CHAPTER 1

Quasi-interpolation

1.1. Introduction

1.1.1. Exercise for a freshman. Suppose we are given the task of drawing the graph of the function

$$f(x) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-m)^2/2}$$

obtained by summation of shifted Gaussians, which are depicted in Fig. 1.1.

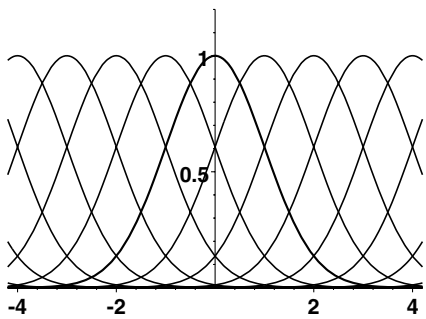

FIGURE 1.1

The function $f(x)$ is, of course, bounded, positive, and smooth. Moreover, $f(x+1) = f(x)$, i.e., it is periodic with period 1. So we expect that the graph of f should look like a nice periodic wavy curve. However, it is quite astonishing to find out that this graph, which can be easily produced with standard plotting software and which is depicted in Fig. 1.2, is a constant.

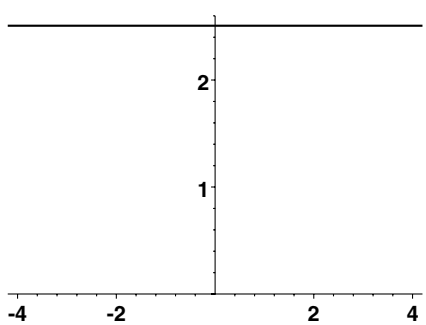

FIGURE 1.2. Graph of $f(x)$

In fact, this superficial impression proves to be wrong. If the scale of the y-axis is changed as in Fig. 1.3, then we see that $f(x)$ is not constant; it oscillates between 2.50662826 and 2.50662829.

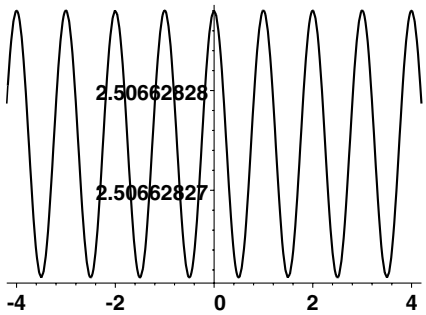

FIGURE 1.3. Zoomed graph of $f(x)$

One obtains the same picture if this procedure is repeated for the sum

$$f_{\mathcal{D}}(x) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-m)^2/\mathcal{D}}$$

with different values of the parameter $\mathcal{D} > 0$. Figs. 1.4 and 1.5 show the graph of $f_{\mathcal{D}}$ for the parameters $\mathcal{D} = 0.5$ and $\mathcal{D} = 4$, respectively. The plot of the function

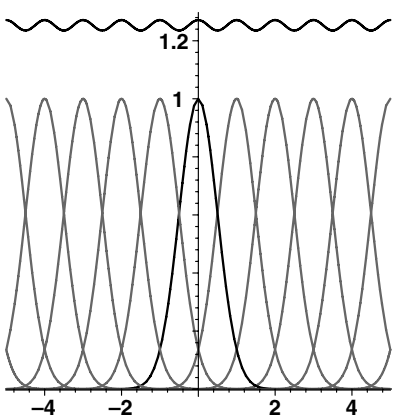

FIGURE 1.4. $f_{1/2}(x)$ and individual terms

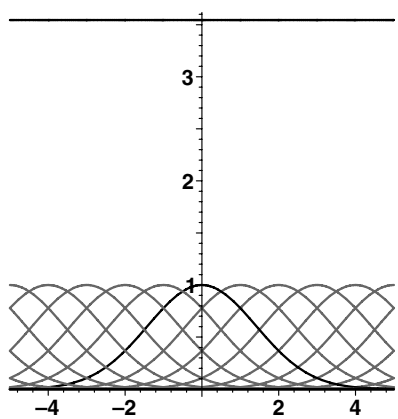

FIGURE 1.5. $f_4(x)$ and individual terms

$f_{1/2}(x)$ shows the oscillating behavior, whereas f_4 looks like a constant. In fact f_4 is also oscillating between $3.54490770181103205 \pm 1.43 \cdot 10^{-15}$, which is very hard to depict. One can conjecture, that the oscillating function $f_{\mathcal{D}}$ tends to a constant if $\mathcal{D}$ increases.

To rigorously explain peculiarities of the graphs, let us consider the Fourier series of the function

$$\theta(x,\mathcal{D}) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-m)^2/\mathcal{D}}\,,\ \mathcal{D}>0\,. \tag{1.1}$$

Its coefficients can be computed as follows:

$$\frac{1}{\sqrt{\pi\mathcal{D}}}\int_0^1 \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-m)^2/\mathcal{D}}\ \mathrm{e}^{-2\pi i\nu x}\ dx = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} \int_m^{m+1} \mathrm{e}^{-x^2/\mathcal{D}}\ \mathrm{e}^{-2\pi i\nu x}\ dx$$

$$= \frac{1}{\sqrt{\pi\mathcal{D}}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/\mathcal{D}}\ \mathrm{e}^{-2\pi i\nu x}\ dx = \frac{\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}}{\sqrt{\pi\mathcal{D}}}\int_{-\infty}^{\infty} \mathrm{e}^{-(x/\sqrt{\mathcal{D}}+i\pi\sqrt{\mathcal{D}}\nu)^2}\ dx$$

$$= \frac{\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}}{\sqrt{\pi\mathcal{D}}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/\mathcal{D}}\ dx = \mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\ .$$

The order of summation and integration can be changed here because of the absolute convergence of the infinite sum. Hence we obtain the Fourier series

$$\theta(x,\mathcal{D}) = \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\ \mathrm{e}^{2\pi i\nu x}\ . \tag{1.2}$$

This representation of the function θ is a special case of the so-called Poisson summation formula

$$\sum_{m=-\infty}^{\infty} u(x+m) = \sum_{\nu=-\infty}^{\infty} \mathcal{F}u(\nu)\ \mathrm{e}^{2\pi i\nu x}\,, \tag{1.3}$$

where $\mathcal{F}u$ denotes the Fourier transform of the function u. The definition of the Fourier transform will be given in Section 2.1, where we also discuss some properties of this important formula.

From (1.2), we have

$$\theta(x,\mathcal{D}) = 1 + 2\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\cos 2\pi\nu x\,,$$

i.e., our function $\theta(x,\mathcal{D})$ differs from 1 by the infinite series

$$2\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\cos 2\pi\nu x\,. \tag{1.4}$$

The coefficients $\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}$, $\nu=1,2,\ldots$, can be very small depending on $\mathcal{D}$, as seen from the relation $\mathrm{e}^{-\pi^2} = 0.000051723\ldots$. In particular, if $\mathcal{D}\geq 1$, then for any x the modulus of (1.4) is less than $1.04\cdot 10^{-4\mathcal{D}}$. Note that in the cases $\mathcal{D}=2$ and $\mathcal{D}=4$ the difference is comparable to the single and, respectively, double precision in the arithmetics of most modern computers, i.e., in these cases where the function $\theta(x,\mathcal{D})$ is numerically the constant function 1. Moreover, the difference $|\theta(x,\mathcal{D})-1|$ can be made less than any prescribed positive tolerance ε by choosing $\mathcal{D}$ large enough. For this it suffices to take

$$\mathcal{D} > \pi^{-2}(\log|\varepsilon| - \log 2)\,.$$

REMARK 1.1. The function θ is closely connected with Jacobi's Theta function ϑ_3, which is defined as (see [**1**, 16.27])

$$\vartheta_3(z|\tau) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{i\pi\tau n^2}\, \mathrm{e}^{2inz}\,, \tag{1.5}$$

by the relation $\theta(x,\mathcal{D}) = \vartheta_3(\pi x|i\pi\mathcal{D})$.

1.1.2. Simple approximation formula. We have seen that for "large" $\mathcal{D}$ the integer shifts

$$\left\{\frac{1}{\sqrt{\pi\mathcal{D}}}\, \mathrm{e}^{-(x-m)^2/\mathcal{D}}, m \in \mathbb{Z}\right\} \tag{1.6}$$

form an approximate partition of unity, i.e., the sum of these functions is approximatively equal to the constant function 1. In addition, the functions in the family (1.6) decay very rapidly if $|x-m| \to \infty$. Hence, in the sum (1.1), one has to take into account only a small number of terms, if one wants to compute the value at a given point x. This leads to the idea of introducing an approximation formula using the usual scaling and translation operations with a "small" parameter h for the family of functions $\mathrm{e}^{-x^2/\mathcal{D}}$

$$\mathcal{M}_{h,\mathcal{D}}u(x) = \frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} u(mh)\, \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}\,. \tag{1.7}$$

Formulas of this type are known as *quasi-interpolants* and we are interested in their behavior as $h \to 0$.

Let us suppose that the function u is twice continuously differentiable with bounded derivatives. The Taylor expansion of u at the point mh has the form

$$u(mh) = u(x) + u'(x)(mh-x) + u''(x_m)\frac{(mh-x)^2}{2}$$

for some x_m between x and mh. Putting this into (1.7), we derive

$$\begin{aligned}\mathcal{M}_{h,\mathcal{D}}u(x) = &\frac{u(x)}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} \\ &+ \frac{u'(x)}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} (mh-x)\, \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} \\ &+ \frac{1}{2\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} u''(x_m)(mh-x)^2\, \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}\,.\end{aligned} \tag{1.8}$$

The sum of the first term on the right-hand side is the function $\theta(x/h,\mathcal{D})$, whereas the sum in the second term can be expressed, for example, by the derivative

$$\theta'\big(\frac{x}{h},\mathcal{D}\big) = \frac{2}{\sqrt{\pi\mathcal{D}}\mathcal{D}h} \sum_{m=-\infty}^{\infty} (mh-x)\, \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2},$$

which provides the relation

$$\frac{1}{\sqrt{\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} (mh-x)\, \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} = -2\pi\mathcal{D}h \sum_{\nu=1}^{\infty} \nu\, \mathrm{e}^{-\pi^2\mathcal{D}\nu^2} \sin 2\pi\nu\frac{x}{h}\,.$$

Therefore, by using (1.2), we can write the quasi-interpolant in the form

$$\mathcal{M}_{h,\mathcal{D}}u(x) = u(x) + C_{\mathcal{D},h}(x) + R_h(x) \tag{1.9}$$

with the function

$$C_{\mathcal{D},h}(x) = 2u(x)\sum_{\nu=1}^{\infty} \mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\cos 2\pi\nu\frac{x}{h} - 2u'(x)\pi\mathcal{D}h\sum_{\nu=1}^{\infty}\nu\,\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\sin 2\pi\nu\frac{x}{h} \tag{1.10}$$

and the remainder term

$$R_h(x) = \frac{1}{2\sqrt{\pi\mathcal{D}}}\sum_{m=-\infty}^{\infty} u''(x_m)(mh-x)^2\ \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2},$$

which obviously satisfies

$$|R_h(x)| \le \max_{t\in\mathbb{R}}|u''(t)|\,\frac{1}{2\sqrt{\pi\mathcal{D}}}\sum_{m=-\infty}^{\infty}(mh-x)^2\ \mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}.$$

The Fourier series of the last sum can be calculated similarly to the case $\theta(x,\mathcal{D})$, and it holds that

$$\frac{1}{\sqrt{\pi\mathcal{D}}}\sum_{m=-\infty}^{\infty}(mh-x)^2\,\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} = \frac{\mathcal{D}h^2}{2}\sum_{\nu=-\infty}^{\infty}(1-2\pi^2\mathcal{D}\nu^2)\,\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\,\mathrm{e}^{2\pi i\nu x/h},$$

which leads to the estimate

$$|R_h(x)| \le \max_{t\in\mathbb{R}}|u''(t)|\frac{\mathcal{D}h^2}{4}\Big(1+2\sum_{\nu=1}^{\infty}\Big|(1-2\pi^2\mathcal{D}\nu^2)\cos 2\pi\nu\frac{x}{h}\Big|\,\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\Big).$$

Hence, the difference between u and the quasi-interpolant $\mathcal{M}_{h,\mathcal{D}}u$ can be estimated for any $x\in\mathbb{R}$ by

$$\boxed{\begin{aligned}\big|\mathcal{M}_{h,\mathcal{D}}u(x)-u(x)\big| \le\ & \frac{\mathcal{D}h^2}{4}\big(1+\sum_{\nu=1}^{\infty}|4\pi^2\mathcal{D}\nu^2-2|\,\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}\big)\max_{t\in\mathbb{R}}|u''(t)| \\ & + \big|C_{\mathcal{D},h}(x)\big|.\end{aligned}} \tag{1.11}$$

This inequality is valid for all values of the positive parameters $\mathcal{D}$ and h. Here we find the special feature of approximate approximations, mentioned in the Preface. The approximation error consists of a term of the order $\mathcal{O}(\mathcal{D}h^2)$ and the term $\big|C_{\mathcal{D},h}(x)\big|$, which is called the *saturation error*, because it does not converge to zero as $h\to 0$. However, we obtain from (1.10) that

$$\big|C_{\mathcal{D},h}(x)\big| \le 2|u(x)|\sum_{\nu=1}^{\infty}\mathrm{e}^{-\pi^2\mathcal{D}\nu^2} + 2\pi\mathcal{D}h|u'(x)|\sum_{\nu=1}^{\infty}\nu\,\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}.$$

Therefore, owing to the rapid decay of $\mathrm{e}^{-\pi^2\mathcal{D}\nu^2}$, $\nu=1,2,\ldots$, for any $\varepsilon>0$ one can fix $\mathcal{D}>0$ such that the saturation error satisfies

$$|C_{\mathcal{D},h}(x)| < \varepsilon\,\big(|u(x)|+h|u'(x)|\big).$$

Since the first term of the right-hand side of (1.11) with a fixed $\mathcal{D}$ converges to zero, we see that $\mathcal{M}_{h,\mathcal{D}}u$ approximates u with the order $\mathcal{O}(h^2)$ as long as the saturation bound $\varepsilon\big(|u(x)|+h|u'(x)|\big)$ is reached. Hence, choosing the parameter $\mathcal{D}$ such that

ε is less than the precision of the computing system, formula $\mathcal{M}_{h,\mathcal{D}}u$ behaves in numerical computations as a usual second-order approximation.

Let us emphasize the structure of $C_{\mathcal{D},h}$, which is the sum of $u(x)$ and $hu'(x)$ multiplied by oscillating functions with period h. For sufficiently large $\mathcal{D}$ the main term of $C_{\mathcal{D}}$ is given by

$$2u(x)\,\mathrm{e}^{-\pi^2\mathcal{D}}\cos 2\pi\frac{x}{h}\,.$$

This is a fast oscillating simple harmonics modulated by the slowly varying value of the approximated function.

In the following we show that formulas of type (1.7), where the Gaussian e^{-x^2} is replaced by more general basis functions, can provide similar or even better approximation properties. We give some one- and multi-dimensional examples of those approximation formulas and define approximate quasi-interpolation on uniform grids in the next section.

1.2. Further examples

1.2.1. Errors of approximate quasi-interpolation. We illustrate here the approximation properties of the quasi-interpolant $\mathcal{M}_{h,\mathcal{D}}u$ defined in (1.7) for the function $u(x) = \sin x$ using different values of the parameters $\mathcal{D}$ and h. Figs. 1.6 and 1.7 show the particular form of the terms

$$\frac{1}{\sqrt{\pi\mathcal{D}}}\,\sin(mh)\;\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} \tag{1.12}$$

and its sum

$$(\mathcal{M}_{h,\mathcal{D}}\sin)(x) = \frac{1}{\sqrt{\pi\mathcal{D}}}\sum_{m=-\infty}^{\infty}\sin(mh)\;\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2} \tag{1.13}$$

for $h = 0.4$ and two different values $\mathcal{D} = 1$ and $\mathcal{D} = 2$. Visually the sums are good approximations of $\sin x$ for this rather large step h.

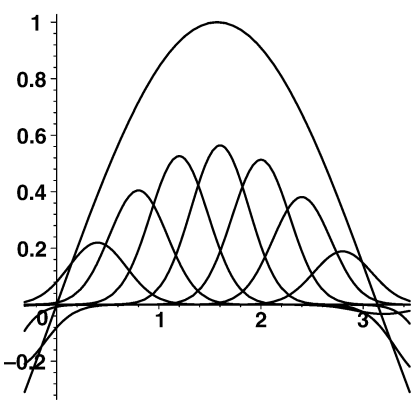

FIGURE 1.6. $\mathcal{D} = 1$, $h = 0.4$

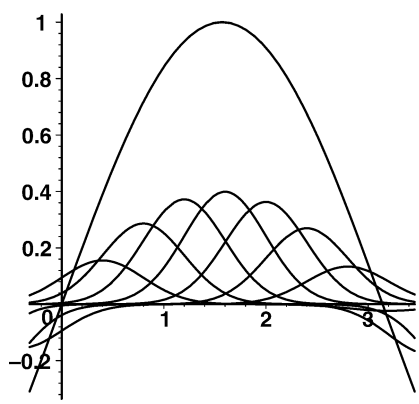

FIGURE 1.7. $\mathcal{D} = 2$, $h = 0.4$

Although the functions $\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}$ are supported by the whole real axis, one needs only a few terms in the sum (1.7) to compute the value of $\mathcal{M}_{h,\mathcal{D}}u$ at a

given point x within a given accuracy. For a fixed tolerance $\delta > 0$ one has to sum only over the integers m for which

$$|m - x/h| \le \sqrt{-\mathcal{D}\log\delta}\,.$$

Hence the number of terms, necessary to compute $\mathcal{M}_{h,\mathcal{D}}u(x)$ for fixed h, increases proportionally to $\sqrt{\mathcal{D}}$.

On the other hand, if $\mathcal{D}$ is fixed, then this number does not depend on h. For example, if $\delta = 10^{-6}$, then one has to sum up 7 and 11 terms in (1.7) if $\mathcal{D} = 1$ and $\mathcal{D} = 2$, respectively.

The differences between $\sin x$ and the quasi-interpolants (1.13) are plotted in Figs. 1.8 and 1.9 for the values $\mathcal{D} = 1, 2$ and $h = 0.4, 0.2$, respectively. The graphs confirm the second-order convergence from estimate (1.11). However, the case of

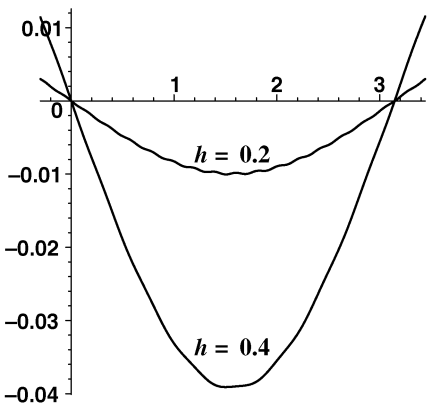

FIGURE 1.8. $(\mathcal{M}_{h,1} - I)\sin x$

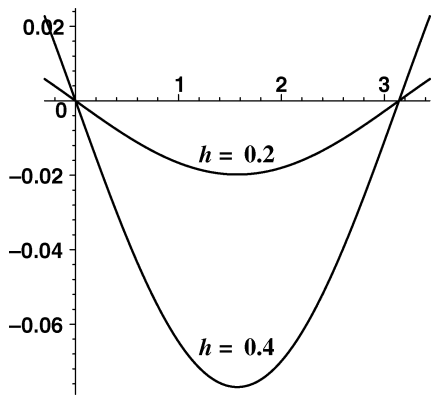

FIGURE 1.9. $(\mathcal{M}_{h,2} - I)\sin x$

smaller step h already gives different pictures. Figs. 1.10–1.15 depict the quasi-interpolation error of $\sin x$ with smaller h and for $\mathcal{D} = 1$ and 2.

The plotted errors confirm the second-order convergence, but the error for $\mathcal{D} = 1$ oscillates very fast, with frequency depending of h. In Figs. 1.10 and 1.12 the saturation error is already visible.

It can be seen from Fig. 1.14, that for $\mathcal{D} = 1$ the quasi-interpolation error has reached its saturation bound, since it does not decrease if h becomes smaller. On the other hand, Fig. 1.15 shows that the approximation with $\mathcal{M}_{h,2}$ for the same values of h is of the second order, that the saturation is not reached, yet.

The behavior of the quasi-interpolants $\mathcal{M}_{h,\mathcal{D}}$, predicted by the estimate (1.11), is confirmed also in Table 1.1, where the quasi-interpolation error in the maximum norm for different h and $\mathcal{D}$ and the convergence rate calculated as

$$\log_2 \frac{\|u - \mathcal{M}_{2h,\mathcal{D}}u\|_{L^\infty}}{\|u - \mathcal{M}_{h,\mathcal{D}}u\|_{L^\infty}} \tag{1.14}$$

are given.

Recall that the main term of the saturation error is $1.04 \cdot 10^{-4\mathcal{D}}\,|u(x)|$. If $\mathcal{D} = 1$, then we have the second-order approximation only for relative large h. In the cases $\mathcal{D} = 2$ and $\mathcal{D} = 4$ the saturation error is still negligible compared to the first term of estimate (1.11).

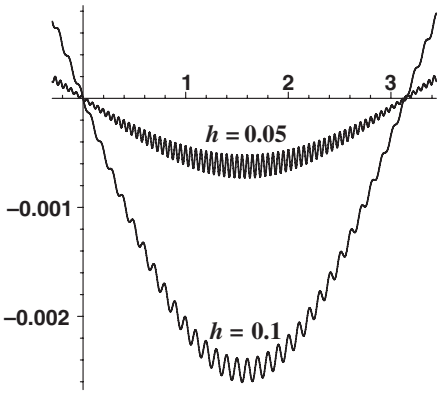

FIGURE 1.10. $(\mathcal{M}_{h,1} - I)\sin x$

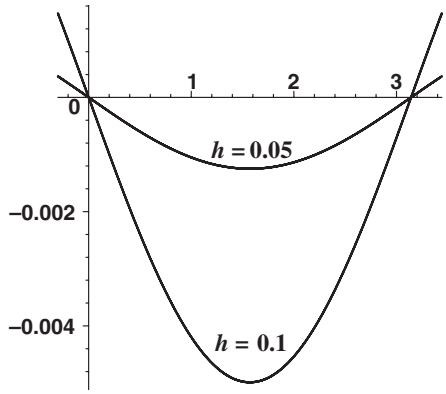

FIGURE 1.11. $(\mathcal{M}_{h,2} - I)\sin x$

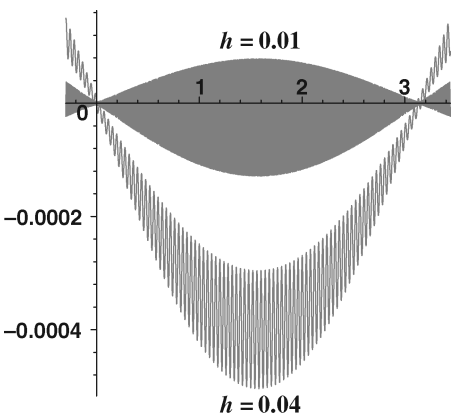

FIGURE 1.12. $(\mathcal{M}_{h,1} - I)\sin x$

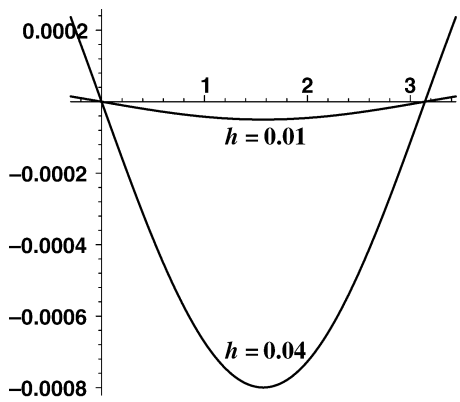

FIGURE 1.13. $(\mathcal{M}_{h,2} - I)\sin x$

1.2.2. A simple application of the approximation formula (1.7)**.** Consider the initial value problem for the heat equation

$$u_t(x,t) - u_{xx}(x,t) = 0\,, \quad t > 0\,, \quad u(x,0) = \varphi(x), \quad x \in \mathbb{R}\,. \tag{1.15}$$

Its solution is given by the *Poisson integral*

$$u(x,t) = \mathcal{P}_t\varphi(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/4t}\, \varphi(y)\, dy\,.$$

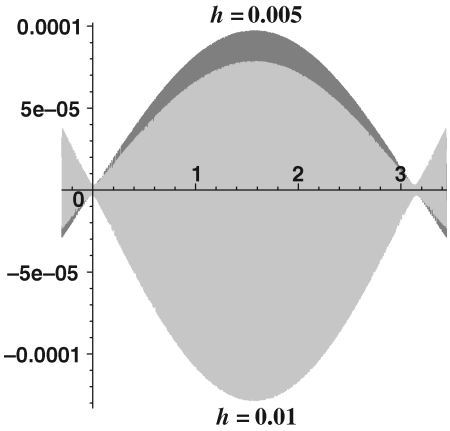

FIGURE 1.14. $(\mathcal{M}_{h,1} - I)\sin x$

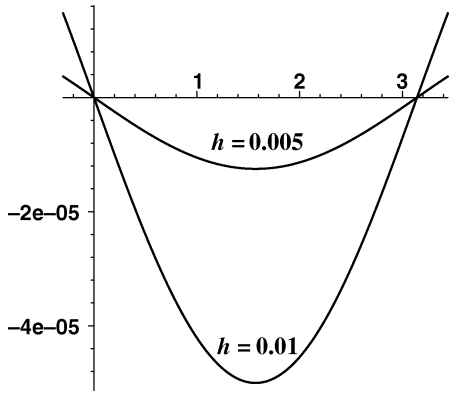

FIGURE 1.15. $(\mathcal{M}_{h,2} - I)\sin x$

h	$\mathcal{D} = 1$	rate	$\mathcal{D} = 2$	rate	$\mathcal{D} = 4$	rate
0.4	$3.91 \cdot 10^{-2}$		$7.69 \cdot 10^{-2}$		$1.48 \cdot 10^{-1}$	
0.2	$1.00 \cdot 10^{-2}$	1.96	$1.98 \cdot 10^{-2}$	1.96	$3.92 \cdot 10^{-2}$	1.91
0.1	$2.60 \cdot 10^{-3}$	1.95	$4.99 \cdot 10^{-3}$	1.99	$9.95 \cdot 10^{-3}$	1.98
0.05	$7.29 \cdot 10^{-4}$	1.84	$1.25 \cdot 10^{-3}$	2.00	$2.50 \cdot 10^{-3}$	1.99
0.025	$2.60 \cdot 10^{-4}$	1.49	$3.12 \cdot 10^{-4}$	2.00	$6.25 \cdot 10^{-4}$	2.00
0.0125	$1.42 \cdot 10^{-4}$	0.87	$7.81 \cdot 10^{-5}$	2.00	$1.56 \cdot 10^{-4}$	2.00
0.00625	$1.11 \cdot 10^{-4}$	0.36	$1.95 \cdot 10^{-5}$	2.00	$3.91 \cdot 10^{-5}$	2.00

TABLE 1.1. Approximation error for the function $u(x) = \sin x$ using the quasi-interpolant (1.7)

This integral cannot be taken in a closed form, in general, but this is possible for some functions φ, for example, for the Gaussian function. In particular,

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/4t}\ \mathrm{e}^{-y^2/\mathcal{D}h^2}\ dy = \frac{\sqrt{\mathcal{D}}h}{\sqrt{\mathcal{D}h^2 + 4t}}\ \mathrm{e}^{-x^2/(\mathcal{D}h^2+4t)}\ . \tag{1.16}$$

Hence, if we replace the initial value φ by the quasi-interpolant $\mathcal{M}_{h,\mathcal{D}}\varphi$ defined by (1.7), then we obtain the exact solution

$$\mathcal{P}_t(\mathcal{M}_{h,\mathcal{D}}\varphi)(x) = \frac{h}{\sqrt{\pi(\mathcal{D}h^2 + 4t)}} \sum_{m=-\infty}^{\infty} \varphi(hm)\ \mathrm{e}^{-(x-hm)^2/(\mathcal{D}h^2+4t)} \tag{1.17}$$

of the heat equation (1.15) with the modified initial condition $u(x,0) = \mathcal{M}_{h,\mathcal{D}}\varphi(x)$. Since

$$\begin{aligned}\big|\mathcal{P}_t\varphi(x) - \mathcal{P}_t(\mathcal{M}_{h,\mathcal{D}}\varphi)(x)\big| &\le \sup_{y\in\mathbb{R}} \big|\varphi(y) - \mathcal{M}_{h,\mathcal{D}}\varphi(y)\big| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/4t}\, dy \\ &= \sup_{y\in\mathbb{R}} \big|\varphi(y) - \mathcal{M}_{h,\mathcal{D}}\varphi(y)\big|\,,\end{aligned}$$

the estimate (1.11) shows that the function $u_h(x,t) = \mathcal{P}_t(\mathcal{M}_{h,\mathcal{D}}\varphi)(x)$ approximates the solution $u(x,t)$ of the original problem (1.15) with the error

$$\begin{aligned}|u(x,t) - u_h(x,t)| \le{}& \frac{\mathcal{D}h^2}{4}\big(1 + 4\mathcal{D}\pi^2\,\mathrm{e}^{-\pi^2\mathcal{D}}\big) \max_{y\in\mathbb{R}} |\varphi''(y)| \\ &+ 2\max_{y\in\mathbb{R}} \big(|\varphi(y)| + |\varphi'(y)|\pi\mathcal{D}h\big)\,\mathrm{e}^{-\pi^2\mathcal{D}} + \mathcal{O}(\mathrm{e}^{-2\pi^2\mathcal{D}})\,.\end{aligned} \tag{1.18}$$

This very simple example is in many ways typical in applying approximate approximations to the solution of partial differential equations; one replaces some function in the original problem by an approximant such that the solution of the equation can be performed very efficiently, either analytically or by some other numerical method.

Let us mention that (1.18) is only a rough error estimate for the approximate solution of the heat equation. This can be seen from Table 1.2 which contains numerical results for the heat equation (1.15) with the initial value $\phi(x) = \mathrm{e}^{-x^2}$. It provides the maximum error

$$\max_x |u(x,t) - u_h(x,t)|\,, \quad t = 10\,,$$

for different values of $\mathcal{D}$ and h.

h	$\mathcal{D}=1$	rate	$\mathcal{D}=2$	rate	$\mathcal{D}=4$	rate
0.4	$3.04\cdot10^{-4}$		$6.06\cdot10^{-4}$		$1.20\cdot10^{-3}$	
0.2	$7.61\cdot10^{-5}$	2.00	$1.52\cdot10^{-4}$	2.00	$3.04\cdot10^{-4}$	1.99
0.1	$1.90\cdot10^{-5}$	2.00	$3.81\cdot10^{-5}$	2.00	$7.61\cdot10^{-5}$	2.00
0.05	$4.76\cdot10^{-6}$	2.00	$9.52\cdot10^{-6}$	2.00	$1.90\cdot10^{-5}$	2.00
0.025	$1.19\cdot10^{-6}$	2.00	$2.38\cdot10^{-6}$	2.00	$4.76\cdot10^{-6}$	2.00
0.0125	$2.98\cdot10^{-7}$	2.00	$5.95\cdot10^{-7}$	2.00	$1.19\cdot10^{-6}$	2.00
0.00625	$7.44\cdot10^{-8}$	2.00	$1.49\cdot10^{-7}$	2.00	$2.98\cdot10^{-7}$	2.00

TABLE 1.2. Numerical error for the initial value problem (1.15) with $\phi(x) = \mathrm{e}^{-x^2}$ and $t = 10$ using the approximate solution (1.17)

In contrast to the quasi-interpolation results, given in Table 1.1, a saturation error cannot be seen. We will show in Subsection 6.2.1 that due to the properties of the Poisson integral and the structure of the saturation error the approximate solution $u_h(x,t)$ converges to $u(x,t)$.

1.2.3. Other basis functions. The simplicity of formulas of the form

$$\mathcal{Q}_h u(x) := \sum_{m=-\infty}^{\infty} u(mh)\,\eta\Big(\frac{x}{h} - m\Big) \tag{1.19}$$

makes them very attractive for approximation processes. Suppose, for example, that η is a *Lagrangian function*, which means that η is subject to

$$\eta(0) = 1 \quad \text{and} \quad \eta(m) = 0 \quad \text{for all } m \in \mathbb{Z}\backslash 0\,.$$

Then the sum (1.19) satisfies $\mathcal{Q}_h u(mh) = u(mh)$, $m \in \mathbb{Z}$, i.e., $\mathcal{Q}_h u$ interpolates u. As two representative examples, we mention here the piecewise linear hat function and the sinc function

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}\,.$$

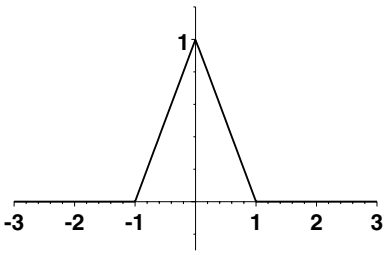

FIGURE 1.16. Hat function

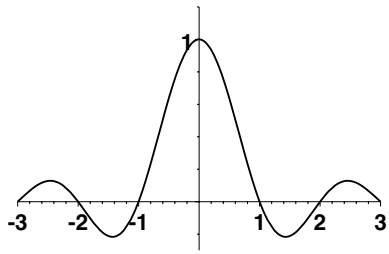

FIGURE 1.17. sinc function

If η is the hat function as shown in Fig. 1.16, then the resulting sum is the polygonal line connecting the points $(hm, u(hm))$. This piecewise linear function approximates u with the order $\mathcal{O}(h^2)$. To find the approximant at a given point x, one has to sum up only two terms of (1.19). But it is visually not very nice to approximate a smooth curve by some piecewise linear function.

On the other hand, the sinc function (depicted in Fig. 1.17) generates an interpolant which is smooth and even provides an exponential order of convergence (see [**91**]). However, since the generating function decreases very slowly, it is practically impossible to compute the approximant (1.19) if the function u is not compactly supported.

Let us mention that in Chapter 7 we introduce another Lagrangian function

$$\Psi_{\mathcal{D}}(x) = \frac{\sin \pi x}{\pi \mathcal{D} \sinh \dfrac{x}{\mathcal{D}}}$$

depending on the parameter $\mathcal{D} > 0$. This function is a small perturbation of the Lagrangian function from the family of shifted Gaussians (1.6). The corresponding interpolant approximates smooth functions with exponential order, but similar to the approximation formula (1.7) only up to a saturation error of the order $\mathcal{O}(\mathrm{e}^{-\pi^2 \mathcal{D}})$. Therefore $\Psi_{\mathcal{D}}$ can be considered as approximate sinc function, providing similar approximation properties but decaying exponentially for $|x| \to \infty$.

There exists, of course, a variety of other basis functions η for interpolation formulas (1.19). However, the Lagrangian functions for those bases have, in general, large supports. For example, the Lagrangian function for the class of smooth cubic splines, which are cubic polynomials on the intervals $(m, m+1)$, $m \in \mathbb{Z}$, and two-times continuously differentiable, is supported on the whole real line.

It turns out that good approximations can also be obtained by replacing the Lagrangian function in (1.19) by some simpler function of the same class. In the case of smooth cubic splines one can choose η as the cubic B-spline

$$b(x) = \frac{1}{12}\big(|x+2|^3 - 4|x+1|^3 + 6|x|^3 - 4|x-1|^3 + |x-2|^3\big) \tag{1.20}$$

depicted in Fig. 1.18, which gives a C^2-approximant of the order $\mathcal{O}(h^2)$.

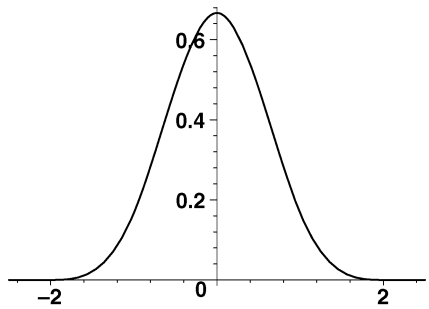

FIGURE 1.18. Cubic B-spline

But clearly the resulting approximant does not interpolate; therefore approximation formulas (1.19) with non-Lagrangian functions η are called *quasi-interpolants*.

Thus, $\mathcal{M}_{h,\mathcal{D}}$ in (1.7) represents a quasi-interpolant with

$$\eta(x) = \frac{\mathrm{e}^{-x^2/\mathcal{D}}}{\sqrt{\pi\mathcal{D}}}\,.$$

We have seen in (1.11) that for a fixed $\mathcal{D}$ the sum $\mathcal{M}_{h,\mathcal{D}}u$ is a smooth approximation to u of order $\mathcal{O}(h^2)$ until the saturation error is reached, which can be neglected in numerical computations if $\mathcal{D}$ is sufficiently large.

It is important that for a quite general class of basis functions the quasi-interpolants have similar properties as in the case of the Gaussian. Take, for example, the function

$$\operatorname{sech} x = \frac{1}{\cosh x}\,.$$

Putting the Taylor expansion of u into

$$\mathcal{M}_h u(x) = \frac{1}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty} u(mh)\operatorname{sech}\frac{x-mh}{\sqrt{\mathcal{D}}h}\,, \tag{1.21}$$

we obtain as in (1.8)

$$\mathcal{M}_h u(x) = \frac{u(x)}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}\operatorname{sech}\frac{x-mh}{\sqrt{\mathcal{D}}h} + \frac{u'(x)}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}(mh-x)\operatorname{sech}\frac{x-mh}{\sqrt{\mathcal{D}}h}$$
$$+\frac{1}{2\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}u''(x_m)(mh-x)^2\operatorname{sech}\frac{x-mh}{\sqrt{\mathcal{D}}h}\,.$$

The infinite sums in the first and second term on the right-hand side can be transformed by using Poisson's summation formula (1.3) and the Fourier transform of $\operatorname{sech} x$,

$$(\mathcal{F}\operatorname{sech})(\lambda) = \pi \operatorname{sech} \pi^2 \lambda \,.$$

Then we obtain the relations

$$\begin{aligned} I_0 &:= \frac{1}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} \operatorname{sech} \frac{x-m}{\sqrt{\mathcal{D}}} = 1 + 2\sum_{\nu=1}^{\infty} \operatorname{sech}(\pi^2\sqrt{\mathcal{D}}\nu)\, \cos 2\pi\nu x \,, \\ I_1 &:= \frac{1}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} \frac{x-m}{\sqrt{\mathcal{D}}} \operatorname{sech} \frac{x-m}{\sqrt{\mathcal{D}}} \\ &= \pi \sum_{\nu=1}^{\infty} \operatorname{sech}(\pi^2\sqrt{\mathcal{D}}\nu) \tanh(\pi^2\sqrt{\mathcal{D}}\nu)\, \sin 2\pi\nu x \,, \end{aligned}$$

which shows that

$$|I_0 - 1| < 2\varepsilon(\mathcal{D}) \quad \text{and} \quad |I_1| < \pi\varepsilon(\mathcal{D}) \,,$$

where we use the notation

$$\varepsilon(\mathcal{D}) := \sum_{\nu=1}^{\infty} \operatorname{sech}(\pi^2\sqrt{\mathcal{D}}\nu) \,.$$

Moreover,

$$\frac{1}{2\pi\sqrt{\mathcal{D}}} \left| \sum_{m=-\infty}^{\infty} u''(x_m) \frac{(mh-x)^2}{\mathcal{D}h^2} \operatorname{sech} \frac{x-mh}{\sqrt{\mathcal{D}}h} \right| \le \frac{5}{4} \sup_{t\in\mathbb{R}} |u''(t)| \,,$$

so that

$$|u(x) - \mathcal{M}_h u(x)| \le \frac{5}{4}\, \mathcal{D}h^2 \max_{\mathbb{R}} |u''| + \varepsilon(\mathcal{D})\big(2\ |u(x)| + \pi\sqrt{\mathcal{D}}h\, |u'(x)|\big) \,. \tag{1.22}$$

As in the example with the Gaussian function the quasi-interpolant (1.21) does not converge to $u(x)$, but the number $\varepsilon(\mathcal{D})$ is an upper bound for the saturation error and can be made arbitrarily small by choosing $\mathcal{D}$ large enough. For example, if $\mathcal{D} = 4$, then $\varepsilon(\mathcal{D}) = 0.000000005351$.

Again, the inequality (1.22) shows that the quasi-interpolant $\mathcal{M}_h u$ approximates any C^2-function u like a second-order approximant above the tolerance

$$\varepsilon(\mathcal{D})\big(2\,|u(x)| + \pi\sqrt{\mathcal{D}}h\,|u'(x)|\big) \,,$$

and that any prescribed accuracy can be reached if $\mathcal{D}$ is chosen sufficiently large. In Table 1.3 we give the L^∞-error of the quasi-interpolation of $\sin x$ with formula (1.22) for different h and $\mathcal{D}$ and the convergence rate obtained using (1.14).

1.2.4. Examples of higher-order quasi-interpolants. There exist approximants with approximation orders larger than 2 up to some prescribed accuracy which have the same simple form as second-order approximate quasi-interpolants. Consider, for example, the quasi-interpolant

$$u_h(x) := \mathcal{D}^{-1/2} \sum_{m=-\infty}^{\infty} u(mh)\, \eta\Big(\frac{x-mh}{\sqrt{\mathcal{D}}h}\Big) \tag{1.23}$$

h	$\mathcal{D}=1$	rate	$\mathcal{D}=2$	rate	$\mathcal{D}=4$	rate
0.4	$1.69\cdot 10^{-1}$		$2.96\cdot 10^{-1}$		$4.73\cdot 10^{-1}$	
0.2	$4.75\cdot 10^{-2}$	1.83	$9.12\cdot 10^{-2}$	1.70	$1.69\cdot 10^{-1}$	1.48
0.1	$1.24\cdot 10^{-2}$	1.93	$2.42\cdot 10^{-2}$	1.92	$4.74\cdot 10^{-2}$	1.84
0.05	$3.32\cdot 10^{-3}$	1.91	$6.14\cdot 10^{-3}$	1.98	$1.22\cdot 10^{-2}$	1.96
0.025	$1.01\cdot 10^{-3}$	1.71	$1.54\cdot 10^{-3}$	1.99	$3.08\cdot 10^{-3}$	1.99
0.0125	$4.37\cdot 10^{-4}$	1.22	$3.89\cdot 10^{-4}$	1.99	$7.71\cdot 10^{-4}$	2.00
0.00625	$2.18\cdot 10^{-4}$	1.01	$9.98\cdot 10^{-5}$	1.96	$1.93\cdot 10^{-4}$	2.00

TABLE 1.3. Error of approximating $u(x)=\sin x$ with formula (1.21)

with one of the two generating functions

$$\eta_1(x)=(3/2-x^2)\,\frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}} \quad \text{or} \quad \eta_2(x)=\sqrt{\frac{\mathrm{e}}{\pi}}\;\mathrm{e}^{-x^2}\cos\sqrt{2}x\,, \tag{1.24}$$

shown in Figs. 1.19 and 1.20.

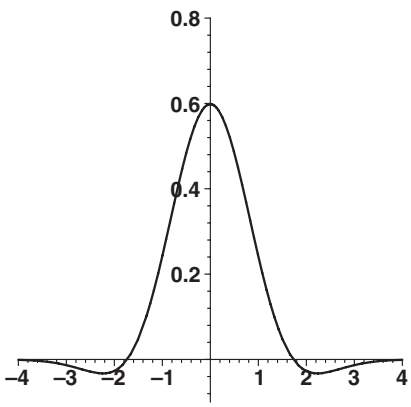

FIGURE 1.19. $\eta_1(x/\sqrt{2})$

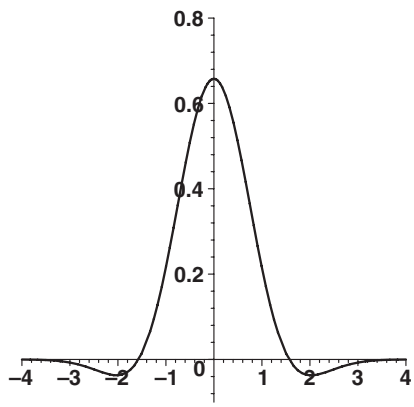

FIGURE 1.20. $\eta_2(x/\sqrt{2})$

Figs. 1.21 – 1.26 repeat the error plots of Subsection 1.2.1 for the approximation of the function $\sin x$ with

$$\mathcal{N}_{h,\mathcal{D}}(\mathbf{x}) := \Big(\frac{\mathrm{e}}{\pi\mathcal{D}}\Big)^{1/2}\sum_{m=-\infty}^{\infty} u(mh)\,\cos\big(\sqrt{2/\mathcal{D}}(x/h-m)\big)\;\mathrm{e}^{-(x-mh)^2/\mathcal{D}h^2}\,, \tag{1.25}$$

where now the values $\mathcal{D}=1.5$ and $\mathcal{D}=2.5$ are used.

The absolute errors given in Figs. 1.21 and 1.22 are much smaller than those plotted in Figs. 1.8 and 1.9. Moreover, the graphs indicate approximation with the order 4.

The visible oscillations of the errors in Fig. 1.21 (the case $\mathcal{D}=1.5$ for quite large steps h) are caused by the relatively large saturation error. The error plots in Figs. 1.23 and 1.25 show clearly that $\mathcal{N}_{0.1,1.5}$ has reached the saturation and that

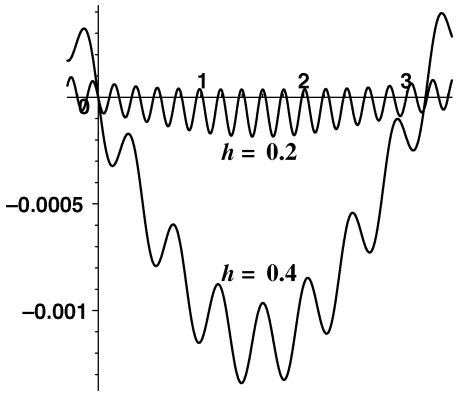

FIGURE 1.21. $(\mathcal{N}_{h,1.5} - I)\sin x$

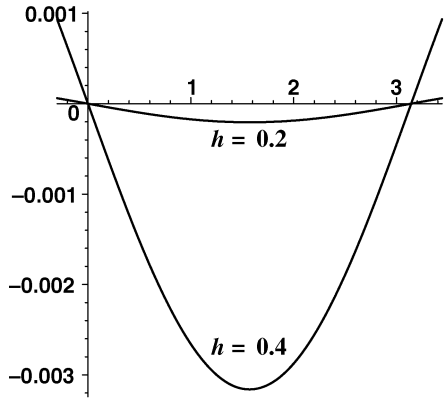

FIGURE 1.22. $(\mathcal{N}_{h,2.5} - I)\sin x$

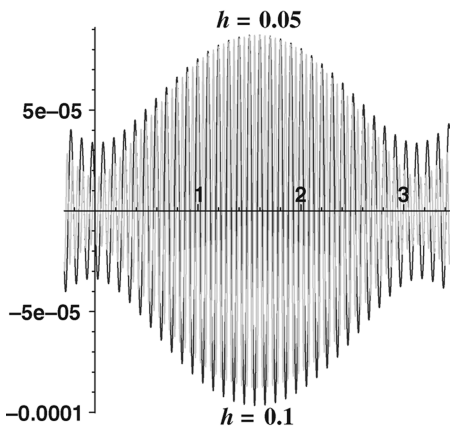

FIGURE 1.23. $(\mathcal{N}_{h,1.5} - I)\sin x$

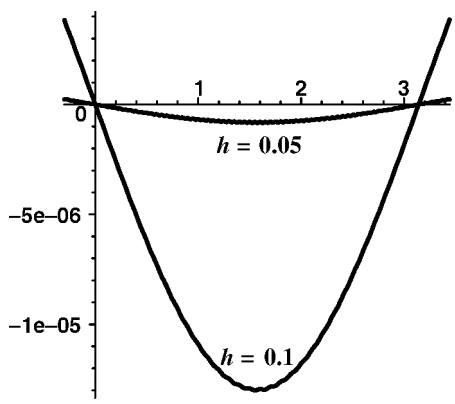

FIGURE 1.24. $(\mathcal{N}_{h,2.5} - I)\sin x$

any h smaller than 0.1 does not give more accurate results for the quasi-interpolant $\mathcal{N}_{h,1.5}$.

The situation is much better for $\mathcal{N}_{h,2.5}$, as indicated in Figs. 1.24 and 1.26. The approximation is much more accurate for small h; the approximation error is dominated by the saturation only if $h \leq 0.01$.

The approximation errors of the function $\sin x$ with the basis functions (1.24) are given in the Tables 1.4 and 1.5 which confirm an approximation with the order 4 up to some saturation error.

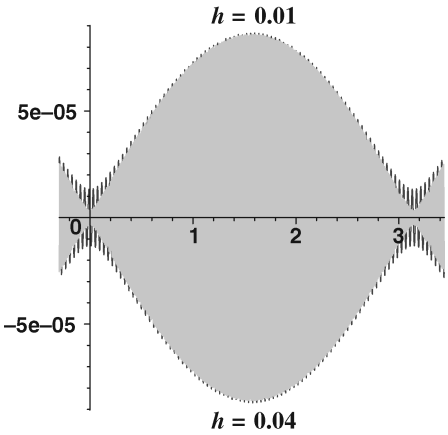

FIGURE 1.25. $(\mathcal{N}_{h,1.5} - I)\sin x$

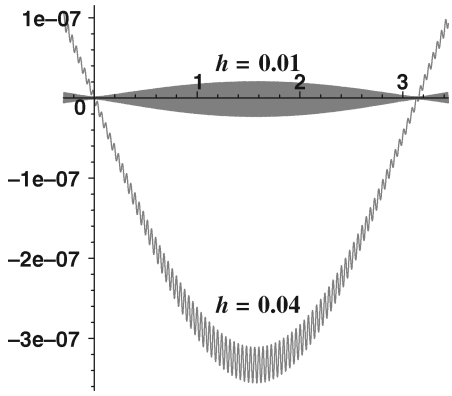

FIGURE 1.26. $(\mathcal{N}_{h,2.5} - I)\sin x$

h	$\mathcal{D} = 2$	ord	$\mathcal{D} = 3$	ord	$\mathcal{D} = 4$	ord
0.4	$3.03 \cdot 10^{-3}$		$6.65 \cdot 10^{-3}$		$1.15 \cdot 10^{-2}$	
0.2	$1.97 \cdot 10^{-4}$	3.85	$4.41 \cdot 10^{-4}$	3.76	$7.79 \cdot 10^{-4}$	3.70
0.1	$1.26 \cdot 10^{-5}$	3.92	$2.80 \cdot 10^{-5}$	3.94	$4.97 \cdot 10^{-5}$	3.92
0.05	$8.96 \cdot 10^{-7}$	3.51	$1.75 \cdot 10^{-6}$	3.99	$3.12 \cdot 10^{-6}$	3.98
0.025	$1.60 \cdot 10^{-7}$	1.39	$1.09 \cdot 10^{-7}$	4.00	$1.95 \cdot 10^{-7}$	4.00
0.0125	$9.72 \cdot 10^{-8}$	0.41	$7.63 \cdot 10^{-9}$	3.98	$1.24 \cdot 10^{-8}$	4.03

TABLE 1.4. Error of approximating $u(x) = \sin x$ with $\eta(x) = \dfrac{3/2 - x^2}{\pi^{1/2}}\,\mathrm{e}^{-x^2}$

h	$\mathcal{D} = 2$	ord	$\mathcal{D} = 3$	ord	$\mathcal{D} = 4$	ord
0.4	$2.04 \cdot 10^{-3}$		$4.50 \cdot 10^{-3}$		$7.84 \cdot 10^{-3}$	
0.2	$1.34 \cdot 10^{-4}$	3.81	$2.95 \cdot 10^{-4}$	3.81	$5.22 \cdot 10^{-4}$	3.75
0.1	$9.94 \cdot 10^{-6}$	3.37	$1.87 \cdot 10^{-5}$	3.95	$3.32 \cdot 10^{-5}$	3.94
0.05	$2.00 \cdot 10^{-6}$	1.24	$1.17 \cdot 10^{-6}$	3.99	$2.08 \cdot 10^{-6}$	3.98
0.025	$1.48 \cdot 10^{-6}$	0.34	$7.43 \cdot 10^{-8}$	3.97	$1.30 \cdot 10^{-7}$	4.00
0.0125	$1.33 \cdot 10^{-6}$	0.28	$5.38 \cdot 10^{-9}$	3.65	$8.42 \cdot 10^{-9}$	4.07

TABLE 1.5. Error of approximating $u(x) = \sin x$ with $\eta(x) = \dfrac{\mathrm{e}^{1/2 - x^2}}{\pi^{1/2}} \cos\sqrt{2}x$

We see in Section 3.3, Example 3.2, that

$$\eta_{10}(x) = \pi^{-1/2}\,\mathrm{e}^{-x^2}\left(\frac{315}{128} - \frac{105}{16}\,x^2 + \frac{63}{16}\,x^4 - \frac{3}{4}\,x^6 + \frac{1}{24}\,x^8\right) \tag{1.26}$$

generates a quasi-interpolant which approximates smooth functions with the order $N = 10$ up to some small saturation. This theoretical result is confirmed in Table 1.6.

h	$\mathcal{D}=3$	ord	$\mathcal{D}=5$	ord	$\mathcal{D}=6$	
0.8	$1.41\cdot 10^{-4}$		$1.41\cdot 10^{-3}$		$3.08\cdot 10^{-3}$	
0.7	$4.17\cdot 10^{-5}$	9.11	$4.33\cdot 10^{-4}$	8.85	$9.74\cdot 10^{-4}$	8.62
0.6	$1.00\cdot 10^{-5}$	9.24	$1.06\cdot 10^{-4}$	9.13	$2.45\cdot 10^{-4}$	8.96
0.5	$1.87\cdot 10^{-6}$	9.22	$1.92\cdot 10^{-5}$	9.38	$4.53\cdot 10^{-5}$	9.25
0.4	$3.03\cdot 10^{-7}$	8.16	$2.26\cdot 10^{-6}$	9.58	$5.44\cdot 10^{-6}$	9.50
0.3	$6.68\cdot 10^{-8}$	5.26	$1.37\cdot 10^{-7}$	9.75	$3.34\cdot 10^{-7}$	9.70
0.2	$2.67\cdot 10^{-8}$	2.26	$2.57\cdot 10^{-9}$	9.80	$6.20\cdot 10^{-9}$	9.83
0.1	$1.39\cdot 10^{-8}$	0.94	$7.52\cdot 10^{-11}$	5.10	$2.07\cdot 10^{-11}$	8.29
0.05	$1.12\cdot 10^{-8}$	0.31	$7.12\cdot 10^{-11}$	0.07	$8.52\cdot 10^{-12}$	1.34

TABLE 1.6. Error of approximating $u(x) = \sin x$ with the basis function (1.26)

1.2.5. Examples of multi-dimensional quasi-interpolants. One important feature of approximate quasi-interpolation is the simplicity of its multi-dimensional generalization. In the next chapters, we shall see that sufficiently smooth and rapidly decaying functions with non-vanishing mean value can be taken as generating functions for quasi-interpolants on uniform grids in $\mathbb{R}^n$. So we have access to a large class of appropriate functions, which generate high-order approximants with simple analytic representations. This is, for example, in contrast to the case of spline functions, where n-dimensional generalizations have quite complicated analytic expressions.

One possibility is, for example, to use the radial counterparts of one-dimensional generating functions. So the n-dimensional analogue of (1.7) is the formula

$$\mathcal{M}_{h,\mathcal{D}}u(\mathbf{x}) = \frac{1}{(\pi\mathcal{D})^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^n} u(h\mathbf{m})\, \mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/\mathcal{D}h^2}\,, \tag{1.27}$$

whereas (1.21) can be extended to the approximation formula

$$\frac{1}{c_n\mathcal{D}^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^n} u(h\mathbf{m}) \operatorname{sech}\frac{|\mathbf{x}-h\mathbf{m}|}{\sqrt{\mathcal{D}}h}\,, \quad c_n = \int_{\mathbb{R}^n} \operatorname{sech}|\mathbf{x}|\, d\mathbf{x}\,.$$

Here and in what follows we make the notational convention that finite-dimensional vectors are denoted by bold face symbols, i.e., $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\mathbf{m} = (m_1,\dots,m_n)$, $m_j \in \mathbb{Z}$. The scalar product of two vectors $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$ in the Euclidean space $\mathbb{R}^n$ is denoted by

$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{j=1}^n x_j y_j\,.$$

For the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$, we use the notation

$$|\mathbf{x}| = |\mathbf{x}|_2 = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle}\,.$$

We shall see in the next chapter that both formulas approximate with order $\mathcal{O}(h^2)$ up to some saturation bound.

A fourth-order approximation up to some small saturation is given by the formula

$$\frac{1}{(\pi\mathcal{D})^{n/2}} \sum_{\mathbf{m}\in\mathbb{Z}^n} u(\mathbf{m}h)\Big(\frac{n+2}{2} - \frac{|\mathbf{x}-\mathbf{m}h|^2}{\mathcal{D}h^2}\Big)\,\mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/\mathcal{D}h^2}$$

whereas the sixth-order can be obtained with the generating function

$$\eta(\mathbf{x}) = \frac{\mathrm{e}^{-|\mathbf{x}|^2}}{2\pi^{n/2}}\Big(\frac{(n+4)(n+2)}{4} - (n+4)|\mathbf{x}|^2 + |\mathbf{x}|^4\Big).$$

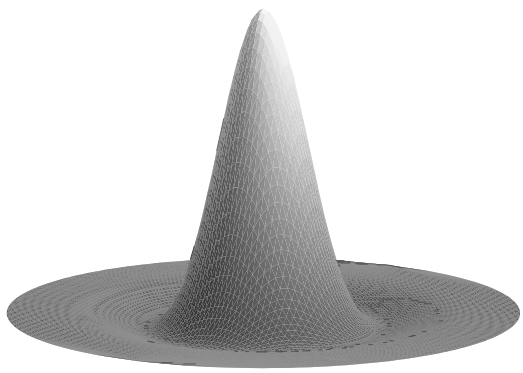

FIGURE 1.27. Sixth-order generating function in $\mathbb{R}^2$

Furthermore, in some applications, we use generating functions of the form

$$\eta(\mathbf{x}) = \phi(\langle A\mathbf{x}, \mathbf{x}\rangle)\,,$$

where A is an $n \times n$ matrix.