Introduction

(A) The original motivation for this work came from the study of certain results in ergodic theory, primarily, but not exclusively, obtained over the last several years. These included: i) Work of Hjorth and Foreman-Weiss concerning the complexity of the problem of classification of ergodic measure preserving transformations up to conjugacy, ii) Various results concerning the structure of the outer automorphism group of a countable measure preserving equivalence relation, including, e.g., work of Jones-Schmidt, iii) Ergodic theoretic characterizations of groups with property (T) or the Haagerup Approximation Property, in particular results of Schmidt, Connes-Weiss, Jolissaint, and Glasner-Weiss, iv) Results of Hjorth, Popa, Gaboriau-Popa and Törnquist on the existence of many non-orbit equivalent ergodic actions of certain non-amenable groups, v) Popa's recent work on cocycle superrigidity.

Despite the apparent diversity of the subjects treated in these works, we gradually realized that they can be understood within a rather unified framework. This is the study of the global structure of the space $A(\Gamma, X, \mu)$ of measure preserving actions of a countable group Γ on a standard measure space (X, μ) and the canonical action of the automorphism group $\mathrm{Aut}(X, \mu)$ of (X, μ) by conjugation on $A(\Gamma, X, \mu)$ as well as the study of the global structure of the space of cocycles and certain canonical actions on it. Our goal here is to explore this point of view by presenting (a) earlier results, sometimes in new formulations or with new proofs, (b) new theorems, and finally (c) interesting open questions that are suggested by this approach.

(B) The book is divided into three chapters, the first consisting of Sections 1–9, the second of Sections 10–18, and the third of Sections 19–30. There are also nine appendices.

In the first chapter, we study the automorphism group $\mathrm{Aut}(X, \mu)$ of a standard measure space (i.e., the group of measure preserving automorphisms of (X, μ)) and various subgroups associated with measure preserving equivalence relations. Note that $\mathrm{Aut}(X, \mu)$ can be also identified with the space $A(\mathbb{Z}, X, \mu)$ of measure preserving $\mathbb{Z}$-actions. Sections 1, 2 review some basic facts about the group $\mathrm{Aut}(X, \mu)$. In Section 2 we also show that the class of mild mixing transformations in $\mathrm{Aut}(X, \mu)$ is co-analytic but not Borel (a result also proved independently by Robert Kaufman). This is in contrast with the well-known fact that the ergodic, weak mixing and (strong) mixing transformations are, resp., G_δ, G_δ and $F_{\sigma\delta}$ sets. In Section

3 we discuss the full group $[E] \subseteq \mathrm{Aut}(X, \mu)$ of a measure preserving countable Borel equivalence relation E on (X, μ). In Section 4 we give a detailed proof of Dye's reconstruction theorem, which asserts that the equivalence relation E is determined up to (measure preserving) isomorphism by $[E]$ *as an abstract group*. In Section 5 we give a new method for proving the turbulence property of the conjugacy action of $\mathrm{Aut}(X, \mu)$ on the set ERG of ergodic transformations in $\mathrm{Aut}(X, \mu)$, originally established by Foreman-Weiss, and use this method to prove other turbulence results in the context of full groups. We also extend the work of Hjorth and Foreman-Weiss on non-classification by countable structures of weak mixing transformations in $\mathrm{Aut}(X, \mu)$, up to conjugacy or unitary (spectral) equivalence, to the case of mixing transformations (and obtain more precise information in this case). In Sections 6, 7 we review the basic properties of the automorphism group $N[E]$ of a measure preserving countable Borel equivalence relation E on (X, μ) and its outer automorphism group, $\mathrm{Out}(E)$, and establish (in Section 7) the turbulence of the latter, when E is hyperfinite (and even in more general situations). Understanding when $\mathrm{Out}(E)$ is a Polish group (with respect to the canonical topology discussed in Section 7) is an interesting open problem raised in work of Jones-Schmidt. In Section 8 we establish a connection between the Polishness of $\mathrm{Out}(E)$ and Gaboriau's theory of costs and use it to obtain a partial answer to this problem. In Section 9 we discuss Effros' notion of inner amenability and its relationship with the open problem of Schmidt of whether this property of a group is characterized by the failure of Polishness of the outer automorphism group of some equivalence relation induced by a free, measure preserving, ergodic action of the group. Some known and new results related to this concept and Schmidt's problem are presented here.

In the second chapter, we start (in Section 10) with establishing some basic properties of the space $A(\Gamma, X, \mu)$ of measure preserving actions of a countable group Γ on (X, μ). For finitely generated groups, we also calculate an upper bound for the descriptive complexity of the cost function on this space and use this to show that the generic action realizes the cost of the group. In Section 11 we recall several known ergodic theoretic characterizations of groups with property (T) or the Haagerup Approximation Property (HAP). In Section 12 we study the structure of the space of ergodic actions (and some of its subspaces) in $A(\Gamma, X, \mu)$, recasting in this context characterizations of Glasner-Weiss and Glasner concerning property (T) or the HAP, originally formulated in terms of the structure of extreme points in the space of invariant measures. In Section 13 we study, using the method introduced in Section 5, turbulence of conjugacy in the space of actions and also discuss work of Hjorth and Foreman-Weiss concerning non-classification by countable structures of such actions, up to conjugacy. We also prove an analogous result for unitary (spectral) equivalence. In Section 14, we present the essence of Hjorth's result on the existence of many non-orbit equivalent actions of property (T) groups as a basic property concerning the topological structure of the conjugacy classes of ergodic actions of such groups. We use Hjorth's method to show that for such groups the set of ergodic actions is clopen in the uniform topology and so is each conjugacy class of ergodic actions. In Section 15 we study connectedness properties in the space of actions, using again the method of Section 5. This illustrates the close connection between local connectedness properties and turbulence. We show, in particular, that the space $A(\Gamma, X, \mu)$ is path-connected in the weak topology. We also contrast this to the work in Section 14 to point out the interesting phenomenon that connectedness properties in the space of actions of a group seem to be related to properties of the group, such as amenability or property (T). In particular, for groups Γ with property (T), we determine completely the path components of the space $A(\Gamma, X, \mu)$ in the uniform topology. In Section 16 we discuss results of Popa concerning the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$. These are used in Section 17, along with other ideas, in the proof of a non-classification result of Törnquist for orbit equivalence of actions of non-abelian free groups. We also briefly discuss very recent results of Ioana, Epstein and Epstein-Ioana-Kechris-Tsankov that extend this to arbitrary non-amenable groups. Finally, Section 18 contains a survey of classification problems concerning group actions.

In the third chapter, we give in Section 19 a short introduction to the properties of the group of group-valued random variables and then in Sections 20, 21 we discuss the space of cocycles of a group action or an equivalence relation and some of the invariants associated with such cocycles, like the associated Mackey action and the essential range. We also discuss cocycles arising from reductions and homomorphisms of equivalence relations. Our primary interest is in cocycles with countable (discrete) targets. The next two Sections 22 and 23 contain background material concerning continuous and isometric group actions and Effros' Theorem. The topology of the space of cocycles is discussed in Section 24 and the study of the global properties of the cohomology equivalence relation is the subject of the final Sections 25–30. Section 25 contains some general properties of the cohomology relation, and Section 26 is concerned with the hyperfinite case. There is a large literature here but it is not our main focus in this work. There is a fundamental dichotomy in the structure of the cohomology relation for the cocycles of a given equivalence relation (or action), which in some form is already present in the work of Schmidt for the case of cocycles with abelian targets. In a precise sense, that is explained in these sections, when an equivalence relation E is *non* E_0*-ergodic* (or *not strongly ergodic*), the structure of the cohomology relation on its cocycles is very complicated. This is the subject in Section 27. On the other hand, when the equivalence relation E is E_0*-ergodic* (or *strongly ergodic*), then the cohomology relation is simple, i.e., smooth, provided the target groups satisfy the so-called *minimal condition on centralizers*, discussed in Section 28 (and these include, e.g., the abelian and the linear groups). If the target groups do not satisfy this condition, in certain situations, like, e.g., when E is given by an action of the free group with infinitely many generators, the cohomology relation is not smooth. Thus in the E_0-ergodic case, there is an additional dichotomy having to do with the structure of the target groups. This is the topic of Section 29. Finally in Section 30 we deal with the special case of actions of groups with property (T) and discuss some recent results of Popa on cocycle superrigidity. We also establish in the above sections some new characterizations of amenable and property (T) groups, that, in particular, extend earlier results of Schmidt and also show that there is a positive link between the E_0-ergodicity of an equivalence relation E and the Polishness of its outer automorphism group, $\mathrm{Out}(E)$, an issue raised by Jones-Schmidt. They have pointed out that E_0-ergodicity does not in general imply that $\mathrm{Out}(E)$ is Polish but we show in Section 29 that one has a positive implication when E is induced by a free action of a group with the minimal condition on centralizers.

Appendices A – I present background material concerning Hilbert spaces and tensor products, Gaussian probability spaces and the Wiener chaos decomposition, several relevant aspects of the theory of unitary representations (including unitary representations of abelian groups and induced representations as well as some basic results about the space of unitary representations of a group) and finally semidirect products of groups. We also include, in Appendix E, a detailed proof of the standard result that any unitary representation of a countable group is a subrepresentation of the Koopman representation associated with some measure preserving action of that group.

(C) I would like to thank Miklós Abért, Scot Adams, Oleg Ageev, Sergey Bezuglyi, Andrés Caicedo, Clinton Conley, Damien Gaboriau, Sergey Gefter, Eli Glasner, Greg Hjorth, Adrian Ioana, John Kittrell, Alain Louveau, Ben Miller, Sorin Popa, Dinakar Ramakrishnan, Sergey Sinelshchikov, Simon Thomas, Todor Tsankov and Vladimir Zhuravlev for their comments on this book or useful conversations on matters related to it. Finally, I would like to thank Leona Kershaw for typing the manuscript.

Research for this work was partially supported by NSF Grant DMS-0455285.