CHAPTER 1

Introduction

Students come into contact with unipotent elements very early in their studies. Indeed, unipotent elements in the general linear group are conjugate to lower triangular matrices with 1's on the diagonal, while nilpotent elements in the matrix algebra are conjugate under the general linear group to lower triangular matrices with 0's on the diagonal. Moreover, the Jordan form determines the conjugacy of such elements. One immediate consequence is that there are only finitely many conjugacy classes of unipotent elements in the general linear group, and precise matrix representatives can be given. An additional piece of information would be to determine the centralizers of the unipotent elements. This is less obvious.

The conjugacy classes of unipotent elements, nilpotent elements, and their centralizers are important for other groups as well. Let K be an algebraically closed field of characteristic p and let G denote a simple algebraic group over K (we allow the possibility that $p = 0$). Then G has a corresponding Dynkin diagram and is either a classical group, with Dynkin diagram of type A_n, B_n, C_n or D_n, or of exceptional type, with Dynkin diagram of type E_6, E_7, E_8, F_4 or G_2. Unipotent elements play a major role in the structure theory of all of these groups. Moreover, each group has an associated Lie algebra where nilpotent elements are of great importance. If $p > 0$, then G admits Frobenius morphisms and the fixed point groups of these morphisms are the finite groups of Lie type. Here too the unipotent classes play a fundamental role.

It is far from a trivial task to determine the conjugacy classes of unipotent elements and nilpotent elements and obtain precise information regarding their centralizers. A satisfactory description of conjugacy classes is fairly smooth when $p = 0$ or a large prime, although even here a precise determination of centralizers requires substantial work. In small characteristics, things become much more complicated. Even the finiteness of the number of unipotent classes is difficult. One might think that at least the classical groups would be straightforward, yet for orthogonal and symplectic groups in characteristic 2, the situation is quite complicated and the results are very different from those in other characteristics. In particular, the number of classes can be dramatically different from what it is in other characteristics.

The goal of this book is to settle these questions completely. Towards that end we provide a new approach to the classification of the conjugacy classes of unipotent elements in G, the nilpotent orbits in $L(G)$ (the Lie algebra of G), and the determination of the structure of their centralizers. We also determine the unipotent classes and centralizers in the corresponding finite groups of Lie type $G(q)$. We give proofs for all types of simple algebraic groups and in all characteristics, largely independent of other results in the literature. The one exception is the paper of

Hesselink [**23**], which we use as a starting point for our analysis of orthogonal and symplectic groups in characteristic 2.

To be sure there is a large literature on this subject spread out over many important papers. The finiteness of the number of unipotent classes in all simple algebraic groups was proved by Lusztig [**40**]. For classical groups, previous work on unipotent classes can be found in papers of Wall [**73**], Springer-Steinberg [**69**], Aschbacher-Seitz [**3**] and Lusztig [**42, 43, 44**]; and for exceptional groups in the work of Dynkin [**16**], Kostant [**30**], Bala and Carter [**5**], Pommerening [**48, 49**], Elashvili [**17**], Alexeevski [**1**], Mizuno [**46, 47**], Shoji [**60**], Shinoda [**59**], Chang [**13**], Stuhler [**72**], Lawther [**31**] and Spaltenstein [**63**]. A summary of some of these results can be found in Chapter 5 of Carter's book [**12**], and Humphreys' book [**27**] is also a good source. Nilpotent orbits are considered also in Hesselink [**23**], Holt and Spaltenstein [**24**], Spaltenstein [**64, 65**] and Premet [**51**], and in Jantzen's article in [**2**].

While some of the information presented in the book can be found elsewhere in the literature, much of our work is new. For example, our results on the precise centralizers of nilpotent elements in bad characteristic are new. Also our material on symplectic and orthogonal groups in characteristic 2 is quite different from what has appeared before. We produce a notion of the "distinguished normal form" of an element (the Jordan form is not sufficient) which is particularly useful for understanding conjugacy classes and centralizers. In addition we obtain information on the reductive parts of centralizers which is new even in characteristic 0; and we obtain close connections between unipotent and nilpotent classes in all characteristics.

We believe that this is a subject of sufficient importance that there should be a single source where the results are established and presented precisely with a consistent notation. For the exceptional groups we present tables giving detailed information on conjugacy classes, centralizers, component groups, and associated labelled Dynkin diagrams. For classical groups we state general theorems on centralizers including the reductive parts and component groups, based on the above-mentioned distinguished normal forms.

The so-called *bad primes* play an important role in this book. These are defined as follows: $p = 2$ is a bad prime for all types except A_n; $p = 3$ is bad for all exceptional types; and $p = 5$ is bad for E_8. In all other cases, including characteristic $p = 0$, we say that p is *good* for G. When p is good there are a number of important results giving general information on representatives and centralizers of nilpotent and unipotent classes. For example, there exist G-equivariant maps (so-called "Springer maps") between the unipotent variety of G and the nilpotent variety of $L(G)$. Information on conjugacy classes is obtained by the results of Bala and Carter [**5**], and the general structure of centralizers by Premet [**51**] (see also the discussion in Section 5 of Jantzen's article in [**2**]). However, even for p good we need to go beyond these results, particularly regarding centralizers, since our aim is to get complete information about the structure and embedding of centralizers of elements.

For classical groups, the analysis when p is good is relatively easy; nevertheless we proceed in a somewhat different manner to other approaches in the literature, and obtain complete information for this case in Chapter 3. When p is bad (that is,

$p = 2$ and G is symplectic or orthogonal) we also obtain complete information on classes and centralizers, but this is much more difficult. Our results on centralizers of nilpotent elements, presented in Chapters 4 and 5 are new, and the results on unipotent elements in Chapter 6 have a somewhat different flavor from what is presented in [**73**]. In Chapter 7, we apply our results to give the classes of unipotent elements and their centralizers in the finite classical groups, and Chapter 8 contains tables illustrating the results for classical groups of low dimension.

Our results for classical groups are an integral tool for the analysis of the exceptional groups, where the situation is much more complicated. Our general approach is first to obtain complete results for the nilpotent elements, where the existence of a certain 1-dimensional torus aids the analysis, and then use the information obtained to derive results for closely related unipotent elements. It is a key feature of our approach that the unipotent analysis is greatly simplified by using results on nilpotent elements. These results are stated in Chapter 9 with the detailed tables appearing in Chapter 22.

If e is a nilpotent element in $L(G)$, then there is a Levi subgroup L of G, such that e lies in the Lie algebra of L', and such that $C_{L'}(e)^0$ is a unipotent group. Such an element e is said to be *distinguished* in the Lie algebra of L'. A similar definition holds for unipotent elements. The first major goal is to construct a sufficient number of distinguished nilpotent elements, even allowing for bad characteristic. This follows the basic philosophy of the Bala-Carter theory, but we do not actually use the results of that theory since we are allowing for bad primes, where the theory does not hold. We produce these distinguished elements in Chapters 13 and 14. The "standard" elements – which exist in all characteristics – are analyzed in Chapter 13, and some further exceptional classes which exist only in characteristics 2 and 3, in Chapter 14.

For a distinguished nilpotent element e in the Lie algebra of L', we produce a co-character of G which has as its image a 1-dimensional torus T contained in L' and acting on $\langle e \rangle$ by weight 2. The existence of such a torus is an important advantage of working first with nilpotent elements. For example, we prove generally that $C_G(e)$ factors as a product of its unipotent radical and $C_G(e) \cap C_G(T)$. Except for certain cases where p is a bad prime, $C_G(e) \cap C_G(T)$ is shown to be the reductive part of $C_G(e)$. For p a good prime we bring into play a pair of explicitly defined reductive groups, J and R. These groups are dual in the sense that each is the G-centralizer of the other. Things are organized so that $e \in L(J)$ and $T < J$. We determine the restriction $L(G) \downarrow JR$ and use this along with additional arguments to show that $R = C_G(e) \cap C_G(T)$. At this point we have the structure of $C_G(e)$ and the precise embedding of R in G, which clarifies the role of the component group of $C_G(e)$.

In the case of exceptional groups, we first settle the case where $G = E_8$, in Chapter 15. The other exceptional groups occur as centralizers of specific subgroups of E_8, and in Chapter 16 we derive information for these groups from the results for E_8. Here is a sketch of our approach for $G = E_8$. We choose a collection of explicitly defined distinguished nilpotent elements in the Lie algebras of Levi subgroups of G, and determine their centralizers as discussed above. To show that our collection is a complete set of class representatives, we use the Lang-Steinberg theorem (see Section 2.4) to determine how the corresponding classes decompose in the finite Lie algebras $L(G)(q)$, and then count to show that the total number of nilpotent elements obtained is precisely the number given by a result of Springer on the number of nilpotent elements (see Lemma 2.16). This gives us our results for nilpotent elements.

In good characteristic the results for unipotent elements follow from those for nilpotent elements, using a Springer map. However this does not apply in characteristics 2, 3 and 5. To cover these cases and also to obtain new information for the good prime case, we produce explicit unipotent elements which are closely related to our chosen nilpotent elements and establish similar, although sometimes different, information on their centralizers. Once again we descend to the finite group to see that all unipotent elements have been accounted for. All this is done in Chapters 17 - 20. Note that the above counting method does not apply in characteristic 0; here we establish a result in Section 2.8 that implies that we have all the classes.

It is worth mentioning that our approach does not assume, a priori, that the number of unipotent or nilpotent classes is finite. This is a consequence of the count, thus providing an alternative proof of the finiteness of the numbers of unipotent and nilpotent classes. As mentioned above, our approach is also independent of the Bala-Carter theory and does not rely on other papers determining unipotent and nilpotent classes. Finally, we mention that while there are many calculations, they are computer-free.

Our results for exceptional groups correct several errors in the literature. Namely, the dimension of the centralizer of the nilpotent class labelled $(A_6)_2$ is different from that given in [**24**], and the component groups of the centralizers of the unipotent elements labelled $E_8(a_2)$ and $(D_4A_2)_2$ are different from those given in the tables of [**47**]. The latter two errors were first noticed by Lusztig and recorded in [**66**, p.329].

As we have already indicated in the above outline, this book is divided into chapters, and the detailed results for the various types of groups will be presented in the appropriate chapter as indicated in the Table of Contents. However, in order to give a flavor of some of the results to follow, we now state two theorems that hold for all types of simple algebraic groups. To avoid complications, we exclude $p = 2$ in the statements. Detailed results for $p = 2$ are presented in the appropriate chapters.

We require a little notation before stating the first result. As mentioned above, for each nilpotent element $e \in L(G)$, we produce an associated 1-dimensional torus T which acts with weight 2 on $\langle e \rangle$. We also produce a unipotent element u corresponding to e. There is a fundamental system of roots for which T acts by a non-negative weight on each of the corresponding fundamental root elements of $L(G)$. In this way T determines a labelling of the Dynkin diagram of G by non-negative integers – each fundamental root is labelled by the corresponding weight. It turns out that the labels are among $0, 1, 2$. In characteristic 0 this is the weighted Dynkin diagram which goes back to Dynkin [**16**]. The torus T also determines a corresponding parabolic subgroup $P = QL$ of G, where $L = C_G(T)$ is the Levi subgroup (corresponding to the zero labels) and the unipotent radical Q is the product of all root subgroups for which the root affords a positive weight of T (see Section 10.1 for a discussion of all this). Let $Q_{\geq 2}$ (respectively, $Q_{>2}$) denote the product of all root groups for which the T-weight is at least 2 (respectively, greater than 2). Our choices then give $u \in Q_{\geq 2}$ and $e \in L(Q_{\geq 2})$.

In the statement of the following theorem, Z_p denotes a cyclic group of prime order p, and for an algebraic group C, $R_u(C)$ denotes the unipotent radical of C – that is, the largest connected unipotent normal subgroup of C. Also $C_G(T, e)$ simply means $C_G(T) \cap C_G(e)$.

THEOREM 1. *Let G be a simple algebraic group over an algebraically closed field of characteristic p. Assume $p \neq 2$ if G is not of type A_n. Then there is a bijective correspondence between the unipotent classes of G and the nilpotent classes of $L(G)$, such that if $u \in G$ and $e \in L(G)$ are representatives of corresponding classes, the following hold, writing $C_e = C_G(e)$, $C_u = C_G(u)$.*

(i) *We have* $\dim R_u(C_u) = \dim R_u(C_e)$.
(ii) *We have $C_u/R_u(C_u) \cong C_e/R_u(C_e)$ or $C_e/R_u(C_e) \times Z_p$. The latter case is only possible if p is a bad prime, and when it does occur the Z_p factor is generated by the image of u.*
(iii) *Let $P = QL$ be the parabolic subgroup of G determined by T, as above. The corresponding labelling has all labels* 0,1 *or* 2. *With the exception of one pair of classes in E_8 and one pair in G_2, both with $p = 3$, we have*
  (a) $C_G(e) \leq P$;
  (b) *e^P is open dense in $L(Q_{\geq 2})$ and $e^Q = e + L(Q_{>2})$; moreover, u can be chosen such that $u \in Q_{\geq 2}$, $C_G(u) \leq P$, and u^P is open dense in $Q_{\geq 2}$ and $u^Q = uQ_{>2}$;*
  (c) *$C_e = R_u(C_e)C_G(T, e)$, a semidirect product, where T is as above and $C_G(T, e)$ is reductive.*

Several remarks are in order regarding the above theorem:

(1) Our results, stated in the appropriate chapters, go far beyond what is stated in Theorem 1. We obtain explicit lists of representatives and the precise structure of their centralizers for all simple algebraic groups in all characteristics.

(2) The assumption that $p \neq 2$ in Theorem 1 is essential for the existence of the stated bijective correspondence. For example, for groups of type B_n, C_n, D_n with $p = 2$ there are many more nilpotent classes than unipotent (the difference tends to infinity as n does). However, for exceptional groups the difference is at most 2, and we provide a natural "almost correspondence" in Chapter 17 (see Theorems 17.3 and 17.4).

(3) The precise cases that occur in the exceptional case of (ii) (i.e. when a Z_p factor occurs) are indicated in Theorem 17.2. This occurs, for example, for the regular unipotent class when G is an exceptional group and p is a bad prime for G.

(4) The two exceptional classes in E_8 in part (iii) of the theorem are the ones labelled A_7 and $(A_7)_3$. And for G_2 they are the classes $\tilde{A}_1$ and $(\tilde{A}_1)_3$. These classes are related in the following way. When $p = 3$, the A_7 class (respectively, the $\tilde{A}_1$ class) of unipotent or nilpotent elements has slightly larger centralizer dimension than is required to satisfy the density condition of (iii) for the corresponding parabolic P. There is a new class, the $(A_7)_3$ class (respectively, $(\tilde{A}_1)_3$), that does satisfy the density condition for P.

The next theorem concerns simple algebraic groups in good characteristic. The result provides information on the reductive parts of centralizers of nilpotent and unipotent elements. The expression in part (ii) for the reductive part of the centralizer is not new. What is new is realizing the reductive part as one member of a pair of reductive groups, each the centralizer of the other. This clarifies the precise structure of the reductive part of the centralizer, its component group, and its embedding in the ambient group.

If G is a simple algebraic group and e is a nilpotent element in $L(G)$, we let T denote a 1-dimensional torus as in the discussion preceding Theorem 1.

THEOREM 2. *Assume that G is a simple adjoint algebraic group in good characteristic. Let e be a nilpotent element of $L(G)$. Then there exist a unipotent element $u \in G$ corresponding to e under a G-equivariant Springer map, and reductive subgroups J, R of G such that the following conditions hold:*

(i) *$u \in J$ and $e \in L(J)$;*
(ii) *$C_G(u) = C_G(e) = UR$, where $U = R_u(C_G(u))$;*
(iii) *$C_G(R) = J$, $C_G(J) = R$ and $N_G(R) = JR$.*

The pair (J, R) satisfying (i)-(iii) *is unique up to conjugacy by elements of $C_G(e) = C_G(u)$, and $R = C_G(T, e)$ is one such choice for R.*

We shall call a pair of subgroups J, R of G satisfying properties (i)-(iii) of the theorem a *dual pair* for the elements e, u.

Finally, we state some corollaries which follow from our results. The first of these was also obtained using a different method by Lusztig [**41**] in response to a question of Serre.

COROLLARY 3. *Let G be a simple algebraic group over an algebraically closed field, and let u be a unipotent element in G. If v is a generator of $\langle u \rangle$, then v is G-conjugate to u.*

COROLLARY 4. *Let G be a simple algebraic group over an algebraically closed field of characteristic p, and let u be a unipotent element in G. If p is good for G, then $u \in C_G(u)^0$; and if p is bad, then $u \in C_G(u)^0$ unless u is in one of the classes in Corollary* 4.3 *(for G classical) or Corollary* 17.8 *(for G exceptional).*

COROLLARY 5. *Let X be a simple algebraic group and $G = X\langle j \rangle$, where j is an involutory graph automorphism if X has type A_n, D_n, or E_6 and $j = 1$ otherwise. Then all elements of X are real in G (i.e. G-conjugate to their inverses).*

Partial results along the lines of Corollary 5 were obtained by Feit and Zuckerman in [**18**]. For $p \neq 2$ it has also been proved by Burness, Guralnick and Saxl.

COROLLARY 6. *Let G be a simple algebraic group of type A_n, $D_n\,(n \neq 4)$ or E_6, and let τ be a graph automorphism of G. Then τ fixes all unipotent classes of G except for the classes in $G = D_n$ consisting of elements which project to a single Jordan block in each factor of a Levi subgroup with derived group $\prod SL_{n_i}$, where all n_i are even and $\sum n_i = n$.*

Note the exclusion of $G = D_4$ in the above result; this is because of the presence of extra graph automorphisms in this case.

The following corollary concerns the parity of the dimension of a centralizer. For unipotent elements or when p is a good prime more precise information has been obtained by Spaltenstein (see Theorem 5.10.2 of [**12**]).

Corollary 7. *Let G be a simple algebraic group of rank r, and let $e \in L(G)$, $u \in G$ be arbitrary nilpotent and unipotent elements. Then*

$$\dim C_G(e) \equiv \dim C_G(u) \equiv r \ \mathit{mod}\ 2.$$

The final corollary concerns double cosets of unipotent element centralizers. It answers a question of Prasad and is used in his paper [**50**]. Additional information can be found in a recent paper of Guralnick, Malle and Tiep [**22**].

Corollary 8. *Let G be a simple algebraic group, and let u be a non-identity unipotent element in G. Then $C_G(u)$ has no dense double coset in G.*

The completions of the proofs of Theorems 1 and 2, and also the deductions of the above corollaries, will be given in Chapter 21.