac{y}{L_y}, \qquad \zeta' = \frac{\zeta}{a_{surf}}, \qquad b' = \frac{b}{a_{bott}}.$$

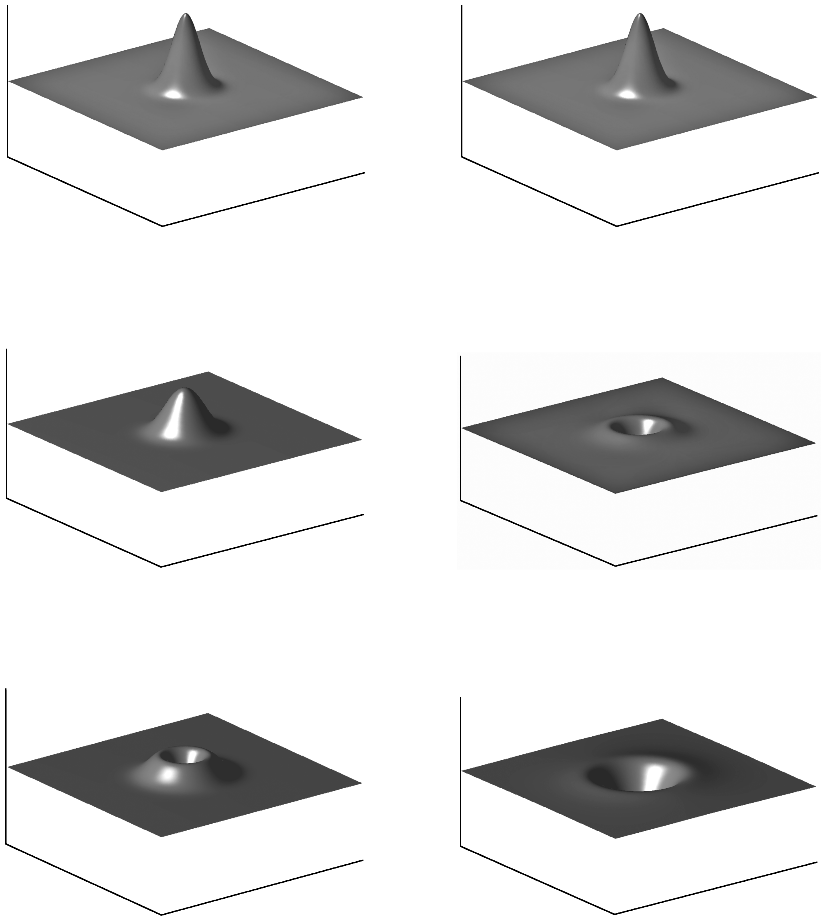

FIGURE 1.5. Same as Figure 1.4 in dimension $d = 2$ (left: shallow water; right: deep water).

In other words, we take L_x and L_y as unit lengths in the x and y direction respectively, and the water and bottom parametrizations have amplitude variations roughly equal to 1.

The nondimensionalization of the variable t and of the velocity potential Φ (and thus of ψ) are less intuitive, and we appeal to the linear analysis performed in §1.3.2. It is quite natural to scale the time variable on L_x/c_0, where c_0 is the wave celerity. Owing to (1.24), this yields

$$t' = \frac{t}{t_0} \quad \text{with} \quad t_0 = \frac{L_x}{\sqrt{gH_0\nu}}.$$

We have already seen that the typical order of magnitude for the velocity potential is given by (1.26). Consequently, we set

$$\Phi' = \frac{1}{\Phi_0}\Phi, \quad \text{with} \quad \Phi_0 = \frac{a_{surf}}{H_0} L_x \sqrt{\frac{gH_0}{\nu}}.$$

REMARK 1.9. Two distinct nondimensionalizations are used in oceanography (e.g., [**125**]) depending on the value of μ: shallow water scaling ($\mu \ll 1$) and Stokes wave scaling for intermediate to deep water ($\mu \not\ll 1$). They correspond to the specializations of the present nondimensionalization in the cases $\nu = 1$ and $\nu = 2\pi\mu^{-1/2}$ respectively, that is, to the asymptotic value of ν as $\mu \to 0$ and $\mu \to \infty$.

Even though the vertical variable z does not appear in (1.3), it is implicitly involved in the definition (1.2) of the Dirichlet-Neumann operator. We therefore need to nondimensionalize it also, but the choice of the scaling length is technically rather than physically motivated. For the convenience of working with a reference depth equal to 1, we set

$$z' = \frac{z}{H_0}.$$

But any other choice (such as $z' = z/L_x$ in infinite depth) would be equivalent. See Remark 1.13.

REMARK 1.10. As a wave propagates towards the beach, its wavelength shortens, but its period[6], denoted T_0, remains constant. It is sometimes convenient to use T_0 rather than L_x to nondimensionalize the equations. The typical wavelength L_x is therefore deduced from the constitutive relation $c_0 = L_x/T_0$, by solving the following equation for L_x,

$$\frac{L_x}{\sqrt{gH_0}}\big(\frac{L_x}{H_0}\frac{1}{2\pi}\tanh(2\pi\frac{H_0}{L_x})\big)^{-1/2} = T_0.$$

For instance, a swell of period T_0 will have a wavelength $L_0 \sim \frac{1}{2\pi}gT_0^2$ offshore (infinite depth limit in the above formula), while this wavelength will shorten as $L_0 \sim \sqrt{gH_0}T_0$ as the waves will approach the shore (take $\tanh(2\pi H_0/L_x) \sim 2\pi H_0/L_x$ in the above formula).

1.3.4. Nondimensionalization of the equations. Before addressing the nondimensionalization of (1.3), let us introduce a few notations for parameter depending operators.

NOTATION 1.11. We define "twisted" horizontal gradient and $\mathbb{R}^{d+1}$-dimensional Laplace operator as follows:

$$\begin{array}{llll} \nabla^\gamma = (\partial_x, \gamma\partial_y)^T & \text{if } d=2 \quad \text{and} & \nabla^\gamma = \partial_x & \text{if } d=1, \\ \Delta^{\mu,\gamma} = \mu\partial_x^2 + \gamma^2\mu\partial_y^2 + \partial_z^2 & \text{if } d=2 \quad \text{and} & \Delta^{\mu,\gamma} = \mu\partial_x^2 + \partial_z^2 & \text{if } d=1. \end{array}$$

In order to write the water waves equations (1.3) in nondimensionalized form let us first focus our attention on the Dirichlet-Neumann operator. One gets readily from (1.2) that

$$\begin{aligned} G[\zeta,b]\psi &= (\partial_z\Phi)_{|_{z=\zeta}} - (\nabla\Phi)_{|_{z=\zeta}}\cdot\nabla\zeta \\ &= \frac{\Phi_0}{H_0}\big(\partial_{z'}\Phi' - \mu\nabla'^\gamma(\varepsilon\zeta')\cdot\nabla'^\gamma\Phi'\big)_{|_{z'=\varepsilon\zeta'}}, \end{aligned}$$

and therefore

$$G[\zeta,b]\psi = \frac{\Phi_0}{H_0}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta',\beta b']\psi', \tag{1.27}$$

[6]Roughly defined as the time interval between two consecutive "crests".

where

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta',\beta b']\psi' = \sqrt{1+|\nabla'(\varepsilon\zeta')|^2}\partial'_{\mathbf{n}}\Phi'_{|_{z'=\varepsilon\zeta'}}$$

and where Φ' solves the nondimensionalized version of (1.1), namely,

$$\begin{cases} \Delta'^{\mu,\gamma}\Phi' = 0 & \text{for} \quad -1+\beta b' \le z' \le \varepsilon\zeta', \\ \Phi'_{|_{z'=\varepsilon\zeta'}} = \psi', & \partial'_{\mathbf{n}}\Phi'_{|_{z'=-1+\beta b'}} = 0; \end{cases}$$

here, $\partial'_{\mathbf{n}}\Phi'_{|_{z'=-1+\beta b'}} = \big(\partial_{z'}\Phi' - \mu\nabla'^{\gamma}(\beta b')\cdot\nabla'^{\gamma}\Phi'\big)_{|_{z'=-1+\beta b'}}$, according to the notation below.

Notation 1.12. As always in these notes, $\partial_{\mathbf{n}}$ refers to the upward *conormal* derivative associated to an elliptic operator $\nabla_{X,z}\cdot P\nabla_{X,z}$ (where P is a $(d+1)\times(d+1)$ coercive matrix). It is defined as

$$\partial_{\mathbf{n}} = \mathbf{n}\cdot P\nabla_{X,z},$$

where $\mathbf{n}$ is the upward normal vector. When $P = I$, the elliptic operator is the standard Laplace operator and the conormal derivative coincides with the normal derivative.

Remark 1.13. As noted in §1.3.3, the choice of a scaling length for the vertical variable is purely technical. Let us for instance consider what happens if the vertical variable z is scaled by L_x (which is usually done in deep or infinite depth), i.e.,

$$z' = \frac{z}{L_x}.$$

Instead of (1.27), one would then obtain

$$G[\zeta,b]\psi = \frac{\Phi_0}{L_x}\tilde{\mathcal{G}}_{\mu,\gamma}[\epsilon\zeta',\beta\sqrt{\mu}b']\psi',$$

where

$$\tilde{\mathcal{G}}_{\mu,\gamma}[\epsilon\zeta',\beta\sqrt{\mu}b']\psi' = \sqrt{1+|\nabla'(\epsilon\zeta')|^2}\partial'_{\mathbf{n}}\Phi'_{|_{z'=\epsilon\zeta'}} = \big(\partial'_z - \epsilon\nabla'^{\gamma}\zeta'\cdot\nabla'^{\gamma}\big)\Phi'_{|_{z'=\epsilon\zeta'}}$$

and where Φ' solves the nondimensionalized version of (1.1), namely,

$$\begin{cases} \Delta^{\gamma}_{X',z'}\Phi' = 0 & \text{for} \quad -\sqrt{\mu}+\beta\sqrt{\mu}b' \le z' \le \epsilon\zeta', \\ \Phi'_{|_{z'=\epsilon\zeta'}} = \psi', & \partial'_{\mathbf{n}}\Phi'_{|_{z'=-\sqrt{\mu}+\beta\sqrt{\mu}b'}} = 0, \end{cases}$$

where $\Delta^{\gamma}_{X',z'} = \partial_z^2 + \gamma^2\partial_y^2 + \partial_z^2$ (if $d = 2$). Because of the change in the scaling of z, the dependence on μ no longer appears in the Laplace operator, but in the bottom parametrization. Both formulations are of course completely equivalent in the sense that they give the same expression for $G[\zeta,b]\psi$. While the first scaling is more convenient to study shallow water regimes, the second scaling is the relevant one to handle the case of infinite depth.

Using (1.27) together with the nondimensionalizations described in §1.3.3, it is easy to deduce the dimensionless version of the water waves equations[7] (1.3), namely (omitting the primes for the sake of clarity),

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2\nu}|\nabla^{\gamma}\psi|^2 - \dfrac{\varepsilon\mu}{\nu}\dfrac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta,\beta b]\psi + \nabla^{\gamma}(\varepsilon\zeta)\cdot\nabla^{\gamma}\psi)^2}{2(1+\varepsilon^2\mu|\nabla^{\gamma}\zeta|^2)} = 0, \end{cases} \tag{1.28}$$

[7]See also (4.65) for the case of infinite depth.

where

$$
\begin{aligned}
\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta,\beta b]\psi &= \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi_{|_{z=\varepsilon\zeta}} \\
&= -\mu\nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\Phi_{|_{z=\varepsilon\zeta}} + \partial_z\Phi_{|_{z=\varepsilon\zeta}},
\end{aligned}
\tag{1.29}
$$

and with

$$
\begin{cases}
\Delta^{\mu,\gamma}\Phi = 0, & -1+\beta b \leq z \leq \varepsilon\zeta, \\
\Phi_{|_{z=\varepsilon\zeta}} = \psi, & \partial_{\mathbf{n}}\Phi_{|_{z=-1+\beta b}} = 0.
\end{cases}
\tag{1.30}
$$

(Recall that ∇^γ and $\Delta^{\mu,\gamma}$ are defined in Notation 1.11 and that $\partial_{\mathbf{n}}$ stands for the upwards conormal derivative. See Notation 1.12.)

REMARK 1.14. It is of course possible to perform the nondimensionalization directly on the Euler equations (H1)′–(H9)′. For instance, (H1)′ reads in dimensionless form,

$$
\begin{cases}
\partial_t V + \frac{\varepsilon}{\nu}(V\cdot\nabla^\gamma + \frac{1}{\mu}w\partial_z)V = -\frac{1}{\varepsilon}\nabla^\gamma P, \\
\partial_t w + \frac{\varepsilon}{\nu}(V\cdot\nabla^\gamma + \frac{1}{\mu}w\partial_z)w = -\frac{1}{\varepsilon}(\partial_z P + 1),
\end{cases}
$$

where, for the sake of clarity, we have split the equation into its horizontal and vertical components, and where $P_0 = \rho g H_0$ (the typical order of the hydrostatic pressure) has been used to nondimensionalize the pressure, while V and w have been naturally[8] scaled with $V_0 = \Phi_0/L$ and $w = \Phi_0/H$.
Similarly, the dimensionless version of (H6)′ is given by

$$
\partial_t\zeta + \frac{\varepsilon}{\nu}\underline{V}\cdot\nabla^\gamma\zeta - \frac{1}{\mu\nu}\underline{w} = 0,
$$

where $\underline{\mathbf{U}} = (\underline{V},\underline{w})$ stands for the velocity evaluated at the surface $\underline{\mathbf{U}} = \mathbf{U}_{|_{z=\varepsilon\zeta}}$.

1.4. Plane waves, waves packets, and modulation equations

The goal of this section is to give a brief and formal introduction to the study of wave packets and modulation equations that will be addressed in Chapter 8.

One easily deduces from the analysis performed in §1.3.2 that the linear water waves equations (1.18) possess *plane waves* solution of the form

$$
\begin{pmatrix} \zeta(t,X) \\ \psi(t,X) \end{pmatrix} = \begin{pmatrix} \zeta_{01} \\ \psi_{01} \end{pmatrix} e^{i(\mathbf{K}\cdot X - \Omega t)} + \text{c.c.}
$$

[8] This nondimensionalization comes from the fact that the velocity can be expressed as the gradient of a potential, $V = \nabla^\gamma\Phi$, $w = \partial_z\Phi$. If one is not interested in working with the potential—for instance, to handle the irrotational case—it is possible to use a different nondimensionalization. Typically, in the shallow water regime ($\mu \ll 1$, and we thus set $\nu = 1$), we know that with the above nondimensionalization, one has $w' = O(\mu)$ (see §5.6.2.3). We can therefore choose to scale the vertical velocity by $\mu w_0 = \frac{a}{L_x}\sqrt{gH}$, so that $w' = O(1)$. The equations are then (with $\nu = 1$)

$$
\begin{cases}
\partial_t V + \varepsilon(V\cdot\nabla^\gamma + w\partial_z)V = -\frac{1}{\varepsilon}\nabla^\gamma P, \\
\partial_t w + \varepsilon(V\cdot\nabla^\gamma + w\partial_z)w = -\frac{1}{\varepsilon\mu}(\partial_z P + 1).
\end{cases}
$$

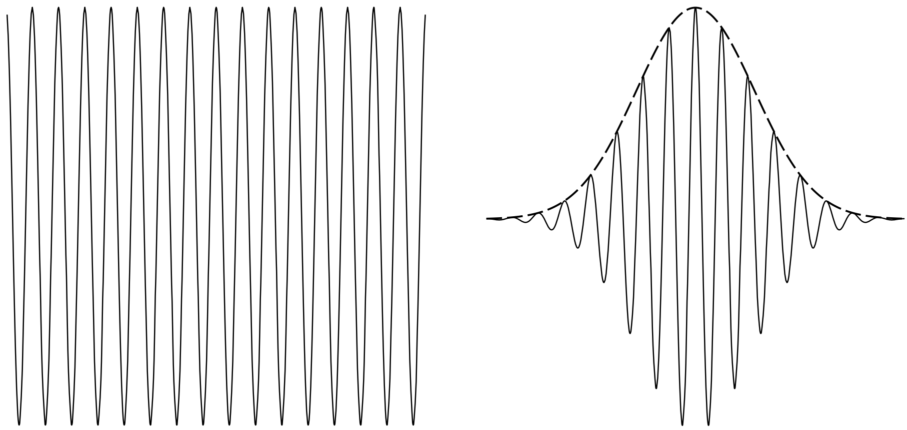

FIGURE 1.6. A plane wave and a wave packet (in dashed line, the envelope of the wave packet).

(c.c. stands for "complex conjugate"), that is, oscillations of constant amplitude ζ_{01} and ψ_{01}, wave number $\mathbf{K}$ and pulsation Ω, provided that

$$\Omega = \omega(\mathbf{K}) \quad \text{and} \quad \zeta_{01} = i\frac{\omega(\mathbf{K})}{g}\psi_{01},$$

where $\omega(\cdot)$ is as in (1.22).
Oscillating waves observed at the surface of an ocean are of course not of infinite extent, and their amplitude is slowly modulated (i.e., it varies with a typical scale much larger than the period of the oscillations). These slowly modulated oscillations are called *wave packets* (see Figure 1.6), and they are of the form

$$\begin{pmatrix} \zeta(t,X) \\ \psi(t,X) \end{pmatrix} = \begin{pmatrix} \zeta_{01}(\epsilon t, \epsilon X) \\ \psi_{01}(\epsilon t, \epsilon X) \end{pmatrix} e^{i(\mathbf{K}\cdot X - \Omega t)} + \text{c.c.},$$

where ϵ is some dimensionless parameter assumed to be small[9]. The difference between wave packets and plane waves is that the amplitudes ζ_{01} and ψ_{01} are not constant anymore but depend on space and time through $X' = \epsilon X$ and $t' = \epsilon t$. (The functions ζ_{01} and ψ_{01} are often called *envelopes* of the oscillations.)

Contrary to what happened with plane waves, one cannot find *exact* solutions of the linear water waves equations under the form of wave packets. We (formally) show below that one can, however, find approximate solutions under the form of wave packets. Let us therefore consider the linear water waves equations (1.18) with initial conditions

$$\begin{pmatrix} \zeta(0,X) \\ \psi(0,X) \end{pmatrix} = \begin{pmatrix} \zeta_{01}^0(\epsilon X) \\ \psi_{01}^0(\epsilon X) \end{pmatrix} e^{i\mathbf{K}\cdot X} + \text{c.c.},$$

and let us assume for simplicity that the Fourier transforms of ζ_{01} and ψ_{01} are compactly supported. From our brief analysis of the plane wave solutions, we naturally impose that

$$\zeta_{01}^0 = i\frac{\omega(\mathbf{K})}{g}\psi_{01}^0.$$

[9]In Chapter 8 where the study of wave packets is addressed, this small parameter is the *steepness* of the wave (see §1.5 below for a definition of this parameter).

Remarking that

$$\mathcal{F}(\zeta_{01}^0(\epsilon X)e^{i\mathbf{K}\cdot X})(\xi) = \frac{1}{\epsilon^d}\widehat{\zeta}_{01}^0(\frac{\xi-\mathbf{K}}{\epsilon})$$

we get from the representation formula (1.21) that

$$\begin{aligned}\zeta(t,X) &= \frac{i}{2}\int_{\mathbb{R}^d} e^{i(\xi\cdot X-\omega(\xi)t)}\frac{\omega(\mathbf{K})+\omega(\xi)}{g}\frac{1}{\epsilon^d}\widehat{\psi}_{01}^0(\frac{\xi-\mathbf{K}}{\epsilon})d\xi\\ &+ \frac{i}{2}\int_{\mathbb{R}^d} e^{i(\xi\cdot X+\omega(\xi)t)}\frac{\omega(\mathbf{K})-\omega(\xi)}{g}\frac{1}{\epsilon^d}\widehat{\psi}_{01}^0(\frac{\xi-\mathbf{K}}{\epsilon})d\xi\end{aligned}$$

and therefore, with a simple change of the integration variable,

$$\begin{aligned}\zeta(t,X) &= e^{i(\mathbf{K}\cdot X-\Omega t)}\frac{i}{2}\int_{\mathbb{R}^d} e^{i(\epsilon\xi\cdot X+(\omega(\mathbf{K})-\omega(\mathbf{K}+\epsilon\xi))t)}\frac{\omega(\mathbf{K})+\omega(\mathbf{K}+\epsilon\xi)}{g}\widehat{\psi}_{01}^0(\xi)d\xi\\ &+ \frac{i}{2}\int_{\mathbb{R}^d} e^{i(\epsilon\xi\cdot X+\omega(\mathbf{K}+\epsilon\xi)t)}\frac{\omega(\mathbf{K})-\omega(\mathbf{K}+\epsilon\xi)}{g}\widehat{\psi}_{01}^0(\xi)d\xi + \text{c.c.},\end{aligned}$$

with $\Omega = \omega(\mathbf{K})$.
Since we assumed that $\widehat{\psi}_{01}$ is compactly supported, a first-order Taylor expansion of $\omega(\mathbf{K}+\epsilon\xi)$ shows that the second term in the above identity is of order $O(\epsilon)$; up to another error term of size $O(\epsilon)$, we can also replace $\omega(\mathbf{K}+\epsilon X)$ by $\Omega = \omega(\mathbf{K})$ in the first term of the right-hand-side. We can therefore write

$$\zeta(t,X) = \zeta_{01}(\epsilon t,\epsilon X)e^{i(\mathbf{K}\cdot X-\Omega t)} + \text{c.c.} + O(\epsilon); \tag{1.31}$$

denoting by $(t',X') = (\epsilon t,\epsilon X)$ the slow time and space variables, and recalling that $\zeta_{01}^0 = i\frac{\Omega}{g}\psi_{01}^0$, the envelope $\zeta_{01}(t',X')$ is explicitly given by

$$\zeta_{01}(t',X') = \int_{\mathbb{R}^d} e^{i(\xi\cdot X' + \frac{\omega(\mathbf{K})-\omega(\mathbf{K}+\epsilon\xi)}{\epsilon}t)}\widehat{\zeta}_{01}^0(\xi)d\xi.$$

Equivalently, ζ_{01} can be defined as the solution of the *linear full-dispersion equation*

$$\partial_t'\zeta_{01} + i\frac{\omega(\mathbf{K}+\epsilon D')-\omega(\mathbf{K})}{\epsilon}\zeta_{01} = 0, \qquad \zeta_{01}\,_{|_{t'=0}} = \zeta_{01}^0, \tag{1.32}$$

with $D' = -i\nabla'$ and where we used Notation 1.7 for Fourier multipliers.

The linear full-dispersion equation (1.32) is a nonlocal equation that might be delicate to handle in some contexts. It is therefore of interest to approximate it by a differential equation. This approximation depends on the time scale of interest:

- Time scale $t = O(1/\epsilon)$ (equivalently, $t' = O(1)$). Up to another error term of size $O(\epsilon)$ in (1.31), one can replace $\omega(\mathbf{K}+\epsilon D')$ by its *first-order* Taylor expansion in (1.32). This leads us to change the definition of ζ_{01} in (1.31) into the solution of the following *transport equation at the group velocity*

$$\partial_t'\zeta_{01} + \nabla\omega(\mathbf{K})\cdot\nabla'\zeta_{01} = 0, \qquad \zeta_{01}\,_{|_{t'=0}} = \zeta_{01}^0. \tag{1.33}$$

- Time scale $t = O(1/\epsilon^2)$ (equivalently, $t' = O(1/\epsilon)$). The approximation error made when replacing (1.32) by (1.33) add up to finally be of order $O(1)$ when $t' = O(1/\epsilon)$. This is why a more accurate approximation of (1.32) is necessary, and $\omega(\mathbf{K}+\epsilon D')$ is therefore replaced by its *second-order* Taylor expansion. Consequently, ζ_{01} in (1.31) is now defined as the solution to the *linear Schrödinger equation*,

$$\partial_t'\zeta_{01} + \nabla\omega(\mathbf{K})\cdot\nabla'\zeta_{01} - \epsilon\frac{i}{2}\nabla'\cdot\mathcal{H}_\omega(\mathbf{K})\nabla'\zeta_{01} = 0, \qquad \zeta_{01}\,_{|_{t'=0}} = \zeta_{01}^0. \tag{1.34}$$

By analogy with optics, we say that the time scale $t' = O(1)$ corresponds to geometric optics for which wave packets travel at the group velocity, while the time scale $t' = O(1/\epsilon)$ corresponds to diffractive optics where dispersive diffractive effects significantly affect the envelope of the wave packet.

The equations (1.32), (1.33), and (1.34) do not directly describe the solution to the (linear) water waves equation, but of the envelope of the wave packet; an approximate solution to the water waves equation is then recovered through (1.31). Such approximations are called *modulation approximations*, and (1.32), (1.33), and (1.34) are *modulation equations.*

This book is mainly focused on the approximation of shallow water models (see §1.5 below) that describe the transformation of waves as they approach the shore, but we nevertheless consider modulation approximations in Chapter 8. Dealing with the full nonlinear water waves equations makes the analysis much more technical than in the above presentation. An important nonlinear effect is the possible creation of a nonoscillating term by quadratic interaction of oscillating wave packets (the equivalent phenomenon in optics is called *optical rectification*). Such nonoscillating terms may interact with the modulation equations governing the evolution of wave packets and therefore there exist many models (such as the nonlinear Schrödinger, the Benney-Roskes, or the Davey-Stewartson equations) that describe the very rich dynamics of wave packets. Most of these models are not fully justified yet.

1.5. Asymptotic regimes

The significance of introducing the dimensionless parameters ε, μ, β and γ in §1.3.3 is that it is often possible to deduce from their values some insight on the behavior of the flow. More precisely, it is possible to derive from (1.30) some (much simpler) asymptotic models more amenable to numerical simulations and whose properties are more transparent. Various asymptotic models, with different properties, can be derived from (1.30); each of them corresponds to a specific *asymptotic regime*, that is, to a specific range of the values of ε, μ, β and γ. Before commenting on these asymptotic regimes, let us briefly discuss the typical values of the dimensionless parameters mentioned earlier.

- The amplitude parameter $\varepsilon = a_{surf}/H_0$. Since the amplitude of the wave is at most of the same order as the depth, it is perfectly reasonable to assume that $0 \leq \varepsilon \leq 1$. (See Figure 1.7 for experimental data near the shore and Figure 1.8 for typical data about the ocean.)
- The topography parameter $\beta = a_{bott}/H_0$. Since the order of the bottom variations does not exceed the typical depth, we also have $0 \leq \beta \leq 1$.
- The shallowness parameter $\mu = H_0^2/L_x^2$. This ratio can be either small or very large. We thus take $\mu \geq 0$.
- The transversality parameter $\gamma = L_x/L_y$. Up to a permutation of the horizontal axis, it is possible to assume that $0 \leq \gamma \leq 1$.
- The steepness of the wave $\epsilon = a_{surf}/L_x = \varepsilon\sqrt{\mu}$. Physical observations show that the steepness ϵ never exceeds a critical value because of the occurrence of wave breaking. We thus assume that $0 \leq \epsilon \leq \epsilon_{max}$ (for some $\epsilon_{max} > 0$).
- The steepness of the bottom variations $\epsilon_{bott} = a_{bott}/L_x = \beta\sqrt{\mu}$. As for the steepness of the surface variations, we impose that $0 \leq \epsilon_{bott} \leq \epsilon_{max}$.

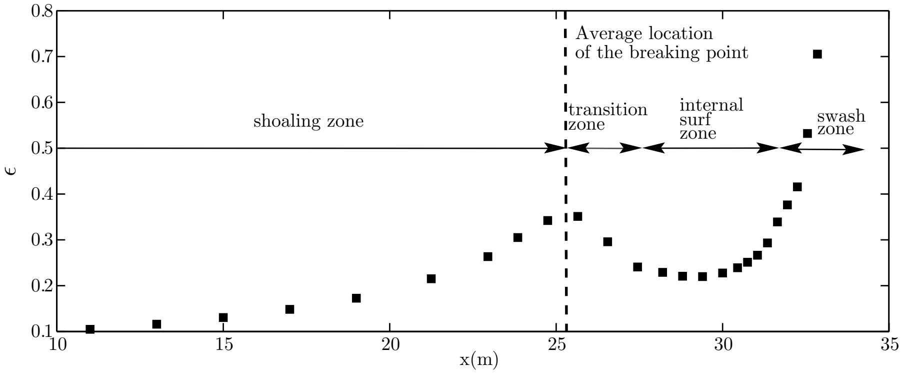

FIGURE 1.7. Variation of the nonlinear parameter ε averaged over several wave groups in terms of the position. Computed by Tissier [**313**] using laboratory experiments by van Dongeren et al. [**127**] on bichromatic waves breaking on a plane beach.

This limitation is essentially due to technical reasons, but is completely harmless for the applications.

To sum up the above discussion, we do not lose much generality if we assume that the parameters ε, μ, β and γ are subject to the following constraints

$$0 \leq \varepsilon, \beta, \gamma \leq 1, \quad 0 \leq \mu, \quad 0 \leq \varepsilon\sqrt{\mu} \leq \epsilon_{max}, \quad 0 \leq \beta\sqrt{\mu} \leq \epsilon_{max}. \tag{1.35}$$

It is possible to work in this framework, as in [**12**]. However, we can simplify the work if we assume further that μ remains bounded (but not necessarily small). This assumption excludes very deep water configurations, but we are mostly concerned here with applications to coastal oceanography where such configurations are not relevant. (For the sake of completeness, several infinite depth models are derived in Chapter 8.) We refer to [**12**] for a general approach that allows for very large values of μ and assume throughout these notes that

$$0 \leq \varepsilon, \beta, \gamma \leq 1, \quad 0 \leq \mu \leq \mu_{max}, \tag{1.36}$$

for some $\mu_{max} > 0$ (not necessarily small). Consequently, we can take $\nu = 1$ in (1.28).

What we call *asymptotic regimes* correspond therefore to some subdomains of the set of all the parameters satisfying (1.36). These asymptotic regimes will be studied in full detail in Chapters 5–8 (see also Appendix C for a reader's digest of all the asymptotic models derived in these notes), but it is worth introducing some terminology at this point:

- Small/large amplitude regimes. It is said that the flow under consideration is in a *small amplitude* regime if $\varepsilon \ll 1$. If no smallness assumption is made on ε (i.e. if $\varepsilon = O(1)$), then the flow is said to be in a *large amplitude* regime.
- Shallow/deep water regime. The *shallow water* regime corresponds to $\mu \ll 1$. If this condition is not true, then we are in *deep water* (the situation $\mu \sim 1$ is sometimes referred to as *intermediate depth*).
- Small/large bottom variations. The bottom variations are of *small amplitude* if $\beta \ll 1$ and large amplitude if $\beta = O(1)$.

Wind Speed (Kts)	Sea State	Significant Wave (Ft)	Significant Range of Periods (Sec)	Average Period (Sec)	Average Length of Waves (FT)
3	0	<.5	<.5 - 1	0.5	1.5
4	0	<.5	.5 - 1	1	2
5	1	0.5	1 - 2.5	1.5	9.5
7	1	1	1 - 3.5	2	13
8	1	1	1 - 4	2	16
9	2	1.5	1.5 - 4	2.5	20
10	2	2	1.5 - 5	3	26
11	2.5	2.5	1.5 - 5.5	3	33
13	2.5	3	2 - 6	3.5	39.5
14	3	3.5	2 - 6.5	3.5	46
15	3	4	2 - 7	4	52.5
16	3.5	4.5	2.5 - 7	4	59
17	3.5	5	2.5 - 7.5	4.5	65.5
18	4	6	2.5 - 8.5	5	79
19	4	7	3 - 9	5	92
20	4	7.5	3 - 9.5	5.5	99
21	5	8	3 - 10	5.5	105
22	5	9	3.5 - 10.5	6	118
23	5	10	3.5 - 11	6	131.5
25	5	12	4 - 12	7	157.5
27	6	14	4 - 13	7.5	184
29	6	16	4.5 - 13.5	8	210
31	6	18	4.5 - 14.5	8.5	236.5
33	6	20	5 - 15.5	9	262.5
37	7	25	5.5 - 17	10	328.5
40	7	30	6 - 19	11	394
43	7	35	6.5 - 21	12	460
46	7	40	7 - 22	12.5	525.5
49	8	45	7.5 - 23	13	591
52	8	50	7.5 - 24	14	655
54	8	55	8 - 25.5	14.5	722.5
57	8	60	8.5 - 26.5	15	788
61	9	70	9 - 28.5	16.5	920
65	9	80	10 - 30.5	17.5	1099
69	9	90	10.5 - 32.5	18.5	1182
73	9	100	11 - 34.5	19.5	1313.5

FIGURE 1.8. The Pierson-Moskowitz sea-state table. Table from Resolute Weather.

- Weakly transverse regime. A flow is said to be *weakly transverse* if $\gamma \ll 1$ (the $1D$ case corresponds formally to $\gamma = 0$).

1.6. Extension to moving bottoms

As discussed in §1.1.1, we consider in most of these notes the motion of a fluid domain delimited below by a fixed bottom and above by a free surface. It is possible to extend the study to the case of a moving bottom. This has been done for instance in [**4**] for the local well-posedness theory and in [**182**] in the shallow water setting. In this case, the bottom parametrization b is now time dependent (but, contrary to the free surface, it is known). Therefore, assumption (H4)′ in §1.1.2 must be replaced by

$$\text{(H4)}' \qquad \forall t \in [0,T), \qquad \Omega_t = \{(X,z) \in \mathbb{R}^{d+1}, -H_0 + b(t,X) < z < \zeta(t,X)\},$$

while the boundary condition at the bottom (H5)′ is now a kinematic condition similar to the one imposed at the surface,

$$\text{(H5)}' \qquad \sqrt{1+|\nabla b|^2}\mathbf{U}\cdot\mathbf{n} = \partial_t b \quad \text{on} \quad \{z = -H_0 + b(t,X)\},$$

or, equivalently, in terms of the velocity potential Φ,

$$(\mathrm{H5})'' \qquad \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi = \partial_t b \quad \text{on} \quad \{z=-H_0+b(X)\}.$$

The first consequence of these modifications is that we recover the velocity potential from its trace at the surface by solving a boundary problem with a *nonhomogeneous* Neumann condition at the bottom. More precisely, (1.1) must be replaced by

$$\left\{\begin{array}{ll} \Delta_{X,z}\Phi = 0 & \text{in } \Omega_t, \\ \Phi_{|_{z=\zeta}} = \psi, & \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{|_{z=-H_0+b}} = \partial_t b, \end{array}\right. \tag{1.37}$$

so that we can decompose Φ into its "fix bottom" and "moving bottom" components,

$$\Phi = \Phi_{fb} + \Phi_{mb}$$

with

$$\left\{\begin{array}{ll} \Delta_{X,z}\Phi_{fb} = 0 & \text{in } \Omega_t, \\ \Phi_{fb\,|_{z=\zeta}} = \psi, & \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{fb\,|_{z=-H_0+b}} = 0, \end{array}\right.$$

and

$$\left\{\begin{array}{ll} \Delta_{X,z}\Phi_{mb} = 0 & \text{in } \Omega_t, \\ \Phi_{mb\,|_{z=\zeta}} = 0, & \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{mb\,|_{z=-H_0+b}} = \partial_t b. \end{array}\right.$$

We therefore have

$$\sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi_{|_{z=\zeta}} = G[\zeta,b]\psi + G^{NN}[\zeta,b]\partial_t b,$$

where the "Neumann-Neumann" operator $G^{NN}[\zeta,b]$ is defined as

$$G^{NN}[\zeta,b] : b_t \mapsto \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi_{mb\,|_{z=\zeta}}.$$

The Zakharov-Craig-Sulem formulation (1.3) can therefore be extended to the case of moving bottoms in the following way,

$$\left\{\begin{array}{l} \partial_t\zeta - G[\zeta,b]\psi = G^{NN}[\zeta,b]\partial_t b, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \dfrac{(G[\zeta,b]\psi + G^{NN}[\zeta,b]\partial_t b + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{array}\right. \tag{1.38}$$

REMARK 1.15. We can generalize the linear analysis of Section 1.3.2 to the case of moving bottoms. With the same kind of Fourier analysis used to compute $G[0,0] = |D|\tanh(H_0|D|)$, we obtain $G^{NN}[0,0] = \frac{1}{\cosh(H_0|D|)}\partial_t b$, so that (1.19) must be replaced by

$$\partial_t^2\zeta + gG[0,0]\zeta = G^{NN}[0,0]\partial_t^2 b. \tag{1.39}$$

We will now give a dimensionless version of (1.38) that generalizes (1.28) to the case of moving bottoms. We need to consider the time scale t_{mb} of the bottom (temporal) variations, which is not necessarily of the same order as the time scale $t_0 = \frac{L_x}{gH_0\nu}$ deduced from the linear wave theory and used in §1.3.3 to get the dimensionless version of the water waves equations. We therefore introduce a new dimensionless parameter δ

$$\delta = \frac{t_{mb}}{t_0} = \frac{t_{mb}}{L_x/gH_0\nu},$$

so that in dimensionless variables, the bottom parametrization is given by $-1+\beta b(t/\delta,X)$, and its time derivative $\dfrac{\beta}{\delta}\partial_\tau b$, with $\tau = t/\delta$. The dimensionless form of the potential equation (1.37) generalizing (1.30) is therefore given by

$$
(1.40)\qquad \begin{cases} \Delta^{\mu,\gamma}\Phi = 0, & -1+\beta b \le z \le \varepsilon\zeta, \\ \Phi_{|_{z=\varepsilon\zeta}} = \psi, & \sqrt{1+\beta^2|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{|_{z=-1+\beta b}} = \mu\nu\dfrac{\beta}{\varepsilon\delta}\partial_\tau b \end{cases}
$$

and the equations (1.38) that generalize (1.28) can be written in the form

$$
(1.41)\qquad \begin{cases} \partial_t\zeta - \dfrac{1}{\mu\nu}\mathcal{G}\psi = \dfrac{\beta}{\delta\varepsilon}\mathcal{G}^{NN}\partial_\tau b, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \dfrac{\varepsilon\mu}{\nu}\dfrac{(\frac{1}{\mu}\mathcal{G}\psi + \frac{\beta\nu}{\delta\varepsilon}\mathcal{G}^{NN}\partial_\tau b + \nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0, \end{cases}
$$

where $\mathcal{G} := \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ is as defined in (1.29)–(1.30) while the dimensionless Neumann-Neumann operator $\mathcal{G}^{NN} := \mathcal{G}^{NN}_{\mu,\gamma}[\varepsilon\zeta, \beta b]$ is given by

$$
(1.42)\qquad \begin{aligned} \mathcal{G}^{NN}_{\mu,\gamma}[\varepsilon\zeta,\beta b]\partial_\tau b &= \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi_{mb\,|_{z=\varepsilon\zeta}} \\ &= -\mu\nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\Phi_{mb\,|_{z=\varepsilon\zeta}} + \partial_z\Phi_{mb\,|_{z=\varepsilon\zeta}}, \end{aligned}
$$

and with

$$
(1.43)\qquad \begin{cases} \Delta^{\mu,\gamma}\Phi_{mb} = 0, & -1+\beta b < z < \varepsilon\zeta, \\ \Phi_{mb\,|_{z=\varepsilon\zeta}} = 0, & \sqrt{1+\beta^2|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{mb\,|_{z=-1+\beta b}} = \partial_\tau b. \end{cases}
$$

(Recall that ∇^γ and $\Delta^{\mu,\gamma}$ are defined in Notation 1.11 and that $\partial_{\mathbf{n}}$ stands for the upwards conormal derivative. See Notation 1.12.)

T. Iguchi studied the system (1.41) in [**182**] with the goal of investigating tsunamis created by submarine earthquakes. He considered a shallow water regime ($\mu \ll 1$, $\nu = 1$) with large surface and bottom deformations ($\varepsilon = \beta = 1$) and fast bottom variations ($\delta \ll 1$). A model frequently used to describe tsunami propagation is the standard Nonlinear Shallow Water system (or Saint-Venant; see Chapter 5) with zero initial velocity and initial surface elevation equal to the permanent shift of the seabed. Iguchi deduced from his study of (1.41) that this is a correct approximation if $\mu = o(\delta)$ but that when $\mu \sim \delta$, the initial velocity field cannot be taken equal to zero.
There are many other asymptotic regimes where (1.41) is of interest; see for instance §5.4 of Chapter 5.

1.7. Extension to rough bottoms

We have already commented in Remark 1.4 that it is implicitly assumed here that we do not consider "rough" bottoms. There can actually be several kinds of rough bottoms, for instance:

(1) The bottom parametrization is not regular (see Figure 1.9).
(2) The bottom parametrization is regular, but its typical scale of variation l_b is much smaller than the typical horizontal length L_x (see Figure 1.10) for the surface wave.

1.7.1. Nonsmooth topographies. Nonsmooth topographies raise two kinds of problems. The first one is the issue of the local well-posedness of the water waves equations (1.3). Indeed the well-posedness theorem provided in Chapter 4 requires a lot of smoothness for the bottom parametrization b. Using the fact that the bottom contribution to the surface evolution is of lower order (it is analytic by

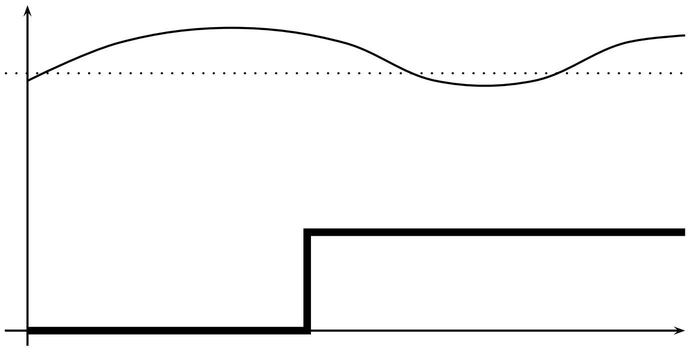

FIGURE 1.9. Rough bottom (1): the bottom parametrization is not smooth.

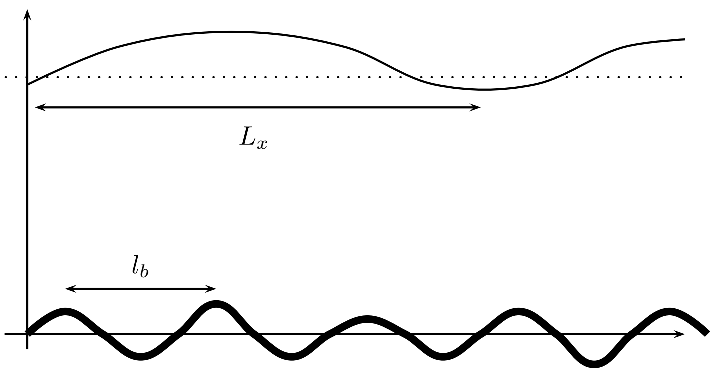

FIGURE 1.10. Rough bottom (2): the typical scale l_b for the bottom variations is much smaller than the typical scale L_x for the surface waves (e.g., distance from crest to crest).

ellipticity; see §A.4 for more comments on this point), Alazard, Burq, and Zuily showed in [**4**] that the water waves equations are locally well posed without any assumption on the bottom regularity. However, the existence time thus obtained shrinks to zero in the shallow water limit $\mu \to 0$ (see §A.4). There is still no result ensuring that the water waves equations are well posed on a time scale compatible with the shallow water limit.

This limitation raises the second issue related to nonsmooth topographies: since no result ensures that the solution to the water waves equations exists on the relevant time scale, no shallow water type model has been rigorously justified when the bottom parametrization is not smooth. For practical purposes, the shallow water models derived in Chapter 5 for smooth topographies are often used with nonsmooth topographies. For instance, the Nonlinear Shallow Water (or Saint-Venant) equations

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot \big((1+\varepsilon\zeta - \beta b)\overline{V}\big) = 0, \\ \partial_t \overline{V} + \nabla^\gamma \zeta + \varepsilon(\overline{V} \cdot \nabla^\gamma)\overline{V} = 0, \end{cases} \tag{1.44}$$

are typically used when the bottom is a step (see Figure 1.9). Here,

$$\overline{V} = \tfrac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla^\gamma \Phi(X,z)dz$$

is the vertically averaged horizontal component of the velocity. While we know from Chapter 6 that the Nonlinear Shallow Water (NSW) equations are fully justified for smooth topographies, there is no reason to believe that such a result holds in the case of a step, for instance. Indeed, while the contribution of the bottom to the surface evolution in (1.44) is singular (it is the divergence of a discontinuous term), we know that it must be infinitely smooth for the full water waves equations. In [**167**] (see also [**67**]), it is argued that in the case of a step, the topography term in (1.44) must indeed be modified if one wants to keep the same $O(\mu)$ precision associated to the NSW model with smooth topographies (see Theorem 6.10). However, the full justification of this new model remains to be done.

1.7.2. Rapidly varying topographies. Rapidly varying topographies do not raise any local well-posedness issues when the bottom parametrization is smooth. The issue is instead about the scale of the existence time. More precisely, if we denote by α the relative size of the bottom variations with respect to the typical horizontal scale L_x,

$$\alpha = \frac{l_b}{L_x},$$

the existence time furnished by standard local existence theorems shrinks to zero as $\alpha \to 0$. Getting an existence time independent of α is an homogenization issue that has not been addressed so far.

In order to fully justify shallow water models with rapidly varying topographies, the existence time should be uniform with respect to α *and* μ, and all the asymptotic models proposed in the literature are, in the best case, only partially justified. The issue is that the shallow water limit $\mu \to 0$ and the homogenization limit $\alpha \to 0$ *do not necessarily commute.*

In their seminal paper [**276**], Rosales and Papanicolaou use standard shallow water models (as those given in Chapter 5) without modifying them when the bottom is rapidly varying ($\alpha \ll 1$). When the bottom variations are periodic, or more generally when they are given by a stationary ergodic process, the techniques of homogenization theory are used to obtain effective shallow water models. The two most important examples are of periodic bottom topography and of topography given by a stationary random process. The approach of [**276**] has been followed in several works, unveiling interesting homogenization mechanisms (such as apparent diffusion). See [**17, 264, 152, 153**].

In the paper by Craig et al. [**105**], another approach is used. The homogenization approximation is performed on the Hamiltonian H given by (1.4), and (Hamiltonian) asymptotic models are deduced from the homogenized Hamiltonian. The periodic case is addressed in [**105**], while random topographies are treated in [**43**].

None of these two approaches brings a complete justification of the homogenized models they propose: The shallow water models used as a starting point in the first approach are *a priori* not valid for rapidly varying topographies, and the homogenization limit performed on the Hamiltonian in the second approach is not fully justified. This observation was the motivation for [**107**], where a shallow water model for rapidly varying periodic bottoms is obtained by handling *simultaneously* the shallow water and homogenization limits ($\mu \to 0$, $\alpha \to 0$). More precisely, the following regime is investigated in [**107**],

$$\beta = \sqrt{\mu} = 1 \ll 1, \tag{1.45}$$

and it is shown that at leading order, the solution solves the Nonlinear Shallow Water equations (1.44) with flat bottom, and that a rapidly oscillating corrector term must be added. It is also pointed out that these corrector terms may grow to destabilize the leading order term when a resonance occurs between the mean (or homogenized) flow and the bottom topography; this resonance can be understood as a nonlinear generalization of the Bragg resonance [**256**]. This approximation is shown to be consistent with the full water waves equations, but the full justification (see §C.1) remains an open problem. Similarly, handling random topography and/or other asymptotic regimes than (1.45) remains to be done.

1.8. Supplementary remarks

1.8.1. Discussion on the basic assumptions. We will first discuss further the relevance of various of the assumptions made in this chapter, namely, (H1)–(H9) in §1.1.1.

- *The fluid is homogeneous and inviscid (H1).*

(i) Homogeneity. The density ρ of seawater is given by an equation of state depending on the pressure P (in bars), the temperature θ (in degree C), and practical salinity S. "The equation of state defined by the Joint Panel on Oceanographic Tables and Standards (UNESCO, 1981) fits available measurements with a standard error of 3.5 ppm for pressure up to 1000 bars, for temperature between freezing and $40C$, and for salinities between 0 and 42" [**160**]. We are mainly interested here in models describing the transformation of the wave at the scale of a beach, for which this equation of state does not exhibit significant variations. A typical value for ρ at the surface is 1025 kg m^{-3}. At larger scales (e.g., the scale of the ocean), density variations of up to 2% are observed. They are mainly due to temperature variations; the region of large temperature gradient is called *thermoclyne* and is generally located at depth smaller than 1500m (a particular case is the *seasonal thermoclyne* observed near the surface in summer and autumn). The corresponding change of density is called *pycnocline*. Density variations are also sometimes due to a large gradient of salinity called *halocline*. Such density variations are responsible for important physical phenomena such as *internal waves*. We refer to [**160**] for a general physical description of these phenomena, to [**170**] for a review on internal waves, and to [**37, 131, 132**] for a mathematical and asymptotic description of these phenomena related to the approach developed in this book for surface waves.

(ii) Inviscidity. Throughout these notes, energy dissipations mechanisms are neglected. In order to take them into account, one should replace the Euler equation (H1)$'$ by the *Navier-Stokes equation*

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z})\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z} P - g\mathbf{e_z} + \frac{v}{\rho}\Delta \mathbf{U} \quad \text{in} \quad \Omega_t, \tag{H1$'_v$}$$

where v is the *viscosity* [10]. One must also replace the boundary condition (H5)$'$ at the bottom by the *no-slip condition*,

$$(\mathrm{H5})'_v \qquad\qquad \mathbf{U} = 0 \quad \text{on} \quad \{z = -H_0 + b(X)\},$$

and the continuity of the pressure at the surface (H7)$'$ by the *continuity of the stress tensor*,

$$(\mathrm{H7})'_v \qquad\qquad (P - P_{atm})\mathbf{n} = 2v(S\mathbf{U})\mathbf{n},$$

where $S\mathbf{U} = \frac{1}{2}(\nabla_{X,z}\mathbf{U} + \nabla_{X,z}\mathbf{U}^T)$ is the symmetric part of the gradient. The dissipation induced by these changes can be roughly decomposed into three categories:

(1) Bottom friction. Tangential motion at the bottom is allowed by the boundary condition (H5)' but not in the viscous case (H5)$'_v$. It is well known that this induces a bottom boundary layer whose size is proportional to $\sqrt{v}$ and whose profile is governed by the Prandtl equation. This raises many difficulties: the Prandtl equation can be ill-posed for nonanalytic data [**154**] and the approximate solution can be unstable [**164, 166**]. This is why the only existing justification of the inviscid limit of the Navier-Stokes equation with a fixed boundary has been done in an analytic framework [**283**]. Handling the bottom boundary layer in the context of water waves therefore remains a widely open problem. If the dissipation effects due to this boundary layer are not significant for large depth (because the fluid is essentially at rest in deep water), they can be important in some situations in a shallow water regime (for which significant horizontal motion exists[11]). Coastal engineers usually model these effects by adding a friction term to the asymptotic models derived in the inviscid case (see §5.6.6). Of course, a complete mathematical justification of this friction term remains at the moment out of reach.

(2) Internal dissipation. This corresponds to the energy dissipated by viscous stresses in the fluid. In shallow water, it is much smaller than the energy dissipated by bottom friction. In deep water, it is the main cause of energy dissipation, but its effects can be considered as negligible. It is indeed shown by qualitative arguments in [**235**] that "the time needed for the energy in waves of length 1m and 10m to be reduced by a factor of e is 8000 periods and 250000 periods respectively".

(3) Surface dissipation. The presence of viscosity also induces a boundary layer near the surface. However, and contrary to what happens at the bottom, the boundary conditions on the velocity are not of Dirichlet type. The boundary conditions (H6)$'$–(H7)$'_v$ are of Navier type and the amplitude of the surface boundary layer profile is therefore smaller than for the bottom; in particular, it is natural to expect a uniform (with respect to the viscosity) control of the velocity in Lipschitz norm near the surface [**251**]. This fact has recently been used in [**252**] to justify the inviscid limit of the free surface Navier-Stokes equations towards the water waves equation in infinite depth. For practical applications, surface dissipation is almost

[10] The quantity v/ρ is often called *kinematic viscosity* (roughly 10^{-6} m^2 s^{-1} for water).

[11] Approximate particle trajectories can be deduced from the expression of the velocity field provided in the Appendix to Chapter 5. We refer to [**91**] and references therein for more refined considerations on particle trajectories.

always neglected (except in certain situations such as water covered with a thin film or surface contaminant [**235**]).

Due to the difficulty of handling rigorous viscous effects on water waves, various simple models have been proposed from the early works of Lamb [**214**] and Boussinesq [**50**]. For a recent reference, see [**122**].
- *Incompressibility assumption (H2).* The sound speed in water is approximatively $c_s = 1400$ m s^{-1}. Since the typical wave celerity is given by $c_0 = \sqrt{gH_0\nu}$, with $\nu \le 1$, we have

$$\frac{c_0}{c_s} \le \frac{\sqrt{gH_0}}{1400} \sim 2.2\,10^{-3}\sqrt{H_0}.$$

We therefore have $c_0 \ll c_s$ (and therefore incompressibility) provided that $H_0 \ll (2.2\,10^{-3})^{-2}$m $\sim$ 200km, which is obviously satisfied on Earth.

- *Irrotationality assumption (H3).* Irrotationality allows one to reduce the problem to equations cast on the interface. If this assumption is not made, an equation on the whole fluid domain must be kept for the vorticity. This makes the construction of an iterative scheme for the construction of solutions more delicate. Among the approaches described in §1.2 the variational approach of Shatah-Zeng [**297**] (see §1.2.3.1), and the Lagrangian approaches of Lindblad [**237**] and Coutand-Shkoller [**99**] (see §1.2.4) do not require the irrotationality assumption (H3). Moreover, Wu's formulation (see §1.2.1.2) has been extended to the rotational case in [**337, 338**], and Zakharov's Hamiltonian formulation (1.3) of the water waves equations can be generalized when $d = 1$ to the case of nonzero, constant vorticity (see also [**244, 90**] for variants in Lagrangian coordinates). When studying steady water waves with vorticity in horizontal dimension $d = 1$, a stream-function formulation or the related Dubreil-Jacotin formulation can be very useful (see for instance [**91**]).
For the applications we have in mind here (coastal oceanography), rotational effects are generally not very important and we therefore consider the irrotational case only.

- *Nonoverhanging waves (and bottoms) assumption (H4).* Our intention here is to describe the wave transformation as it approaches the shore, and it is therefore natural to assume that it is parametrized by a graph (when it ceases to be a graph, the wave enters the breaking zone that is not described here). However, it would be possible to describe the wave through a parametrized curve ($d = 1$) or surface ($d = 2$), thus allowing overhanging of waves; see [**326, 327, 14, 15, 97**]. In particular, all the methods described in §1.2 do not require the surface to be a graph. This is necessary, for instance, to study the formation of "splash" singularities as in [**65, 66, 100**] (see Figure 1.11).

The assumption that the bottom is a graph is not necessary either. In fact, it can be shown that the bottom only contributes to lower order to the dynamics of the surface and does not affect the well-posedness theory. In [**4**], the authors show that it is possible to take very wild bottoms (as far as the depth does not vanish). However, in the shallow water regime, the contribution of the bottom is very important (even if it remains of lower order in terms of regularity), and it is necessary to make stronger assumptions on the bottom in order to describe correctly the dynamics of the waves in this very important (for the applications) regime. The assumption that the bottom is a graph is standard in coastal oceanography [**125**].

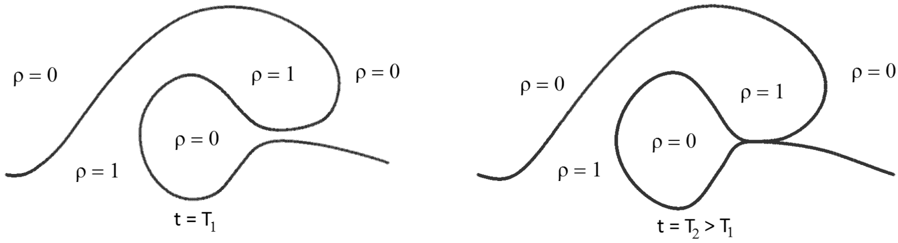

FIGURE 1.11. A splash singularity at time T_2: the interface between water ($\rho = 1$) and vacuum ($\rho = 0$) remains smooth but intersects itself.

- *No surface tension and constant external pressure (H7).* As discussed in §1.1.1, surface tension is in general irrelevant in coastal oceanography and we do not take it into account in the central part of these notes. For the sake of completeness, and because surface tension is relevant in other situations (such as ripples), we show in Chapter 9 how to take it into account, both at the level of the well-posedness of the water waves equations and for their description by asymptotic models.
In order to deal with a nonconstant external pressure, we could replace (H7)′ by

$$(\mathrm{H7})'_{ext} \qquad P = P_{atm} + P_{ext}(t, X) \quad \text{on} \quad \{z = \zeta(t, X)\},$$

for some nonconstant external pressure $P_{ext}(t, X)$; the equations (1.3) should then be replaced by

$$(1.46) \qquad \begin{cases} \partial_t\zeta - G[\zeta, b]\psi = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(G[\zeta, b]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = -\frac{1}{\rho}P_{ext}. \end{cases}$$

- *The fluid is at rest at infinity (H8).* This assumption is very natural but could be removed without major difficulty (for wave-current interactions, see for instance [**91**]).

- *Nonvanishing water depth (H9).* This is a serious limitation for applications, but it remains a completely open mathematical problem. Many of the shallow water models derived in this book (and in particular the nonlinear shallow water equations and the Green-Naghdi equations) are used for applications up to the shoreline where the water depth vanishes (see for instance [**41**] and the review [**40**]). The use of these asymptotic models in such configurations is of course far from being rigorously justified.

1.8.2. Related frameworks. We are interested here in applications to coastal oceanography where the typical scale of interest is the size of the wave or the size of the beach. Shallow water models are also derived in oceanography for much larger scales for which *rotation effects* should be included. If we identify the vertical axis with the direction of gravity, we must consider that the equations are written in a rotating frame, and therefore add a Coriolis force. In full generality (i.e., including viscous effects), the Euler equation (H1)′ should then be replaced by

$$(\mathrm{H1})'_{v,\Omega} \qquad \partial_t\mathbf{U} + (\mathbf{U}\cdot\nabla_{X,z})\mathbf{U} + 2\boldsymbol{\Omega}\times\mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}P - \nabla_{X,z}\Phi_\Omega - g\mathbf{e_z} + \frac{v}{\rho}\Delta\mathbf{U},$$

where $\mathbf{\Omega}$ is the rotation vector of the Earth (assumed to be constant for the scales under consideration) and $\Phi_\Omega = -\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$ ($\mathbf{r}$ being the position vector) is the centrifugal potential. This latter term does not modify the analysis since it can be included in the pressure term,[12] and it is the Coriolis term $2\mathbf{\Omega} \times \mathbf{U}$ that induces the effects of rotation on the fluid motion.

- *Effects of the rotation.* The effects of the Earth's rotation can be estimated by comparing the typical frequency of the phenomenon under study to the Coriolis frequency (also called the Coriolis parameter) $f_{cor} := 2\Omega \sin\phi$, where $\Omega = |\mathbf{\Omega}| = 7.3\,10^{-5}\mathrm{s}^{-1}$ is the Earth's angular velocity, and ϕ is the latitude.
  - At mid-latitudes, one has $f_{cor} \sim 10^{-4}$ Hz.
  - The typical frequency of the wave motion under consideration is given by $f = V_0/L_x$, where V_0 is the typical horizontal velocity.

In order to compare these two quantities, it is convenient to introduce the *Rossby number* defined as

$$\mathrm{Ro} = \frac{f}{f_{cor}} = \frac{V_0}{f_{cor}L_x}.$$

Coriolis effects will have a significant role if $\mathrm{Ro} < 1$, that is, if $L_x > \frac{V_0}{f_{cor}}$. For a typical[13] horizontal velocity $V_0 \sim 1$m/s, the horizontal scale L_x must therefore be larger than 10km in order to notice Coriolis effects. For slower motions $V_0 \sim 0.1$m/s, Coriolis effects become relevant at a shorter scale $L_x > 1$km. These scales are much larger than the typical scale of a beach; this is the reason why we neglected Coriolis effects in this book.

- *Eddy viscosity.* The same reasoning used as in §1.8.1 can be applied to the effect of the viscosity. However, at large scales, an artificial viscosity, called *eddy viscosity*, is often introduced to model the dissipation caused by various phenomena such as white caps, local wave breaking, small-scale dissipation mechanisms, etc. Since these dissipation processes are much more important, this artificial eddy viscosity is anisotropic. Typically, vertical eddy viscosity is 10^3 the molecular value, while horizontal ones can be 10^{10} or 10^{11} times this value [**160**]. Introducing eddy viscosity is of course only a rough approximation of the dissipation processes involved at large scales, and much effort is made in oceanography to improve the modeling of these phenomena.

The large-scale models obtained after taking the above effects into consideration share many common points with the shallow water models derived in this book (see for instance [**155, 249, 54, 250**]). The pioneering work of Laplace[14] on tides [**226**],

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[12]We should consequently modify the boundary condition $(\mathrm{H7})'$ on the pressure. However, the centrifugal potential is generally considered as constant at the surface and since the pressure is defined up to a constant, this does not affect the analysis.

[13]As previously seen in Remark 1.14, a typical scale for the horizontal velocity is $V_0 = \Phi_0/L_x = \frac{a_{surf}}{H_0}\sqrt{\frac{gH_0}{\nu}}$ and therefore $V_0 \sim \frac{a_{surf}}{H_0}\sqrt{gH_0}$ in shallow water and $V_0 \sim \frac{a_{surf}}{L_x}\sqrt{gL_x}$ in deep water. For a wave of amplitude $1m$, with a depth $H_0 \sim 10m$, and of wave length $L_x \gg 1$, therefore one has $V_0 \sim 1$m/s.

[14]In this work, Laplace not only derived the Coriolis force due to the Earth's rotation more than fifty years before Coriolis; he also took into account the Earth's curvature, an effect that should be added to $(\mathrm{H1})'_{v,\Omega}$ at very large scales. For such scales, the variations of the rotation vector $\mathbf{\Omega}$ cannot be neglected as in $(\mathrm{H1})'_{v,\Omega}$ and this has important consequences, such as, for instance, the specific dynamics observed in tropical zones (see for instance [**207, 149, 150**]).

has motivated many mathematicians to study the effects of the Earth's rotation. (See [**72, 137, 149, 150, 238, 247, 312**]).

Shallow water models have also been derived in other contexts, such as the motion of a thin layer of viscous down an inclined plane, avalanches, etc. (see for instance [**318, 57, 43, 44, 45, 51**] and the references in the handbook by D. Bresch [**54**]).

APPENDIX C

Asymptotic Models: A Reader's Digest

We present here a brief review of the various asymptotic models derived in these notes. We refer to Figure C.1 for a graphical presentation of the shallow water models and to the following comments for a reminder of the elements that can be found in these notes concerning the range of validity, the derivation, and the justification of these asymptotic models.

C.1. What is a *fully justified* asymptotic model?

Let us first recall that the dimensionless parameters ε, μ, β and γ (respectively called nonlinearity, shallowness, topography, and transversality parameters) are defined in (1.17). The physical configurations considered in most of these notes range from shallow to moderately deep water, which amounts to assuming that

$$0 < \mu < \mu_{max},$$

for some $\mu_{max} > 0$ not necessarily small, but finite. The dimensionless water waves equations are then given by

$$\text{(C.1)} \qquad \begin{cases} \partial_t \zeta - \dfrac{1}{\mu}\mathcal{G}\psi = 0, \\ \partial_t \psi + \zeta + \dfrac{\varepsilon}{2}|\nabla^\gamma \psi|^2 - \dfrac{\varepsilon}{\mu}\dfrac{(\mathcal{G}\psi + \varepsilon\mu\nabla^\gamma\zeta\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0. \end{cases}$$

See also (1.28) for a more general nondimensionlazation allowing very deep water configurations, and (4.65) in infinite depth.

Further assumptions on the size of ε, μ, β and γ correspond to asymptotic regimes, to which one can associate one or several asymptotic models. Given a particular asymptotic regime, the main issue regarding the validity of an asymptotic model is then the following:

(1) Do the solutions of the water waves equations exist on the relevant time scale?
(2) Do the solutions of the asymptotic model exist on the same time scale?
(3) Are these solutions close to the solution of the water waves equations with corresponding initial data? And how close?

Full justification. When the answer to these three questions is positive, we say that the asymptotic model is *fully justified*. Theorem 4.16 provides an existence theorem[1] for the water waves equations (C.1) that basically always brings a positive answer to the first question. Answering the second question requires a specific

[1] See Theorem 4.36 for infinite depth and Theorem 9.6 in the presence of surface tension.

analysis[2] of the asymptotic model under consideration. It would have required a lot of place and energy to perform this analysis for all the models derived here, and we therefore focused our attention on a few models that cover the most important physical configurations. It is also this analysis of the asymptotic model that allows us to give a positive answer to the third question and therefore to give a full justification of these models. **Almost full justification**. For the asymptotic models for which we did *not* provide local existence and stability results, we cannot provide a full justification. However, the strategy proposed here is quite robust and a full justification would follow directly from the proof of such well-posedness results (which are sometimes straightforward and sometimes more involved). Indeed, for these models, our analysis allows us to say that solutions to these models, if they exist, provide good approximations to the water waves equations[3]. We then say that these models are *almost fully justified* (see §6.2.4 for more details).

Below is a list of the models derived in these notes, with a reminder of their range of validity and their justification status (i.e., fully or almost fully justified).

C.2. Shallow water models

C.2.1. Low precision models. When μ is small but without further assumption on the parameters ε, β and γ, the roughest model, obtained by dropping all $O(\mu)$ terms in (C.1), is the **Nonlinear Shallow Water** (or Saint-Venant) equations (5.7). This model is *fully justified* in Theorem 6.10: for times $t \in [0, \frac{T}{\varepsilon \vee \beta}]$ (with $T > 0$ independent of μ), its precision is $O(\mu t)$. In particular, denoting by (ζ, ψ) the solution to (C.1) and by $(\zeta_{SW}, \overline{V}_{SW})$ the solution to (5.7) with corresponding initial data, one has

$$\forall t \in [0, \frac{t}{\varepsilon \vee \beta}], \qquad \big|(\zeta, \nabla\psi) - (\zeta_{SW}, \overline{V}_{SW})\big|_{L^\infty([0,t]\times\mathbb{R}^d)} \leq C\mu t.$$

C.2.2. High precision models. A better precision is obtained when the $O(\mu)$ terms are kept in the equations and only $O(\mu^2)$ terms are dropped. The relevant time scale is still $t \in [0, \frac{T}{\varepsilon \vee \beta}]$, but the precision of the models is improved into $O(\mu^2 t)$. It is possible to work in full generality (no assumption on ε, β and γ) or to make some extra physical assumptions (e.g., small topography variations $\beta = O(\mu)$). This leads to many different models[4] (we refer to Figure C.1 for a graphical presentation):

♯1 *Large surface and topography variations:* $\varepsilon = O(1)$, $\beta = O(1)$. The **Green-Naghdi** equations (also called Serre or fully nonlinear Boussinesq equations) are the most general (but most complicated) of the models presented here. The

[2]More precisely, a local existence theorem on the relevant time scale, and a stability property with respect to perturbations (see, for instance, Propositions 6.1 and 6.3 for the particular case of the nonlinear shallow water equations).

[3]One has to prove only that the solutions to these models are consistent with other models that are fully justified (e.g., solutions to the Green-Naghdi equations with improved frequency dispersion are consistent at order $O(\mu^2)$ with the standard Green-Naghdi equations, or with the water waves equations themselves (as for the deep water model (8.2)). The result then follows by the stability property already proved for the latter equations (e.g., Proposition 6.5 for the GN equations, Theorem 4.18 for the water waves equations).

[4]These models are different because they are derived under different assumptions, but the relevant time scale and the precision of the approximation is the same for all of them.

standard form of these equations is given in (5.11), or equivalently (5.15). Variants with improved frequency dispersion are given in (5.36) and §5.2.2.2. The *full* justification of the standard Green-Naghdi equations (5.11) is given in Theorem 6.15. Its variants are *almost fully* justified (with the terminology introduced above).

♯2 *Small surface and large topography variations:* $\varepsilon = O(\mu)$, $\beta = O(1)$. To this regime corresponds the **Boussinesq-Peregrine** model (5.23), which requires a small amplitude assumption (namely, $\varepsilon = O(\mu)$) that is not required in ♯1. Variants with improved frequency dispersion are (5.28) and (5.31). All these models are *almost fully* justified.

♯3 *Medium surface and topography variations:* $\varepsilon = O(\sqrt{\mu})$, $\beta = O(\sqrt{\mu})$. In this regime, the Green-Naghdi equations (5.11) can be simplified into the **medium amplitude Green-Naghdi** equations (5.21). This model is *almost fully* justified.

♯4 *Large surface and small topography variations:* $\varepsilon = O(1)$, $\beta = O(\mu)$. Somehow symmetric to the Boussinesq-Peregrine model, the **Green-Naghdi equations with almost flat bottom** (5.18) allow for large amplitude waves, but over small amplitude topography variations. This model is *almost fully* justified[5].

♯5 *Medium surface and small topography variations:* $\varepsilon = O(\sqrt{\mu})$, $\beta = O(\mu)$. The Green-Naghdi equations can then be simplified into (5.22); this model is *almost fully* justified.

♯6 *Small surface and topography variations:* $\varepsilon = O(\mu)$, $\beta = O(\mu)$. This is the well-known long waves regime for which the **Boussinesq** system (5.25) can be derived. Many variants with different frequency dispersion are given in (5.34), and variants with symmetric nonlinearity are also given in (5.40) and (5.41). The Boussinesq systems with symmetric dispersion and nonlinearity are *fully justified* in Theorem 6.20; all the other Boussinesq systems are *almost fully justified.*

C.2.3. Approximation by scalar equations. When the surface elevation is of medium or small amplitude ($\varepsilon = O(\sqrt{\mu})$ or $\varepsilon = O(\mu)$) and for flat bottoms[6] ($\beta = 0$), it is possible in some cases to approximate the water waves equations (C.1) by scalar equations rather than systems. The relevant time scale is $O(1/\varepsilon)$ for all the models described below, but the precision may vary.

♯7 *Medium amplitude wave in* 1*d:* $\varepsilon = O(\sqrt{\mu})$, $d = 1$. For well-prepared initial data, the **Camassa-Holm type equations** (7.47) (velocity based approximation) and (7.51) or (7.53) (surface elevation based approximation) are *fully* justified in Theorems 7.24 and 7.26 respectively. The precision of the approximation is $O(\mu^2 t)$. See also (7.72) for a version of these equations with full dispersion.

♯8 *Small amplitude waves in* 1*d:* $\varepsilon = O(\mu)$, $d = 1$. When $\varepsilon = \mu$ (KdV regime) then the **KdV/BBM equations** (7.7) are *fully* justified and furnish an approximation of precision $O(\mu\sqrt{t})$ in general and $O(\mu)$ under a decay assumption on the initial data (see Corollaries 7.2 and 7.12). Under the weaker assumption $\varepsilon = O(\mu)$, but for well-prepared initial data only, the precision is $O(\mu^2 t)$ (see Corollary 7.37). See also (7.71) for the KdV equation with full dispersion.

[5]In the case of flat bottoms, this model coincides, however, with the standard Green-Naghdi equations (5.11) and is therefore *fully* justified.

[6]We have given in Chapter 7 some references where this assumption is weakened.

♯9 *Small amplitude, weakly transverse 2d waves:* $\varepsilon = \mu$, $d = 2$ *and* $\gamma = \sqrt{\varepsilon}$. The **KP/KP-BBM equations** (7.29) are *fully* justified in Theorem 7.16 under strong assumptions on the initial data and with a $o(1)$ precision only. See also (7.73) for the KP equation with full dispersion.

C.3. Deep water and infinite depth models

Shallow water expansions are irrelevant when μ is not small, but small amplitude expansions are possible. More precisely, asymptotic models can be derived with respect to the *steepness* of the wave

$$\epsilon = \varepsilon\sqrt{\mu} = \frac{a}{L_x},$$

where we recall (see §1.3.1) that a is the typical amplitude and L_x the typical horizontal length of the wave.
With a formal $O(\epsilon^2)$ precision, we derived in this context the so-called **full dispersion model** (8.2) in finite depth (see also (8.4) for nonflat bottoms and (8.10) for infinite depth). This model is *almost fully*[7] justified in Theorem 8.4.

C.4. Modulation equations

See Figure C.2 for a graphical presentation of the comments presented below and devoted to modulation equations in finite and infinite depth.

C.4.1. Modulation equations in finite depth. Modulation equations are generally used in rather deep water. We therefore use the "deep water" nondimensionalization of the water waves equations

$$\text{(C.2)}\qquad \begin{cases} \partial_t\zeta - \dfrac{1}{\sqrt{\mu}}\mathcal{G}\psi = 0, \\ \partial_t\psi + \zeta + \dfrac{\epsilon}{2}|\nabla\psi|^2 - \epsilon\dfrac{(\frac{1}{\sqrt{\mu}}\mathcal{G}\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\epsilon^2|\nabla\zeta|^2)} = 0. \end{cases}$$

The so-called *modulation equations* are evolution equations of slowly modulated *wave packets* of the form

$$\begin{pmatrix} \zeta(t,X) \\ \psi(t,X) \end{pmatrix} = \begin{pmatrix} i\omega\psi_{01}(\epsilon t, \epsilon X) \\ \psi_{01}(\epsilon t, \epsilon X) \end{pmatrix} e^{i(\mathbf{k}\cdot X - \omega t)} + \text{c.c.} + O(\epsilon),$$

with $\omega = \omega(\mathbf{k}) = \big(|\mathbf{k}|\tanh(\sqrt{\mu}|\mathbf{k}|)\big)^{1/2}$.
The evolution of such wave packets on a time scale $t = O(1/\epsilon)$ is governed by a transport equation at the group velocity $c_g = \nabla\omega(\mathbf{k})$. Over the larger time scale $t = O(1/\epsilon^2)$, more complex nonlinear and dispersive effects must be taken into account. The equations below have a formal $O(\epsilon^3)$ precision.

♯10 *Full dispersion and standard Benney-Roskes equations.* These equations couple the evolution of the amplitude ψ_{01} of the leading order term of the wave packet to a nonoscillating mode created by the nonlinearities. A version with full dispersion[8] is proposed in (8.32). The classical version of the Benney-Roskes equation is then recovered by a simple Taylor expansion of the dispersive terms

[7] The obstruction to a full justification is that there is no local well-posedness result for these deep water models.

[8] This means that its linear dispersive properties are the same as those of the water waves equations.

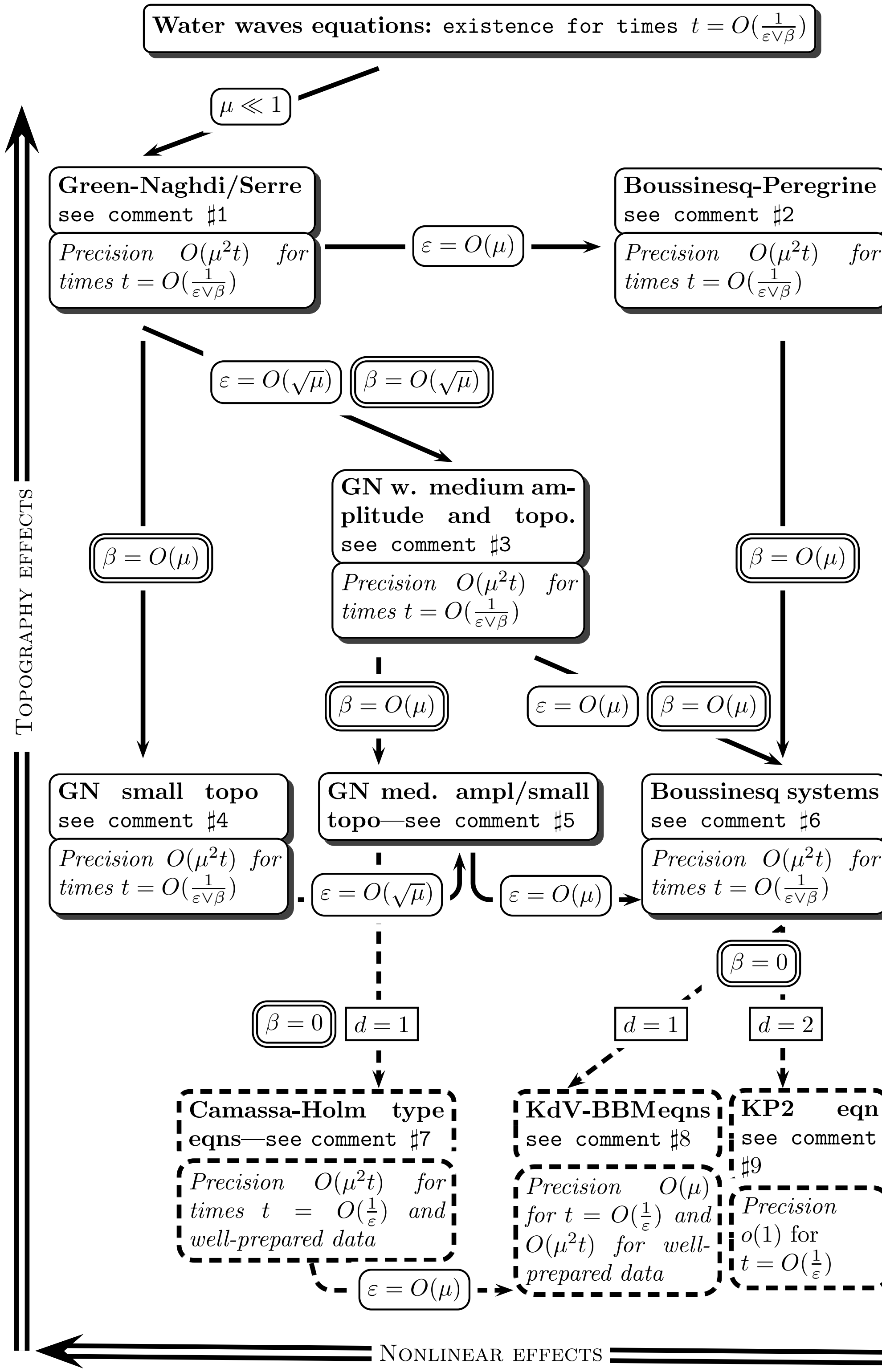

FIGURE C.1. Shallow water models. Approximations by systems correspond to solid boxes; approximations by scalar equations correspond to dashed boxes. See §C.2.2 and §C.2.3 for comments.

in (8.34). Assuming[9] that these equations are well-posed on the relevant time scale, a consistency result is given in Propositions 8.13 and 8.17 respectively.

♯11 *Davey-Stewartson, cubic Schrödinger equations, and variants.* For well-prepared initial data, one can write the Benney-Roskes equations in a frame moving at the group velocity and approximate them by the Davey-Stewartson equations (8.36) in dimension $d = 2$. In dimension $d = 1$, these equations degenerate into a cubic nonlinear Schrödinger equation (8.25). For both models, the mean mode is now slaved to the oscillating mode[10]. Consistency results[11] for the approximations based on these equations are given in Propositions 8.22 and 8.25 respectively. Variants of these equations with full or improved dispersion are given in §8.5.2 and §8.5.3 respectively.

C.4.2. Modulation equations in infinite depth. In infinite depth, the water waves equations are given by (4.65)

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\epsilon\zeta]\psi = 0, \\ \partial_t \psi + \zeta + \dfrac{\epsilon}{2}|\nabla\psi|^2 - \epsilon\dfrac{(\mathcal{G}[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{2(1+\epsilon^2|\nabla\zeta|^2)} = 0. \end{cases}$$

In infinite depth, the creation of a nonoscillating mode by nonlinear interaction of the oscillating modes (rectification) is less important.

♯12 *Nonlinear Schrödinger equations and variants.* Since the nonoscillating modes are of lower order in infinite depth, the Benney-Roskes equation degenerate into the cubic nonlinear Schrödinger equation (8.49) in both dimensions $d = 1$ and $d = 2$. Variants of these equations with full or improved dispersion are given in §8.5.2 and §8.5.3 respectively. The same kind of consistency results as in the finite depth case can be established[12].

♯13 *Higher order model.* A higher order model with formal $O(\epsilon^4)$ precision is provided by the Dysthe equation (8.58).

C.5. Influence of surface tension

C.5.1. On shallow water models. See §9.2, §9.3, and §9.4 for the modifications that need to be made to the shallow water models in the presence of surface tension. These modifications are essentially changes in the third order dispersive terms of the models. Strong qualitative differences are observed in three situations (of limited physical interest):

(1) *Critical Bond number, 1d waves:* $\frac{1}{\mathrm{Bo}} = \mu\frac{1}{\mathrm{bo}}$*, with* $\mathrm{bo} \sim 3$*, and* $d = 1$*.* Dispersive effects disappear from the KdV equation and can be observed only in a less nonlinear regime where a good approximation is provided by the **Kawahara** equation (9.46).

(2) *Critical Bond number, weakly transverse 2d waves:* $\frac{1}{\mathrm{Bo}} = \mu\frac{1}{\mathrm{bo}}$*, with* $\mathrm{bo} \sim 3$*, and* $d = 2$*,* $\gamma = \sqrt{\varepsilon}$*.* The natural extension of the previous situation is provided by the **weakly transverse Kawahara equation** (9.54).

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[9] Such a property is not known. There is a local well-posedness result for (8.34) in [**273**], but not on the relevant time scale.

[10] It is given by an evolution equation coupled to the equation on the oscillating mode in the Benney-Roskes model.

[11] For the NLS equation, a full justification has recently been established in [**134**].

[12] In the one-dimensional case $d = 1$, the NLS approximation has been fully justified in [**315**].

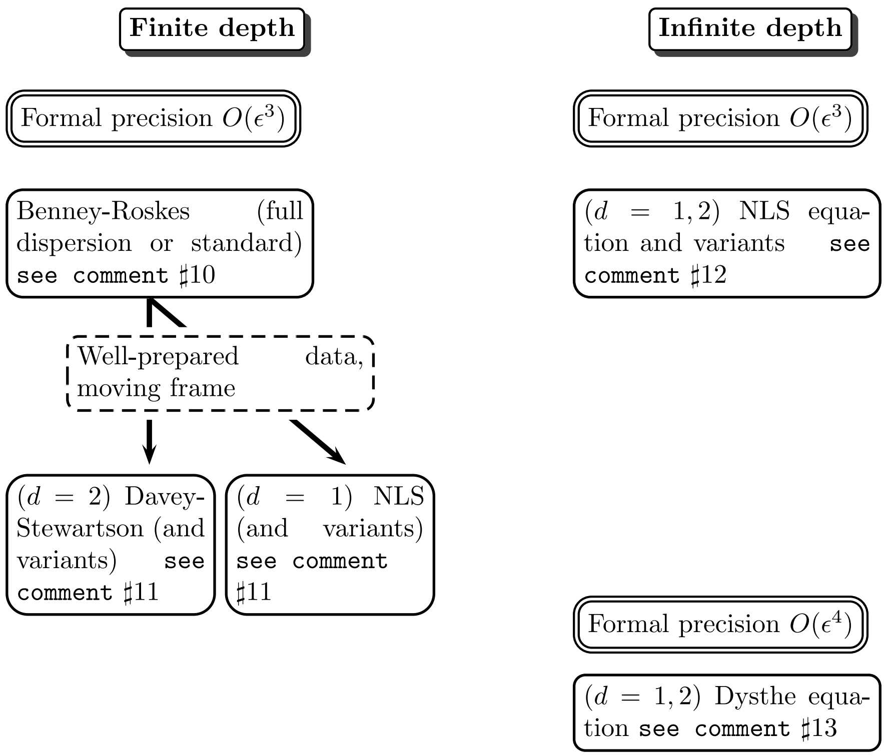

FIGURE C.2. Modulation equations in finite and infinite depth.

(3) *Strong surface tension, weakly transverse* 2*d waves:* bo < 3, $d = 2$, $\gamma = \sqrt{\varepsilon}$. The sign of the third-order dispersive terms in the KP equations (7.29) changes: it is now a **KP1 equation** (as opposed to the KP2 equation [**208**]).

C.5.2. On deep water models and modulation equations. Deep water models with surface tension are derived in §9.4, and capillary effects on modulation equations are mentioned in §8.5.6.