Preface to the Second Edition

The study of Toeplitz operators, Hankel operators, and composition operators has witnessed several major advances since the first edition of the book was published over fifteen years ago. So I decided to undertake a substantial revision when I was offered the opportunity to publish a second edition with the AMS.

I rewrote several existing sections completely and added several sections to reflect new developements in the field. Also, the present Chapter 6 is new. It consists of several results from the old Chapter 6 and several new results that have appeared since the publication of the first editon of the book. As a result, the old Chapters 6, 7, 8, 9, and 10 became the new Chapters 7, 8, 9, 10, and 11, respectively.

Except for a few minor corrections and improvements, the contents of Chapters 1, 2, 5, 9, and 10 remain mostly unchanged.

Chapter 3 has pretty much been rewritten. I added a section to cover several useful versions of Hölder's inequality that are needed elsewhere in the book. I also added a section about integral operators on the unit disk whose kernels are built from the Bergman kernel. Such integral operators have proven useful in a number of problems studied in the book as well as in other related problems in the literature. The section on Schur's theorem has been rewritten, with the theorems here more general and more applicable.

Chapter 4 has also been revised substantially. The original text only covered Bergman spaces whose integral exponent p is greater than or equal to 1. Since all earlier results remain true when $0 < p < 1$, I decided to make a little extra effort to cover all integral exponents p as well as all weight parameters α, so the results are now more general and more complete. The atomic decomposition for functions in Bergman spaces is more general now and has a new proof. The proof given in the first edition was based on duality arguments, so it did not work for Bergman spaces whose integral exponent p is less than 1. On the other hand, a complete proof that covers all exponents p and all weight parameters α requires a very elaborate, two-step partition of the unit disk into hyperbolically small pieces. I decided to strike a balance and present a new proof here that only requires a one-step partition of the disk but works only for exponents $p > 2/3$. Readers who are really interested in atomic decomposition can find a complete proof in my book [**438**] or in Coifman and Rochberg's original paper [**107**].

As far as Bergman spaces are concerned, all results in the first edition were stated and proved for the unweighted case. As a result of the additional effort made in Chapter 4, these results have all been generalized to the weighted case. It turned out that the size estimates for Toeplitz operators T_f on the Bergman space $L^2_a(dA_\alpha)$ with nonnegative symbol f are actually independent of the weight parameter. The same is true for the simultaneous size estimates of the Hankel operators H_f and $H_{\overline{f}}$.

As was mentioned earlier, Chapter 6 of this edition is new. In addition to the material about the Berezin transform from various sections (mostly Section 2) of the old Chapter 6, it includes an elegant recent result of Coburn's which states that the Berezin transform of any bounded linear operator satisfies a sharp Lipschitz condition involving the pseudo-hyperbolic metric. It also contains a description of the fixed-point set of the α-Berezin transform in $L^p(\mathbb{D}, dA_\alpha)$ for $1 \le p \le \infty$. In particular, a function $f \in L^\infty(\mathbb{D})$ is fixed by the α-Berezin tranform if and only if it is harmonic.

Chapter 7 is the result of a substantial revision of the old Chapter 6. In particular, I added two new results here. First, I added the characterization of compactness for Toeplitz operators on the Bergman space in terms of the Berezin transform, a significant result due to Axler and Zheng [**42**] that was not yet available when the book was first published. Second, I chose to include the very elegant description of commuting Toeplitz operators with harmonic symbols by Axler and Cuckovic [**37**] because this result has generated a flurry of activity in the area. I do realize that this second result is somewhat incompatible with the flavor of the rest of the book (namely, size estimates for various operators). Finally, the original presentation concerning Toeplitz operators in the Schatten classes S_p was only for the case $p \ge 1$, but the new edition now covers the full range $0 < p < \infty$. In fact, I believe the precise range of p in part (d) of Theorem 7.18 was obtained only recently by the author in [**439**].

Another significant result that appeared after the publication of the first edition of the book was Luecking's approach to Hankel operators on the Bergman space [**264**]. This has been added in the new edition as Section 8.6. It is still not clear if this approach can be made to work for all weighted Bergman spaces $L^2_a(dA_\alpha)$.

As a result of the improvements made in Chapter 7, composition operators on the Hardy space that belong to the Schatten classes S_p are studied in Chapter 11 for the full range $0 < p < \infty$, while the old text only covered the case $p \ge 1$. Section 11.3 is completely rewritten; it has a new proof for the characterization of compact composition operators on Bergman spaces, and a new result concerning Schatten class composition operators on Bergman spaces has also been added in this section.

In addition to the various changes in content, the style of the book has changed as well. This is mandated by the new publisher, the American Mathematical Society. The new book series has a different format. Therefore, the second edition is set in AMS-LaTeX, while the first edition was set in plain TeX. In particular, results and equations are numbered and referenced differently now. The new text consists of chapters and sections, while the old one was further divided into subsections.

Additional comments and references are added to accompany the new material introduced. However, because the number of papers dealing with Toeplitz operators, Hankel operators, composition operators, and functions spaces that have appeared since the appearance of the first edition is probably in the hundreds, so the updated bibliography is by no means exhaustive. When updating the bibliography, I paid more attention to the area of Toeplitz and Hankel operators on Bergman spaces, as this book is the only one in the market that covers this area of analysis.

In the area of composition operators, several monographs have since appeared, most notably the books by Cowen-MacCluer [**119**] and Shapiro [**351**]. The recent book of Peller [**293**] gives a detailed account of the theory of Hankel operators on the Hardy space as well as the various applications of Hankel operators.

The function theory of Bergman spaces has experienced several breakthroughs during the last fifteen years or so. This includes Seip's description of interpolating and sampling sequences for Bergman spaces [**342**], Hedenmalm's study of contractive zero divisors [**184**] in the Bergman space, and the work on invariant subspaces of the Bergman space by Aleman, Richter, and Sundberg [**8**]. For readers interested in this area of analysis, several related books have appeared, including Duren-Schuster [**134**], Hedenmalm-Korenblum-Zhu [**187**], and Zhu [**438**]. There are some obvious overlaps between the present book and the books mentioned above, but this book presents a unique perspective and emphasizes several useful techniques about size estimates of Hankel, Toeplitz, and composition operators that cannot be found in other books. In particular, none of the other books covers Toeplitz and Hankel operators on Bergman spaces.

I am grateful to Dr. Ina Lindemann, senior editor in the publications department of the American Mathematical Society, for her enthusiasm and encouragement to publish this second edition with the AMS. I wish to thank my former student Eric Grossmann, who studied the first edition of the book very carefully and pointed out several mistakes to me. I am also grateful to Miroslav Englis, who read an early version of this second edition and spotted numerous misprints.

As always, I have been able to count on my wife Peijia and our sons, Peter and Michael, for their constant and continuous support during the preparation of the new manuscript. Thank you for bearing with me!

Kehe Zhu, Albany, 2007

Preface to the First Edition

This book deals with three types of operators: Toeplitz operators, Hankel operators, and composition operators. We treat these operators on both the Bergman space and the Hardy space of the open unit disk in the complex plane. The main emphasis of the book is on the size estimates of these operators: boundedness, compactness, and membership in the Schatten classes.

Toeplitz and Hankel operators on the Hardy space have been studied intensively for a long time. However, the study of these operators on the Bergman space began only a few years ago. In particular, the Bergman space theory has never appeared in book form before. Also, this book is the first to deal with composition operators on both the Bergman space and the Hardy space.

We choose to develop the theory on the open unit disk because maximal clarity can be achieved in this case without losing many techniques of the subject. Almost all of the results in this book about Bergman spaces can be generalized to the open unit ball or even to a bounded symmetric domain in $\mathbb{C}^n$. We will comment on this matter in detail at the end of each chapter (in the section entitled "Notes"). Another reason for choosing the open unit disk is that composition operators in several variables are not yet well understood. Thus, concentrating on the disk will enable us to achieve some uniformity.

This book is intended for both research mathematicians and graduate students in complex analysis and operator theory. The prerequisites for the book are kept to a minimum. A graduate course in each of the following subjects should sufficiently prepare the reader for the book: functions of one complex variable, functional analysis, integration and measure theory. Basic facts in general operator theory are collected in the first two sections of the first chapter.

Some exercises are provided at the end of each chapter. Most of these problems are workable. When a relatively difficult problem appears in the exercises, appropriate references are given to the reader. These exercises should be suitable for homework assignments if the book is used as a textbook.

The book can be divided into three parts. Part one of the book provides the necessary preliminaries on operator theory. This includes Chapters 1, 2, and 3. Part two is about operators on the Bergman space, including Chapters 4, 5, 6, 7, 8, and Section 3 of Chapter 11. Part three of the book, including Chapters 9, 10, and most of Chapter 11, deals with operators on the Hardy space.

I wish to thank Lewis Coburn, Richard Rochberg, and Donald Sarason for encouragement and indispensable help during the writing of the book. Carl Cowen provided the author with most of the references on composition operators. Finally, I am grateful to my wife, Peijia Tan, for her patience and understanding. It was not easy for her to take care of our new-born son alone while I spent long hours at the computer working on the manuscript.

Kehe Zhu, Albany, 1989