Preface

This book summarizes recent developments in the theory of classical Painlevé equations based on the so-called Riemann-Hilbert, or Isomonodromy, method. The emphasis is on the use of this method for the global asymptotic analysis of the Painlevé transcendents. Specifically, we will study the *connection problem* for the Painlevé transcendents, i.e., the problem of finding explicit formulae relating the relevant asymptotic parameters at different critical points. In more detail we will present:

(i) A complete description of all types of possible asymptotic behavior of the solutions of the important particular cases of Painlevé II and III equations on the real line.
(ii) The derivation of the associated connection formulae.
(iii) A complete description of the asymptotic behavior in the complex plane; quasi-linear and nonlinear Stokes phenomena.
(iv) The distributions of zeros and poles of the solutions.

In addition, we will show how the isomonodromy approach can be used for the study of algebraic aspects of the Painlevé theory. These include the study of the symmetry properties of Painlevé equations (Bäcklund transformations) and the derivation of their elementary solutions and the solutions expressed in terms of classical special functions. In other words, although the asymptotic analysis is the primary theme of this book, several other aspects of modern Painlevé theory will be presented as well. One of our methodological goals is to stress the unique role of the Riemann-Hilbert approach as a universal tool which provides the most effective treatment of many aspects of the theory. Another methodological message is that the Riemann-Hilbert method is, essentially, the only tool available for the study of the problem which is analytically the most challenging, namely the problem of global asymptotics.

It should be emphasized that before the emergence of the Riemann-Hilbert approach, it was thought that the explicit global analysis of the solutions was possible only for the linear differential equations associated with the classical special functions of the hypergeometric type. We will show that the Riemann-Hilbert formalism brings both the *nonlinear special functions*, i.e., the Painlevé functions, and the *linear special functions*, such as Bessel and Airy functions, under the same analytic umbrella.

The Riemann-Hilbert method is based on the remarkable observation, which goes back to the classical works of Garnier and Fuchs, that the Painlevé equations describe the isomonodromy deformations of certain systems of linear differential equations with rational coefficients. This implies that solving a Painlevé equation is equivalent to solving the inverse monodromy problem, i.e., *the Riemann-Hilbert* *problem* for that particular linear system which is associated to the given Painlevé equation and whose monodromy data are the first integrals of the latter. What was apparently missed in the classical works is that this monodromy relation provides a powerful tool for analyzing the Painlevé transcendents. Indeed, as will be demonstrated in this book, these *Riemann-Hilbert representations* of the Painlevé functions play the same efficient role that the contour integral representations play in the theory of linear special functions.

This book is the second attempt in the literature to present systematically the theory of Painlevé equations from the Riemann-Hilbert point of view. The first attempt was made by A. Its and V. Novokshenov in 1986 in the monograph *Isomonodromic Method in the Theory of Painlevé Transcendents*. Since then the field has developed considerably. The three most important and interrelated advances are: (1) the asymptotic analysis of the Painlevé equations has been extended to the entire complex plane, (2) the Riemann-Hilbert methodology has been made rigorous, (3) the method has been enhanced by the powerful *nonlinear steepest descent* asymptotic techniques introduced in the beginning of the 1990s by P. Deift and X. Zhou. Also, many new concrete asymptotic results have been obtained and many new exciting applications, notably in random matrix theory, have appeared. In the present book we will attempt to present some of these new developments as fully as possible.

We have tried to make the book self-contained so that no prior knowledge of the Painlevé equations or of the Riemann-Hilbert problems is needed. In fact, we begin the book with a detailed introduction to the monodromy theory of systems of linear ordinary differential equations with rational coefficients and with a discussion of the associated Riemann-Hilbert problems. In this context, the Painlevé equations appear as the isomonodromy deformations of the first nontrivial examples of linear systems. This point of view will allow us to put the theory of Painlevé transcendents from the beginning into the proper analytic context. Indeed, in this approach the Painlevé equations will appear simultaneously with the Riemann-Hilbert representations for their solutions. It is these representations, i.e., the representations in terms of solutions of certain matrix Riemann-Hilbert problems, that will allow us to obtain all the results mentioned earlier.

We also present some of the recent applications of Painlevé transcendents in physics and mathematics. We hope that this, together with the lists of the asymptotic formulae obtained and discussed in the book, will make the monograph valuable, not only for the readers interested in theoretical aspects of the Painlevé equations and of the Riemann-Hilbert method, but also for the broad scientific community as a source of reference material on Painlevé transcendents. A small part of this material is included in the introduction so that the reader can quickly appreciate the type of results contained in this book.