CHAPTER 1

Geometry and topology of systoles

1.1. From Loewner to Gromov via Berger

The *systole* of a compact metric space X is a metric invariant of X, defined to be the least length of a noncontractible loop in X (more detailed definitions appear in Chapter 5). We will denote it $\mathrm{sys}\pi_1$, following M. Gromov's notation [**Gro96**]. When X is a graph, the invariant is usually referred to as the *girth*, ever since the 1947 article [**Tu47**] by W. Tutte (1917-2002), see Subsection 2.4.2 below. Possibly inspired by Tutte's article, C. Loewner started thinking about systolic questions on surfaces in the late 1940's, resulting in a 1950 thesis by his student P.M. Pu.

1.1.1. Thom. This line of research was, apparently, given further impetus by a remark of the venerable René Thom, in a conversation with Marcel Berger in the library of Strasbourg University during the 1961-62 academic year, shortly after the publication of the papers of R. Accola [**Ac60**] and C. Blatter [**Bl61a**]. Referring to these systolic inequalities, Thom reportedly exclaimed: *"Mais c'est fondamental!"* [These results are of fundamental importance!] Subsequently, M. Berger popularized the subject in [**Berg65, Berg93, Berg03**]. For an intuitive description of the problem, see [**Berg70**]. M. Berger's personal account of systolic history appears in Section 2.1.

C. Loewner's seminal ideas in what is known today as systolic geometry have had a slow start, with only a pair of articles appearing in the 1950's: by J. Hersch [**Her55**] and P. Pu [**Pu52**]; three articles in the 1960's: by R. Accola [**Ac60**], M. Berger [**Berg65**], and C. Blatter [**Bl61a**]; and four in the 1970's: by Berger [**Berg72a, Berg72b**], P. Buser [**Bus78**], and I. Chavel [**Cha72**]. Over a dozen publications appear in the 1980's: by W. Abikoff [**Ab86**], C. Bavard [**Bav86, Bav88**], section in a book by Y. Burago and V. Zalgaller [**BuraZ80**, p. 43], C. Croke [**Cr88**], M. Gromov [**Gro81a, Gro83**], R. Harvey and H. B. Lawson [**HarL82**], J. Hebda [**Heb81, Heb86**], F. Jenni [**Je84**], S. Kodani [**Kod87**], T. Sakai [**Sak88**], and others.

The systolic geometry website currently lists over a hundred articles. The website at http://www.math.biu.ac.il/∼katzmik/sgt.html is regularly updated, see Section 2.4.

Systolic geometry is a rapidly developing field, featuring a number of recent publications in leading journals. Recently, an intriguing connection has emerged with the Lusternik-Schnirelmann category, which may signal the genesis of a subject that could be called "systolic topology", alluded to in the book title.

1.1.2. Flavor. To give a preliminary idea of the flavor of the field, one could make the following observations.

The main thrust of Thom's remark to Berger quoted in Subsection 1.1.1 above, appears to be the following. Whenever one encounters an inequality relating geometric invariants, such a phenomenon in itself is interesting; all the more so when the inequality is sharp (*i.e.* optimal). The elegance of the inequalities of C. Loewner, P. Pu, and M. Gromov (see below) is indisputable, *cf.* [**Gro99**, Systolic reminiscences, p. 271].

In systolic questions about surfaces, integral-geometric identities play a particularly important role. Roughly speaking, there is an integral identity relating area on the one hand, and an average of energies of a suitable family of loops, on the other. By the Cauchy-Schwarz inequality, energy is an upper bound for length squared, hence one obtains an inequality between area and the square of the systole. Such an approach works both for the Loewner inequality for the torus,

$$\mathrm{sys}\pi_1(\mathbb{T}^2)^2 \leq \gamma_2 \, \mathrm{area}(\mathbb{T}^2), \tag{1.1.1}$$

where $\gamma_2 = \frac{2}{\sqrt{3}}$, *cf.* (5.4.1), and for Pu's inequality for the real projective plane,

$$\mathrm{sys}\pi_1(\mathbb{RP}^2)^2 \leq \tfrac{\pi}{2} \, \mathrm{area}(\mathbb{RP}^2), \tag{1.1.2}$$

cf. (5.2.2). Biographical notes on C. Loewner and P. Pu appear in Sections 2.2 and 2.3, respectively.

Similar inequalities exist for the Klein bottle [**Bav86**], as well as for the Möbius band [**Bl61b**]. A number of new inequalities of this type have recently been discovered, including universal volume lower bounds, *cf.* [**CrK03**].

1.1.3. Gromov's spectacular inequality. The inequality of M. Gromov [**Gro83**, (0.1), p. 3] for the homotopy 1-systole of an essential n-manifold M,

$$\mathrm{sys}\pi_1(M)^n \leq C_n \, \mathrm{vol}_n(M), \tag{1.1.3}$$

cf. (12.2.1), is the deepest result in the field. I would compare it to Gromov's theorem on almost flat manifolds [**Gro78**], which also originally appeared about a quarter century ago.

The difference between these two results of Gromov's is that in the latter case, P. Buser and H. Karcher almost immediately set to work, and produced a splendid monograph [**BusK81**] to explain the theorem. In the case of Gromov's systolic inequality, no such monograph has appeared, neither immediately afterwards, nor any time later. Gromov's 1999 book [**Gro99**] explains the general idea behind the proof, but does not present the full proof. To my knowledge, the original dense 35 page proof in [**Gro83**] is the only reference where the inequality is proved in full.

Gromov's proof of (1.1.3) involves a version of what is known in minimal surface theory as the monotonicity formula, but in a situation when the ambient space is a Banach space, combined with an application of the deformation theorem from geometric measure theory. Gromov's proof passes via a number of auxiliary inequalities (*e.g.* cone inequality, isoperimetric inequality, coarea formula), and involves additional invariants, such as diameter and filling volume. A presentation of a full proof would fall outside the scope of the present monograph, which focuses on the research of the past four years. A summary of a proof, based on recent results in geometric measure theory by S. Wenger [**Wen05**, Lemma 3.4], building upon earlier work by L. Ambrosio and B. Kirchheim [**AmK00**], appears in Section 12.2 below.

A recent application of (a generalisation of) Gromov's 1983 inequality, can be found in an article by G. Besson, G. Courtois, and S. Gallot [**BesCG03**], *cf.* Remark 12.2.5 below.

1.1.4. Gromov's optimal inequality for complex projective space. A significant difference between 1-systolic invariants (defined in terms of lengths of loops) and the higher, k-systolic invariants (defined in terms of areas of cycles, *etc.*) should be kept in mind. While a number of optimal systolic inequalities, involving the 1-systoles, have by now been obtained and will be detailed in this book, just about the only optimal inequality involving purely the higher k-systoles is M. Gromov's optimal stable 2-systolic inequality for complex projective space,

$$\mathrm{stsys}_2(\mathbb{CP}^n, \mathcal{G})^n \le n! \, \mathrm{vol}_{2n}(\mathbb{CP}^n, \mathcal{G}), \qquad (1.1.4)$$

cf. (13.2.4), where the optimal bound is attained by the Fubini-Study metric.

Just how exceptional inequality (1.1.4) is, only became clear recently. Namely, we discovered that, contrary to expectation, the symmetric metric on the quaternionic projective plane $\mathbb{HP}^2$ is *not* its systolically optimal metric (see [**BaKSS06**]), in contrast with the complex projective case, see Subsection 1.3.3 below for more details. As of this writing, no optimal inequality is available for the quaternionic or the Cayley two-point homogeneous spaces, *cf.* Section 13.3.

Similarly, just about the only nontrivial *lower* bound for a k-systole with $k \ge 2$, namely

$$\sup_{\mathcal{G}} \, \mathrm{confsys}_2 \left(\mathbb{CP}^2 \# n \overline{\mathbb{CP}}^2, \mathcal{G} \right)^2 \ge \sqrt{\tfrac{n}{2\pi e}}(1 + o(1)), \text{ for } n \to \infty, \qquad (1.1.5)$$

cf. (20.1.2), results from recent work in gauge theory and J-holomorphic curves.

1.1.5. Schottky problem. Perhaps one of the most striking applications of systoles is in the context of the Schottky problem, by P. Buser and P. Sarnak [**BusS94**], who distinguished the Jacobians of Riemann surfaces among principally polarized abelian varieties. They showed that the locus of the Jacobians is small because their systoles are severely restricted (by the right-hand bound of (1.1.6) below) compared to the general case. Furthermore, they exhibited Riemann surfaces Σ_g of genus g, satisfying also the lower bound below:

$$C^{-1} \log g < \mathrm{confsys}_1(\Sigma_g)^2 < C \log g, \text{ for } g \to \infty \qquad (1.1.6)$$

cf. (20.6.1).

1.1.6. Tools and techniques. Asking systolic questions often stimulates questions in related fields. Thus, a notion of "systolic category" of a manifold is defined and investigated in Chapter 12, where we exhibit a connection to the Lusternik-Schnirelmann (LS) category. Note that the systolic category (as well as the LS category) is, by definition, an integer (not to be confused with the foundational notion of "category theory"). Once the connection is established, the influence is mutual: known results about LS category stimulate systolic questions, and vice versa.

Asymptotic phenomena for the systole of surfaces of large genus have been shown to be related to interesting ergodic phenomena in Chapter 11, and to properties of congruence subgroups of arithmetic groups in [**KatzSV07, Vis07**].

The study of lower bounds for the conformal 2-systole of 4-manifolds has led to a simplified proof of the density of the image of the period map, *cf.* Appendix A.

To illustrate the variety of the tools involved, we mention a few connections to other areas:

- Banaszczyk's functional analytic results on the successive minima of a pair of dual lattices;
- dynamical systems notion of volume entropy and Katok's inequality;
- the topological techniques of Lusternik-Schnirelmann category;
- quaternion algebras and congruence subgroups of arithmetic Fuchsian and Kleinian groups;
- J-holomorphic curves and Seiberg-Witten invariants.

1.1.7. Statement of problem. Let M^n be a closed smooth n-dimensional manifold, and let $k > 0$ be an integer no greater than n. We study "curvature-free" inequalities satisfied by the various flavors of k-systolic invariants of M (detailed definitions appear in Chapters 5 and 12). Here "curvature-free" means that the coefficients in the inequality are independent of the metric, in the spirit of the Loewner inequality (5.4.1) for the two-torus.

In the first part of the book (Chapters 3 through 11), we start with a brief presentation of classical and semi-classical preliminaries, and continue by presenting systolic inequalities in dimension 2. In the second part (Chapters 12 through 20), we present higher dimensional generalisations. At the cost of a slight redundancy of definitions and basic inequalities, the individual chapters are fairly self-contained. Furthermore, in each chapter we tried to give an indication of the interrelationship of various results, with a further minor increase in redundancy.

Let us now describe the contents of each chapter in more detail.

1.2. Contents of Part 1

Following five chapters (3 through 7) which deal mostly with introductory material, Chapters 8, 9, and 10 explore systolic ramifications of hyperellipticity.

1.2.1. Filling area theorem. We prove M. Gromov's filling area conjecture in the hyperelliptic case (Chapter 8). These results first appeared in [**BaCIK05**]. In particular, we establish the conjecture for all genus 1 fillings of the circle, extending P. Pu's result in genus 0. We translate the problem into a question about closed ovalless real Riemann surfaces. The conjecture then results from a combination of two ingredients. On the one hand, we exploit an integral geometric result of J. Hersch [**Her55**]. The result permits an optimal comparison of the given metric, with orbifold metrics of constant positive curvature on real surfaces of even positive genus. Here the singular points are Weierstrass points. On the other hand, we exploit an analysis of the combinatorics on unions of closed curves, arising as geodesics of such orbifold metrics.

1.2.2. Loewner inequality and hyperellipticity. It turns out that the Loewner inequality for the torus is also satisfied by metrics, on higher genus surfaces Σ, whose underlying conformal structure is hyperelliptic (Chapter 9). These results first appeared in [**KatzS06a**]. In genus 2, we first construct the Loewner loops on the (mildly singular) companion tori, locally isometric to Σ away from Weierstrass points. The loops are then transplanted to Σ, and surgered to obtain a Loewner loop on Σ. In higher genus, we exploit M. Gromov's area estimates for ε-regular metrics on Σ.

1.2.3. Genus two, CAT(0) metrics, and Bolza surface. The optimal systolic ratio of the genus 2 surface is still unknown, inspite of sustained efforts of a number of researchers, *e.g.* [**Cal96, Bry96**]. Meanwhile, a precise answer can be obtained in the context of nonpositively curved metrics (Chapter 10). These results first appeared in [**KatzS06b**]. We prove an optimal systolic inequality for an arbitrary CAT(0) surface Σ_2 of genus 2, namely

$$\mathrm{sys}\pi_1(\Sigma_2)^2 \leq \tfrac{1}{3} \cot \tfrac{\pi}{8} \ \mathrm{area}(\Sigma_2), \qquad (1.2.1)$$

cf. (10.1.1). We use a Voronoi cell technique, introduced by C. Bavard [**Bav92**] in the hyperbolic context. The optimal bound (1.2.1) is attained by a flat singular metric in the conformal class defined by the smooth completion of the affine complex algebraic curve $y^2 = x^5 - x$. Thus, among all CAT(0) metrics, the one with the best systolic ratio is composed of six flat regular octagons centered at the Weierstrass points of the Bolza surface.

1.2.4. Katok's inequality, volume entropy, and systole. Chapter 11 examines asymptotic bounds for the optimal systolic ratio of a surface whose genus tends to infinity. Asymptotic questions in systolic geometry can be studied via the volume entropy. These results first appeared in [**KatzS05**]. We find an upper bound for the entropy of a systolically extremal surface, in terms of its systole. We combine the upper bound with A. Katok's lower bound [**Kat83**] in terms of the volume, to obtain a simpler alternative proof of M. Gromov's asymptotic estimate for the optimal systolic ratio of surfaces of large genus. Furthermore, we improve the multiplicative constant in Gromov's estimate. Namely, we show the following bound:

$$\mathrm{sys}\pi_1(\Sigma_g)^2 \leq \frac{\log^2 g}{\pi g} \, \mathrm{area}(\Sigma_g)(1 + o(1)) \quad \text{when } g \to \infty, \qquad (1.2.2)$$

cf. (11.1.4). We show that every surface of genus at least 20 is Loewner.

Note that asymptotic *lower* bounds for the systole were studied in 1994 by P. Buser and P. Sarnak [**BusS94**]. They constructed hyperbolic surfaces of large genus, whose systole behaves logarithmically in the genus. The Fuchsian groups in their examples are principal congruence subgroups of a fixed arithmetic group with rational trace field. Their construction can be generalized to principal congruence subgroups of arbitrary arithmetic surfaces [**KatzSV07**], by exploiting a new trace estimate valid for an arbitrary ideal in a quaternion algebra. The estimate yields a particularly sharp bound for a principal congruence tower of Hurwitz surfaces (PCH), *i.e.* $(2,3,7)$ triangle surfaces, namely

$$\mathrm{sys}\pi_1(\Sigma_{\mathrm{PCH}}) \geq \frac{4}{3} \log g(\Sigma_{\mathrm{PCH}}). \qquad (1.2.3)$$

Similar results are obtained in [**KatzSV07**] for the systole of hyperbolic 3-manifolds, compared to their simplicial volume, *cf.* [**AdR00**].

1.3. Contents of Part 2

The second part of the book deals with higher dimensional generalisations, with regard to both the ambient manifold, and the systolic invariants themselves.

1.3.1. Systolic category and Lusternik-Schnirelmann category.   We open the second part of the book, by defining the homotopy systole, homology systoles, stable systoles, and systolic category in arbitrary dimension (Chapter 12). Note that both systolic and LS categories are integers, not to be confused with the foundational notion of "category theory".

The invariant called *systolic category*, denoted $\mathrm{cat}_{\mathrm{sys}}(M)$, was first defined in [**KatzR06**]. It is a Riemannian analogue of the Lusternik-Schnirelmann category $\mathrm{cat}_{\mathrm{LS}}(M)$. The invariant $\mathrm{cat}_{\mathrm{sys}}(M)$ is defined in terms of the existence of systolic inequalities satisfied by every metric $\mathcal{G}$ on M, as initiated by C. Loewner and later developed by M. Gromov. Such an approach provides a topological framework in which systolic questions can be studied. We compare the two categories. In all our examples, the inequality

$$\mathrm{cat}_{\mathrm{sys}}\, M \leq \mathrm{cat}_{\mathrm{LS}} M \qquad (1.3.1)$$

is satisfied, which typically turns out to be an equality, *e.g.* in dimensions 2 and 3. We show that a number of existing systolic inequalities can be reinterpreted as special cases of such equality. The comparison with the value of $\mathrm{cat}_{\mathrm{LS}}(M)$ leads us to prove or conjecture new systolic inequalities on M.

1.3.2. Optimal inequality for $\mathbb{CP}^n$, Massey products. Given a compact Riemannian manifold $(M, \mathcal{G})$, its real homology group $H_*(M, \mathbb{R})$ is naturally endowed with the stable norm. Briefly, if $h \in H_k(M, \mathbb{R})$ then the stable norm of h is the infimum of the Riemannian k-volumes of real cycles representing h. The stable k-systole, denoted $\mathrm{stsys}_k(\mathcal{G})$, is the minimum of the stable norm over nonzero elements in the lattice of integral classes in $H_k(M, \mathbb{R})$. In Chapter 13, we prove Gromov's optimal stable systolic inequality (1.1.4) for complex projective space. The proof relies on the Wirtinger inequality.

1.3.3. Exceptional Lie algebra E_7 and systoles. We undertake the study of optimal curvature-free inequalities of the type discovered by C. Loewner and M. Gromov, using a generalisation of the Wirtinger inequality for the comass, *cf.* [**BaKSS06**]. Using a model for the classifying space BS^3 built inductively out of BS^1, we prove that the symmetric metrics of certain 2-point homogeneous manifolds turn out not to be the systolically optimal metrics on those manifolds. Thus, we obtain the following surprising bound for the optimal systolic ratio of the quaternionic projective plane:

$$6 \leq \mathrm{SR}(\mathbb{HP}^2) \leq 14, \qquad (1.3.2)$$

whereas its symmetric metric has a systolic ratio of 10/3. We point out the unexpected role played by the exceptional Lie algebra E_7 in systolic geometry, via the calculation of Wirtinger constants, see Appendix B below.

1.3.4. Massey systoles. We show that, like the Lusternik-Schnirelmann category, systolic category is sensitive to Massey products (Chapter 14). For example, we show that every metric $\mathcal{G}$ on the 19-dimensional manifold $SU(6)/SU(3) \times SU(3)$ satisfies the bound

$$\mathrm{stsys}_4(\mathcal{G})^2 \, \mathrm{stsys}_6(\mathcal{G})^2 \leq 19!\, \mathrm{IQ}(\mathcal{G})\, \mathrm{vol}_{19}(\mathcal{G}), \qquad (1.3.3)$$

cf. (14.4.2), where IQ is a suitable isoperimetric quotient.

1.3.5. Systoles and cup-length. We study the constraints imposed by multiplicative relations in the cohomology ring of a manifold, upon its stable systoles (Chapter 15). These results first appeared in [**BaK03**]. Relying on results from the geometry of numbers due to W. Banaszczyk [**Bana93**], and extending work by M. Gromov [**Gro83**] and J. Hebda [**Heb86**], we prove metric-independent inequalities for products of stable systoles. Thus, we show that if the fundamental cohomology class of an orientable manifold M^n is a cup product of classes of dimensions $k_1, \ldots, k_m$, then every Riemannian metric $\mathcal{G}$ on M satisfies the inequality

$$\prod_{j=1}^{m} \operatorname{stsys}_{k_j}(\mathcal{G}) \leq C(n) \left(\prod_j b_{k_j}(M) \left(1 + \log b_{k_j}(M) \right) \right) \operatorname{vol}_n(\mathcal{G}), \qquad (1.3.4)$$

cf. (15.2.1). Here the product can be as long as the real cup-length of M, which therefore constitutes a lower bound for the systolic category of M.

Chapters 16 and 17 contain two distinct higher-dimensional optimal generalisations of the Loewner inequality (1.1.1).

1.3.6. Systolic inequalities, dual-critical lattices, and volume preserving flows. We present a new optimal systolic inequality for a closed Riemannian manifold M^n, which generalizes a number of earlier inequalities, including those of C. Loewner and J. Hebda (Chapter 16). These results first appeared in [**BaK04**]. Namely, we show that

$$\operatorname{confsys}_1(M) \operatorname{sys}_{n-1}(M) \leq \gamma'_b \operatorname{vol}_n(M)^{\frac{n-1}{n}}, \qquad (1.3.5)$$

cf. (16.2.1), where $b = b_1(M)$ is the first Betti number, while γ'_b is the Bergé-Martinet constant (5.3.3). We characterize the boundary case of equality in terms of the geometry of the Abel-Jacobi map, or more specifically the Lichnerowicz harmonic map $\mathcal{A}_M$, of M. A metric attaining the optimal bound (1.3.5) has the property that the map $\mathcal{A}_M$ is a Riemannian submersion with minimal fibers, onto a flat torus. Here the base of $\mathcal{A}_M$ possesses a group of deck transformations which is a dual-critical lattice. The latter arise in the context of an extremal problem for Euclidean lattices, studied by A.-M. Bergé and J. Martinet [**BeM89**]. Given a closed manifold M that admits a submersion F to its Jacobi torus $\mathbb{T}^{b_1(M)}$, we construct all metrics on M that realize equality in our inequality. While one can choose arbitrary metrics of fixed volume on the fibers of F, the horizontal space is chosen using a multi-parameter version of J. Moser's method of constructing volume-preserving flows.

1.3.7. Abel-Jacobi maps and optimal inequalities. We generalize optimal inequalities due to C. Loewner and M. Gromov, by proving optimal lower bounds for the total volume in terms of the homotopy systole and the stable systole (Chapter 17). These results first appeared in [**IvK04**]. Namely, we show that every Riemannian n-manifold M with Betti number $b = b_1(M) > 0$ satisfies the inequality

$$\deg(\mathcal{A}_M) \operatorname{stsys}_1(M)^b \leq (\gamma_b)^{\frac{b}{2}} \operatorname{vol}_n(M). \qquad (1.3.6)$$

cf. (17.2.3). Here γ_b is the Hermite constant, while $\mathcal{A}_M$ is the Abel-Jacobi map. Our main tool is the construction of an area-decreasing map to the Jacobi torus, streamlining and generalizing the construction of D. Burago and S. Ivanov [**BuraI95**]. It

turns out that one can successfully combine this construction with the coarea formula, to yield new optimal inequalities. As a consequence, we obtain a lower bound for systolic category which is stronger than the real cup-length lower bound studied in Chapter 15.

We study a certain generalisation of the stable systole called the conformal systole in Chapters 18 and 20.

1.3.8. Conformal systolic inequalities. We prove certain optimal systolic inequalities for a closed Riemannian manifold $(M, \mathcal{G})$, depending on a pair of parameters, n and b, where n is the dimension of M, while b is its first Betti number (Chapter 18). These results first appeared in [**BaCIK07**]. The proof of the inequalities involves constructing Abel-Jacobi maps from M to its Jacobi torus $\mathbb{T}^b$, which are area-decreasing (on b-dimensional areas), with respect to suitable norms. These norms are the stable norm of $\mathcal{G}$, the conformally invariant norm, as well as other L^p-norms. Here we exploit L^p-minimizing differential 1-forms in cohomology classes. We characterize the case of equality in our optimal inequalities, in terms of the criticality of the lattice of deck transformations of $\mathbb{T}^b$, while the Abel-Jacobi map is a harmonic Riemannian submersion. That the resulting inequalities are actually nonvacuous follows from an isoperimetric inequality of Federer and Fleming, under the assumption of the nonvanishing of the homology class of the lift of the typical fiber of the Abel-Jacobi map to the maximal free abelian cover.

1.3.9. Conformal 2-systole and the surjectivity conjecture. The asymptotic behavior of the conformal 2-systole of four-manifolds is studied in Chapter 20. These results first appeared in [**Katz03**]. In 1994, P. Buser and P. Sarnak showed that the maximum, over the moduli space of Riemann surfaces of genus g, of the least conformal length of a nonseparating loop, is logarithmic in g. We present an application of (polynomially) dense Euclidean packings, to estimates for an analogous 2-dimensional conformal systolic invariant of a 4-manifold M with indefinite intersection form. The estimate turns out to be polynomial, rather than logarithmic, in $b_2(M)$:

$$\sup_{\mathcal{G}} \operatorname{confsys}_2 \left(\mathbb{CP}^2 \# n \overline{\mathbb{CP}}^2, \mathcal{G} \right)^2 \geq \sqrt{\tfrac{n}{2\pi e}} (1 + o(1)), \text{ for } n \to \infty. \qquad (1.3.7)$$

It relies on the results of T. Li and A. Liu [**Li01**] concerning the density of the image of of the period map, *cf.* Appendix A. These results allow one to insert suitable lattices with metric properties prescribed in advance, into the second de Rham cohomology group of M, as its integer lattice. The idea is to adapt the well-known Lorentzian construction of the Leech lattice, by replacing the Leech lattice by the Conway-Thompson unimodular lattices which define asymptotically dense packings. The final step can be described, in terms of the successive minima λ_i of a lattice, as deforming a λ_2-bound into a λ_1-bound.

1.3.10. Image of period map and J-holomorphic curves. J. Solomon wrote an appendix for the book (Apendix A), proving the density of the image of the period map for blow-ups of the complex projective plane, a result originally due to T. Li and A. Liu [**Li01**]. Such density is proved using the correspondence between Seiberg-Witten and Gromov-Witten invariants developed by C. Taubes, as well as recent results in gauge theory, symplectic geometry, and J-holomorphic curves.

To summarize the relation of this result to systoles, note that Chapter 20 proved a relative result, to the effect that *if* the image is dense, *then* one has a certain lower bound for the conformal 2-systole. Now that one has such density, the relative result becomes an absolute theorem. The theorem is the only non-trivial lower bound in the literature for k-systoles with $k \geq 2$, in contrast with the abundance of such bounds for $k = 1$, some of which have been described in the book.

The existence of metrics asserted by this theorem poses a challenge to the community of Riemannian geometers. The gauge-theoretic proof of the existence of such metrics gives no inkling of what they might look like, or how to construct them using traditional tools of Riemannian geometry.

We conclude with a final Appendix B containing a list of open problems in systolic geometry, topology, and arithmetic.

Historical remarks

2.1. A la recherche des systoles, by Marcel Berger

This section contains Marcel Berger's personal story concerning systoles, as told by the master in the summer of 2005. We have limited editorial changes to a bare minimum, so as to preserve the flavor of the original account.

2.1.1. Thom on Blatter. "To the best and fairness of my knowledge, here is my recollection of my relations with systoles.

"For me things started in the very small [room of the] Strasbourg Mathematics Department Library, I think in the fall of 1961 or in the spring of 1962, I cannot remember [exactly]. [René] Thom was there, looking at the newly arrived journals, I was close to him in the small room. Of course he was my hero. Then he came upon the Blatter paper (Blatter 1961), and spoke to me, saying, 'You know Berger (still now most French mathematicians call between them or quoting their [last] names not by the first name), this paper is very interesting because it concerns the general problem to find universal (independent of the metric) relations between the volumes of various homology, etc. classes.'

"Coming from Thom, I considered that question as good (and natural also, which I like). I started looking at various generalizations. I was really getting nowhere. The only thing I did was to learn [P.M.] Pu's proof, and then to make propaganda for the topic in my Bombay lectures (Berger 1965).

2.1.2. Chern, Wirtinger inequality, and calibration. "I then remember talking about Thom's problem with [S.S.] Chern (I think in Berkeley where I was spending the summer quarter 1968 or the summer of 1969, I am not sure) about the question. He told me two things, the first was that his feeling was that such a universal inequality could be true in the middle dimension (for even dimensions). The second thing he did when thereafter I asked him about the complex projective space, since I was trying to look at Pu generalization to other projective spaces. He told me for the complex case that this followed from Wirtinger inequality (I did not know Wirtinger stuff at that time).

"I kept working with the topic, and I could really get nowhere. The only thing not trivial I could prove was that the inequality was OK for the quaternionic projective space and the Cayley plane (for their standard metric). To do this I had only to prove that the 4-form (respectively, the 8-form) was what is called today calibrating after (Harvey and Lawson 1982). The quaternionic case was easy, but I had a hard time to prove the Cayley case. To make a living, I published the two papers (Berger 1972a) and (Berger 1972), but you perfectly know that there is nothing in them, except propaganda for the subject, and the two calibrations. Then I [made] no contribution, except having kept making propaganda. You can

look at the wonderful introduction book (Morgan 1988-2000), page 75, section 6.5: history of calibration. I kept making propaganda, in lectures, and by some writing (in part).

"Then came [M.] Gromov, and his followers; I still continue propaganda, but as you know the subject started soon to be popular.

2.1.3. Three notes. "The first is the explanation for the two 'funny' titles, in case you are not familiar with the famous French novel 'A la recherche du temps perdu', by Marcel Proust. The year 1971 was Proust centenary, so I used that pretext to imitate the first two sections of his novel which are called 'Du côté de chez Swann', and 'A l'ombre des jeunes filles en fleur'.

"The second thing is the origin of the word 'systole', in case I have not already told you. I was looking at [the] time for a word of the type 'iso-???-ic' both for the systoles and for the injectivity radius. I looked at Greek language dictionaries (French to Greek, basically there are no good ones) and found various wording. [Luckily], I was doing physical education together with a Greek literature colleague; he told me what I was proposing was 'low Greek', of course I explained to him, in ordinary words, what were a systole and the injectivity radius. The next week physical education session he came back with the two proposal: 'isoclysteric' and 'isosphincteric'. I was still young in the bad sense, say provocative, I was [amused]; but I told these two wording to Besse seminar next week, they were horrified and told me 'Marcel, you cannot use that'. So at the next session I asked him to find a less bad taste 'ic' stuffs. Next week he came back with "isosystolic" and 'isoembolic'. Seminar people were happy; you understand that in short I switched from proctology to cardiology. Both wording were immediately accepted by Gromov.

"The third thing is more important. In the same library, Thom asked me another question: 'Berger, I give you a compact manifold: does there exist a preferred metric on it?' This started my propaganda for Einstein manifolds. So that in conclusion, after my dissertation and the pinching stuff, all my interests were given to me by Thom."

2.1.4. Bibliography for this section. The above account refers to the following texts.

(1) Berger, M. (1965). Lecture on Geodesics in Riemannian Geometry, Tata Institute of Fundamental Research.
(2) Berger, M. (1972). A l'ombre de Loewner. Ann. Scient. Ec. Norm. Sup. 5: 241-260.
(3) Berger, M. (1972a). Du côté de chez Pu. Ann. Scient. Ec. Norm. Sup. 5: 1-44.
(4) Blatter, C. (1961). Über Extremallänge auf geschlossenen Flächen. Comment. Math. Helvetici 35: 153-168.
(5) Harvey, R. and Lawson, H.B. (1982). Calibrated geometries. Acta Mathematica 148: 48-157.
(6) Morgan, F. (1988-2000). Geometric measure theory: a beginner's guide, third edition in 2000, Academic Press.

2.2. Charles Loewner (1893-1968)

In this section, and in Section 2.3 below, we include biographical notes for the two pioneers of systolic geometry.

A considerable amount has been written about Loewner's Mathematics in the biographies available in the literature and on the internet (see also Subsection 2.2.12 below). On the human side, what one finds in the biographies is invariably brief. This side of him is perhaps as much his heritage as the Mathematics. To this day, some colleagues who knew him personally, regret that there are not more mathematicians like him nowadays. Others still remember fondly their contacts with Loewner, nearly half a century later. We aim to bring out this aspect of Loewner the man in the notes below.

2.2.1. Childhood. Originally named Karel (later Karl) Löwner, he was born into an Orthodox Jewish family in the village of Lány, near Prague, which was at the time the capital of Bohemia.

Karel's father, Sigmund (or Zygmund) Löwner, owned a general store in Lány which supplied the needs of all the villagers, as well as those of the local Count, whose castle dominated the village. Inspite of this, the family was on a strict budget, as the goods were sold at a very small markup. Thus caring for others and helping them, the elder Löwner planted the seeds of the *middah* (character trait) of *rachmanus* (see below) that would later blossom in his offspring. *Rachmanus* can be loosely translated as concern for others, pity, or lovingkindness; see gemara Talmud Babli, maseches Beitzah, page 32b; as well as Maseches Yevamos, page 79a. Karel lost his father close to bar mitzvah age, six years prior to the onset of World War 1. In accordance with Jewish law, he said the traditional Kaddish prayer for his father every day for a year following his loss.

In line with Sigmund's wishes, Karel studied in a Prague gymnasium where German was not only taught, but was also the language of instruction. The attendant infringement upon traditional Jewish learning, may have seemed a small price to pay for the entry ticket to Europe's most refined ivory towers. Sigmund's infatuation with all things German has been noted by biographers, *cf.* editor's introduction to [**Lo88**].

2.2.2. The Holocaust. Whether or not the infatuation went sour, a generation later, as many as four of Sigmund's children were destined to perish in the Holocaust. Meanwhile, his son Viktor (Karel's only sibling to survive the war) married out (well before the war), fabricated a genealogical tree (perhaps to save his life) purporting to prove that his father's family was not Jewish in the first place, and switched to an unrelated surname (derived from the name of his hometown).

A helpful website on mathematicians under the Third Reich is maintained by Prof. Thomas Huckle at Munich, detailing also Karel's narrow escape from the Nazis' clutches. See also Rachel Pomerantz's highly informative book *The world in flames. A short history of the Holocaust* [**Po93**] for background, root causes, and lessons of the Holocaust.

Karel Löwner held several positions in Germany in the 1920's, and later in Prague, before fleeing for his life from the Nazis. According to his *curriculum vitae* from 1939, his students in Europe were E. Lammel, F. Kraus, O. Dobsch, and Ch. Bers. The latter is Lipman Bers, destined to become his colleague, neighbor, and ultimately editor [**Lo88**].

2.2.3. Berlin. We will spare the reader the litany of mathematical celebrities who surrounded Löwner at Berlin University, typically found in internet McBiographies. A name conspicuous through its absence from such lists is that of Ludwig

Bieberbach (see below). It needs to be mentioned that among Löwner's colleagues in Berlin in the 1920's was Albert Einstein, who played a pivotal role in securing his first job in the US, see Subsection 2.2.7.

Bieberbach was a major influence on Löwner. He may have been the source of the invitation for Löwner to leave his central European university, where he defended his thesis under Georg Pick in 1917, and come to Berlin in 1922. Löwner's famous proof of the Bieberbach conjecture in the first highly nontrivial case, that of the third coefficient, was published the following year [**Lo23**]. The manuscript was sent to Bieberbach for confirmation when it was submitted for publication. In accepting the work, Bieberbach wrote in the margin that it was an outstanding contribution. Löwner's *Habilitation* was mainly reviewed by Bieberbach in 1923 [**Bie88**, pp. 203, 367].

2.2.4. Aryan *versus* Jew. Bieberbach's reputation as a mathematician is solid. Alas, he was also a Nazi sympathizer, capable of arriving to a university lecture while sporting a brownshirt uniform, of harrassing his Jewish colleagues during the 1930's, as well as of pursuing a sinister thesis concerning the existence of a dichotomy of Aryan *versus* Jewish Mathematics. Ostensibly, his dichotomy centers on purely didactic and pedagogical qualities; yet the Aryan aversion to the Jewish trait of *rachmanus* (pity; *cf.* Subsection 2.2.1) is well-known [**Po93**].

The disappointment with Bieberbach is particularly acute for those who wish to believe that toiling in Mathematical pursuits provides a vaccine of sorts against indecency [**Go83**, vol. 2, p. 267, lines 19-25].

As a reaction against Bieberbach's indecency (*cf.* Subsection 2.2.12), his name is routinely doctored out of lists of mathematical celebrities at Berlin University during Löwner's employment there. The name only appears as an afterthought at the end of Bers' introductory essay in [**Lo88**], in the context of a brief discussion of his conjecture.

2.2.5. Paradox. Yet, there is a bit of a paradox in attempting to obscure Bieberbach's mathematical influence on fellow mathematicians. The fact is that O. Teichmuller's name has been canonized in the name of the space covering the moduli space of Riemann surfaces, inspite of his Nazi sympathies (which were even more extreme than Bieberbach's). As it happens, Bers himself actually took part in such a canonisation, although the author of these lines personally heard him express regrets about his role in it, in a lecture at Columbia University in the 1980's.

Louis de Branges succeeded in proving Bieberbach's 1916 conjecture, over half a century later [**Bra85**]. He feels that Löwner would not have undertaken the extremely difficult project of carrying out the necessary estimates for the case of the third coefficient, were it not for the guiding and directing force of Bieberbach's intuition. De Branges further wrote in the fall of 2006:

> [Löwner] uses an original construction which was not appreciated by his colleagues, and indeed is not even cited in the 1978 book by H. Grunsky [**Grun78**]. No one seems to know how Löwner had the confidence to undertake the proof of what was generally considered to be a wild and improbable conjecture by a mathematician who was noted for imprecision in his publications.

An introduction to the proof of the conjecture may be found in [**Kor86**].

2.2.6. Emigration. In an effort to leave the continent and escape its Nazi menace, Löwner applied for university positions both in England and in America. In a testimony to a trait of decency and unselfishness that will have characterized an entire lifetime, Richard von Mises wrote from Istanbul, in a 1939 letter of recommendation for Löwner:

> During his activity at the University of Berlin, [Löwner] was, among all the instructors in Mathematics, the one who had the strongest influence upon the students, stimulated them to independent research, and helped them in his unselfish way. Much of the work involved in the published theses of his pupils is not only due to his influence, but can in a true sense be considered as his work.

See also Subsection 2.2.11 below.

2.2.7. America. Loewner's first job in the United States, already under the suitably Americanized spelling of the surname *sans umlaut*, was at the University of Louisville, Kentucky, arranged by John von Neumann in 1939. However, the deal was not clinched until Loewner's former colleague Einstein agreed to transfer an offprint of the original edition of his renowned work on relativity theory [**Ei13**] to a lawyer and collector at the University of Louisville named W. Bullitt. The autographed offprint is currently in the Bullitt collection of the University of Louisville [**Dav89**].

Loewner's phenomenal generosity immediately made itself felt in Louisville. Thus, he conceded to a request by undergraduate students to offer an advanced Mathematics course, which had to be taught without remuneration. The only available location turned out to be the local beer brewery. Loewner did in fact teach his course there, at a vigorous 7 a.m. time slot, before the arrival of the first shift of brewery workers.

2.2.8. Character. Loewner's daughter recalls that one of his main character strengths was the ability to listen to people and empathize with their viewpoint, even when disagreeing with them. This sometimes led people of many different persuasions to assume, by projection, that he was in agreement with them, whether or not this may have been the case. At the same time, he would always seek out a point of agreement, and back them up on it. This may have been one of the secrets of his popularity, and certainly helped navigate the political tensions of the postwar decade.

2.2.9. Syracuse. After a period of employment at Brown University during World War 2, Loewner obtained a permanent position at Syracuse University. One of his students here was P.M. Pu, see Section 2.3.

When a nephew, Paul Gráf, a Holocaust survivor, arrived from Europe in 1947, the Loewners not only welcomed him into their home, but adopted him.

For a time, it seemed as though Syracuse would become one of the leading research departments in Mathematics. However, certain tensions arose within the department, which may have been exacerbated by the parallel political tensions of the decade. Some of the faculty were called in for questioning by the House of Representatives' Committee on Un-American Activities (HUAC). Certain appointments were made against the wishes of the star mathematicians in the department.

In the end, many of the star faculty ended up leaving Syracuse. Loewner left for Stanford University, where he would stay for the rest of his life.

2.2.10. Identity. Though unobservant as an adult, Loewner "deeply felt his Jewish identity" [**Lo88**, p. ix, line 18]. "A person of Jewish faith", as Loewner is revealingly described by his colleague A. Gelbart [**Gelb53**, p. 1571] in his testimony before the HUAC, he was a decent man and *baal rachmanus* (someone possessing the trait of *rachmanus*, *cf.* Subsection 2.2.1), by all accounts.

A telling circumstance is that, inspite of professional links with leftist academics, Loewner himself was never cross-examined by the HUAC. This circumstance suggests that, contrary to an editorial claim in [**Lo88**, p. ix, line 20], he was a largely apolitical man, as well. Artfully planted in the midst of a rhetorical flourish, there is a claim in [**Lo88**, p. ix, line 20] that Loewner was "a man of the left". Yet, Loewner himself would have likely shunned the Procrustean bed of the left-right political dichotomy, his sympathy for H. Wallace's ideas, and his appeal for clemency for E. and J. Rosenberg, notwithstanding.

The aforementioned editorial claim in [**Lo88**, p. ix, line 20] appears to be a personal projection by an admiring student, see Subsection 2.2.8 above. Indeed, the teacher, unlike the student, never called himself a *menshevik*, *cf.* [**AlAR90**, p. 20], [**AlR87**]. Nor is there any evidence that Loewner shared Bers' sense of belonging to a strain of marxism euphemistically described as "the [J.] Martov tradition", in an interview Bers gave to Constance Reid, of *Hilbert–Courant* [**Re86**] fame, see [**AlAR90**, p. 21].

While the aforementioned editorial claim has been recycled by internet biographies (of both men), one of his colleagues at the time feels today that Loewner was never engaged politically.

2.2.11. Students. Loewner felt particular resposibility toward students who were not among his strongest ones. His personal investment in his students was proverbial; a colleague of his reports, only partly in jest, that "it became generally known that the quality of his students' dissertations tended to vary inversely as the quality of the student". Some of his students' theses were almost entirely Loewner's own work, in America as in Europe, see Subsection 2.2.6 above.

With consistent generosity, the Loewners always had Charles' graduate students for dinner on holidays, while his wife Elisabeth (née Alexander) was still alive. He lost her in 1956. Their daughter recalls that her father insisted on always keeping the only telephone in the house, in his bedroom, "so as to be able to discuss his students' and colleagues' problems whenever they called, even after he had gone to bed". When a Chinese student found himself without suitable housing, Loewner took him into his home in Syracuse for a while.

2.2.12. Mathematics. One of Loewner's central scientific contributions is his proof of the case $n = 3$ of the celebrated Bieberbach conjecture, exploiting what is known today as the Loewner differential equation. The latter turned out to be instrumental in the proof of the conjecture [**Bra85**], see Subsection 2.2.3. The stochastic Loewner equation provided the inspiration for the work of Wendelin Werner, a 2006 Fields medalist.

Two of Loewner's most prolific students, Adriano Garsia and Lipman Bers, have each on the order of a hundred publications, in widely divergent fields. Both have started their own schools, each numbering dozens of students, *cf.* [**Gar06**].

Loewner's seminal role in systolic geometry is discussed in Section 1.1 above.

As early as 1955, Bers urged Loewner to publish a book based on his course at the University of California at Berkeley, and even proposed a publisher. The fact that the book [**Lo71**] was only completed posthumously, is surely a reflection of Loewner's dedication to his students, whom he continued to supervise even after retiring officially from Stanford.

In addition to his published articles, invariably path-breaking, a lot of his work was published, with his encouragement, by his students, for which he never claimed credit. Thus, it would be difficult to gauge the full extent of his influence as one of last century's great mathematicians. But first and foremost, he was a decent man and a *baal rachmanus*, *cf.* Subsections 2.2.4 and 2.2.10.

2.2.13. Acknowledgments. The above notes drew upon interviews and/or email exchanges with Loewner's children, Mrs. Marian Tracy and Mr. Paul Loewner; with his colleagues, Prof. Robert Finn, Prof. Gene Golub, and Prof. Zvi Ziegler; with his colleague's son, Prof. Steve Gelbart; with Prof. Reinhard Siegmund-Schultze; with Mrs. Delinda Buie, curator, Ekstrom Library, University of Louisville; with Prof. Louis de Branges; and with Rodney Ross, Center for Legislative Archives (NARA), Washington, D.C. The author is grateful to all of them for invaluable help.

2.3. Pu, Pao Ming (1910-1988)

2.3.1. Thesis under Loewner. The first name is spelled Bao Ming in the recently modified system of transliteration. Pu received his Ph.D. at Syracuse University in 1950 under the supervision of C. Loewner, resulting in the publication of the seminal paper [**Pu52**] containing both inequalities (1.1.1) and (1.1.2). The listing at the website of the Mathematical Genealogy Project indicates that his first name, according to Syracuse University records, was *Frank*.

Following Chiang Kai-shek's ouster from the mainland in 1949, there was apparently a wave of recalls of Chinese academics working in the West. One such case was Wu, Wen Tsün, a student of Cartan's in Paris, who one day disappeared from Paris without saying a word to anyone.

2.3.2. Mainland. Pu may have also been forced to return to the mainland by the communist authorities. He was professor at Sichuan University, Chengdu. Perhaps because someone who had spent time in the West was automatically suspect, Pu was unable to advise students for most of his scientific career. According to the official obituary at [**Lu90**], he was finally granted the coveted status of *boshi sheng dao shi* (supervisor of graduate students) at the ripe age of 71, four years after the end of the infamous cultural revolution. Most of his later papers concern fuzzy topology.

2.4. A note to the reader

2.4.1. Website. The systolic geometry website currently lists over a hundred articles and over fifty contributors, and is regularly updated. One can visit the website at

http://www.math.biu.ac.il/∼katzmik/sgt.html

(suggestions and corrections are welcome). Please email the author at the address katzmik@math.biu.ac.il with any comments, which will be used to improve a possible new edition of the book.

2.4.2. What is not in this book. One particularly significant omission is the vast literature, of over a thousand articles, on the girth of graphs, originating with Tutte's article [**Tu47**] (see the first paragraph of Section 1.1 above), and particularly with the influential "solution to advanced problem" he published under the charming pseudonym of Blanche Descartes [**UD54**].

The techniques used in this subfield of graph theory are somewhat disjoint from those used in the present book. A discussion of girth, Moore graphs, and related literature may be found in B. Bollobás [**Bol78**, Chapter 3]. See also A. Lubotzky [**Lub94**].

Similarly, no effort is made to account for the extensive literature on dense sphere packings and lower bounds for the Hermite constant. A related bibliography may be found in [**RT90**] and [**ConS99**].