INTRODUCTION

Painlevé Transcendents as Nonlinear Special Functions

The six classical Painlevé transcendents were introduced at the turn of the twentieth century by Paul Painlevé and his school, as the solution of a specific classification problem for second order ODEs of the type

$$u_{xx} = F(x, u, u_x),$$

where F is a function meromorphic in x and rational in u and u_x. The problem was to find all equations of this form, which have the property that their solutions are free from movable critical points, i.e., the locations of possible branch points and essential singularities of the solution do *not* depend on the initial data. The motivation for posing this problem is quite clear: the absence of movable critical points means that every solution of the equation can be meromorphically extended to the entire universal covering of a punctured complex sphere, determined only by the equation. This implies that such equations share one of the fundamental properties of linear equations.

It was shown by Painlevé and Gambier (1900, 1910), that within a Möbius transformation,

$$u \mapsto \frac{\alpha(x)u + \beta(x)}{\gamma(x)u + \delta(x)}, \qquad x \mapsto \varphi(x),$$

$$\alpha, \beta, \gamma, \delta, \varphi - \text{are meromorphic in } x,$$

there exist only fifty such equations (see monograph [**In**]). Each of them either can be integrated in terms of known functions, or can be mapped to a set of six equations, which cannot be integrated in terms of known functions. These six equations are called Painlevé equations, and their general solutions are called Painlevé functions or Painlevé transcendents. The canonical forms for the Painlevé equations are:

1. $u_{xx} = 6u^2 + x$
2. $u_{xx} = xu + 2u^3 - \alpha$
3. $u_{xx} = \dfrac{1}{u}{u_x}^2 - \dfrac{u_x}{x} + \dfrac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \dfrac{\delta}{u}$
4. $u_{xx} = \dfrac{1}{2u}{u_x}^2 + \dfrac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \dfrac{\beta}{u}$
5. $u_{xx} = \dfrac{3u - 1}{2u(u-1)}{u_x}^2 - \dfrac{1}{x}u_x + \dfrac{(u-1)^2}{x^2}(\alpha u + \dfrac{\beta}{u}) + \dfrac{\gamma u}{x} + \dfrac{\delta u(u+1)}{u-1}$

6. $$u_{xx} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right){u_x}^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)u_x + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right)$$

Here, α, β, γ, δ are complex parameters. During the rest of the twentieth century a great deal of facts about these equations were discovered: the structure of movable singularities, families of explicit solutions, their transformation properties, etc. In connection with these developments, we mention the works of N. P. Erugin, N. L. Lukashevich, A. I. Yablonsky, V. I. Gromak (see the reviews of N. P. Erugin [**Er**], and more recent works by A. S. Fokas and M. J. Ablowitz [**FoA3**], A. S. Fokas and Y. Yortsos [**FoY**], L. A. Bordag [**Bor2**], V. I. Gromak [**GroL1**], [**GroL2**]).

A new surge of interest in Painlevé functions occurred in the late 1970s after the pioneering work of M. J. Ablowitz and H. Segur [**AS**], and of B. M. McCoy, C. A. Tracy and T. T. Wu [**McTW**]. It was shown in [**AS**] that some of the Painlevé equations appear as exact ODE reductions of integrable nonlinear PDEs. Using this fact together with the powerful analytical technique of inverse scattering (and in particular of the Gelfand-Levitan-Marchenko equations), these authors were able to characterize, for the first time, classes of solutions which are not expressible in terms of elementary functions, as well as to solve a certain connection problem for this class of solutions. The paper [**McTW**] was the first rigorous study on the Painlevé connection formulae. In this work, for the first time, a complete connection formulae, in the case of a specific one-parameter family of solutions of the third Painlevé equations, was derived. This one-parameter family, as it was discovered in the earlier work of E. Barouch, B. M. McCoy, C. A. Tracy and T. T. Wu [**Bar**], plays a very important role in the theory of 2D Ising model.

At the same time, Painlevé equations began to appear in a wide range of physical applications[1]. It is now becoming clear that the Painlevé transcendents play the same role in nonlinear mathematical physics that the classical special functions, such as Airy functions, Bessel functions, etc., play in linear physics. During the last twenty to twenty-five years, great progress in the theory of Painlevé equations themselves has been achieved. Just as for their linear counterparts, it is now possible to derive explicit connection formulae for the Painlevé transcendents relating the relevant asymptotics parameters at different critical points. This fact, apparently unknown to P. Painlevé and his contemporaries, is based on the Isomonodromy Method. This method was introduced in 1980 by H. Flaschka and A.C. Newell [**FN1**], and by M. Jimbo, T. Miwa and K. Ueno [**JMU**], and it is based on the intrinsic relation of the Painlevé functions to *the monodromy theory of systems of linear ODE with rational coefficients*. It seems it was R. Garnier [**Gar1**] who first discovered this connection. Other classical references on this subject are the works by R. Fuchs [**Fu**] and by L. Schlesinger [**Sch**].

Let us outline (for details see the main text) the isomonodromy formulation of the Painlevé equations. Consider the generic case of a linear system with rational coefficients, i.e., the *Fuchsian system*,

$$\frac{d\Psi}{d\lambda} = \sum_{j=1}^{n} \frac{A_j}{\lambda - a_j}\Psi, \qquad \Psi, A_j - N \times N \text{ matrices.}$$

[1]The above mentioned paper [**Bar**] was perhaps the first example of these applications.

The *monodromy group* of this system is defined as a representation of the fundamental group of the punctured Riemann sphere (more exactly, a conjugate class of representations),

$$\rho : \pi_1(\mathbb{C}\mathrm{P}^1 \setminus \{a_1, \ldots, a_n, \infty\}) \mapsto \mathbf{GL}(N, \mathbb{C}),$$

generated by encircling the singular points $a_1, \ldots, a_n, a_\infty = \infty$:

$$\Psi(\lambda)|_{(\lambda - a_j)\mapsto(\lambda - a_j)e^{2\pi i}} = \Psi(\lambda)\mathsf{M}_j; \qquad \lambda - a_\infty \equiv \lambda^{-1},$$

$$\mathsf{M}_\infty \mathsf{M}_n \ldots \mathsf{M}_1 = I.$$

The matrices $\mathsf{M}_1, \ldots, \mathsf{M}_n, \mathsf{M}_\infty$ are called monodromy matrices, and the set $\mathfrak{m}=\{\mathsf{M}_1, \ldots, \mathsf{M}_n\}$, monodromy data. This set completely defines (up to a conjugation) the monodromy group $\mathfrak{M}$ of the above Fuchsian equation. Following [**JMU**] we will call the set $\mathbb{A} = \{a_1, \ldots, a_n; A_1, \ldots, A_n\}$ singular data of the Fuchsian system and the set $\mathbb{M} = \{a_1, \ldots, a_n; \mathsf{M}_1, \ldots, \mathsf{M}_n\} \equiv \{a_1, \ldots, a_n; \mathfrak{m}\}$ its extended monodromy data. We will also use the notations

$$\mathcal{A} \equiv \{\mathbb{A}\}, \quad \mathcal{M} \equiv \{\mathfrak{m}\}, \quad \text{and} \quad \mathcal{M}_e \equiv \{\mathbb{M}\},$$

for the sets of singular, monodromy, and extended monodromy data, respectively.

The associated *Riemann-Hilbert problem* consists of proving the existence of a Fuchsian system with given singular points and monodromy group, i.e., with the given set $\{a_1, \ldots, a_n; \mathfrak{M}\} \equiv \mathbb{M}$. More generally, one has to analyze the direct and inverse monodromy maps, i.e., the maps

$$\mathcal{A} \mapsto \mathcal{M}$$

and

$$\mathcal{M}_e \mapsto \mathcal{A}\ ,$$

respectively. This constitutes the central question of the *global* theory of Fuchsian systems.

The Riemann-Hilbert (RH) problem has a long and illustrious history. A comprehensive solution was obtained only relatively recently by A.A. Bolibruch [**Bol2**]. Rather than discussing this problem in general, we will only limit our discussions to how the monodromy problem can be used to study the Painlevé equations. In this respect we will first discuss the appearance of the Painlevé functions in the monodromy theory.

Let us study the above RH problem for the first nontrivial case, i.e., for the matrix size $N = 2$, increasing successively the number of singular points.

(1) TWO REGULAR SINGULAR POINTS, $n = 1$.
After proper gauge, scaling and conformal transformations, it can be shown that:
   (a) $\dim\mathcal{A} = \dim\mathcal{M}$.
   (b) Both the inverse and direct monodromy problems can be solved explicitly in terms of elementary functions (details - in Chapters 2 and 3).
(2) THREE REGULAR SINGULAR POINTS, $n = 2$.
In this case we again have:
   (a) $\dim\mathcal{A} = \dim\mathcal{M}$.

(b) Both the inverse and direct monodromy problems can be solved explicitly in terms of hypergeometric functions, i.e., in terms of elementary functions and of a finite number of contour integrals of these functions (details - in Chapters 2 and 3).

(3) FOUR REGULAR SINGULAR POINTS, $n = 3$.
In the generic situation, the system can be written as follows:

$$\frac{d\Psi}{d\lambda} = \left(\frac{A}{\lambda} + \frac{B}{\lambda - 1} + \frac{C}{\lambda - x}\right)\Psi \equiv \mathsf{A}(\lambda)\Psi, \tag{1}$$

with

$$\mathrm{tr}A = \mathrm{tr}B = \mathrm{tr}C = 0,$$
$$A + B + C = \mathrm{diag}\,(A + B + C).$$

Thus

$$\dim\mathcal{A} = 8.$$

In this case it can be shown that

$$\dim\mathcal{M} = 7 = \dim\mathcal{A} - 1.$$

Hence, one expects a one-parameter family of equations, i.e., a curve in the space $\mathcal{A}$, with a given monodromy group $\mathfrak{M}$. This is in fact the point where the Painlevé equations appear. Indeed, if one writes (cf. [**JMU**]) the (1,2)-th element of matrix $\mathsf{A}(\lambda)$ as

$$\mathsf{A}_{12}(\lambda) = \frac{w(\lambda - u)}{\lambda(\lambda - 1)(\lambda - x)},$$

then,

$$\mathfrak{m} \equiv \mathrm{const} \quad \Longleftrightarrow \quad u = u(x) \text{ satisfies the PVI equation.}$$

The proof of the last statement is based on the fundamental observation that the x-independence of the monodromy matrices $\mathsf{M}_1, \dots, \mathsf{M}_3$, implies that the logarithmic derivative

$$\frac{d\Psi}{dx}\Psi^{-1}$$

is a rational function on the λ-plane which has (in the generic case) a simple pole at the point $\lambda = x$. In fact, one can see (details are given in Chapter 4) that

$$\frac{d\Psi}{dx}\Psi^{-1} = -\frac{C}{\lambda - x}.$$

Therefore, one concludes that the matrix function $\Psi \equiv \Psi(\lambda, x)$ satisfies the following overdetermined linear system (cf. system (4.2.6) of Chapter 4),

$$\left\{\begin{array}{lllll} \dfrac{\partial\Psi}{\partial\lambda} & = & \left(\dfrac{A}{\lambda} + \dfrac{B}{\lambda - 1} + \dfrac{C}{\lambda - x}\right)\Psi & \equiv & \mathsf{A}(\lambda)\Psi, \\ \dfrac{\partial\Psi}{\partial x} & = & -\dfrac{C}{\lambda - x}\Psi & \equiv & \mathsf{U}(\lambda)\Psi\ . \end{array}\right. \tag{2}$$

The compatibility condition of this system, i.e.,

$$\Psi_{\lambda x} = \Psi_{x\lambda},$$

implies

$$\mathsf{A}_x(\lambda) - \mathsf{U}_\lambda(\lambda) + [\mathsf{A}(\lambda), \mathsf{U}(\lambda)] = 0 \quad \text{(identically in } \lambda\text{)}, \tag{3}$$

which in turn yields the following system of nonlinear ODEs on the matrices A, B, C (cf. system (4.2.8) of Chapter 4),

$$\begin{cases} \dfrac{dA}{dx} = \dfrac{1}{x}[C, A], \\ \dfrac{dB}{dx} = \dfrac{1}{x-1}[C, B], \\ \dfrac{dC}{dx} = -\dfrac{dA}{dx} - \dfrac{dB}{dx} \; . \end{cases} \tag{4}$$

Restricting the last system on its invariant orbit, which is determined by the relations

$$A + B + C = \begin{pmatrix} -\theta_\infty & 0 \\ 0 & \theta_\infty \end{pmatrix},$$

$$\text{the eigenvalues of } A,\, B,\, C \;=\; \pm\theta_0, \quad \pm\theta_1, \quad \pm\theta_3, \tag{5}$$

one arrives (details in [**JMU**], [**JM**]) at the Painlevé VI equation for the function $u(x)$, with the parameters

$$\alpha = \frac{1}{2}(2\theta_\infty - 1)^2, \quad \beta = -2\theta_0^2,$$

$$\gamma = 2\theta_1^2, \quad \delta = \frac{1}{2}(1 - 4\theta_3^2).$$

Although, it is technically nontrivial to pass from (4) to PVI, the fact that system (4) leads to a Painlevé equation can be predicted from general principles. Indeed, this system is a particular case of a *Schlesinger equation* (these equations will be discussed in detail in Chapter 4), which is known to posses the Painlevé property (which in turn follows from the meromorphic, w.r.t. x, solvability of the associated Riemann-Hilbert problem; see [**Mal**], [**Miw**]). Also the fact that the ODE for $u(x)$ is of second order can be predicted by counting the dimension of the spectral orbit (5).

According to the terminology of integrable systems (see [**AC**], [**FT**], [**NMPZ**]), the linear system (2) and the nonlinear equation (3), are the *Lax pair* and the *zero-curvature* (or *Lax*) *representation*, respectively, of the nonlinear ordinary differential equation PVI.

In order to put the other Painlevé equations into a similar context, we consider more general linear systems, namely systems with irregular singularities. The first nontrivial case is the system with only one irregular singularity of order 3 at $\lambda = \infty$:

$$\frac{d\Psi}{d\lambda} = \mathsf{A}(\lambda)\Psi, \qquad \mathsf{A}(\lambda) = A_3\lambda^2 + A_2\lambda + A_1, \qquad A_i \in \mathbf{Mat}(2, \mathbb{C})$$

(if $A_3 = 0$ the system can be solved in terms of the confluent hypergeometric functions; if $A_3 = A_2 = 0$, the system can be solved in terms of elementary functions). In the generic situation one can always reduce this system to the normal form

$$\mathsf{A}(\lambda) = -4i\lambda^2\sigma_3 + 4i\lambda \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix} + \begin{pmatrix} a & -2w \\ -2z & -a \end{pmatrix}, \tag{6}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding *canonical solutions* $\Psi_k(\lambda)$ (details are in Chapters 1 and 2) in the neighborhood of $\lambda = \infty$ are characterized by their asymptotics

$$\Psi_k(\lambda) = (I + O(1/\lambda)) \exp\left\{-\frac{4}{3}i\lambda^3\sigma_3 - ix\lambda\sigma_3 - \nu\log\lambda\sigma_3\right\},$$

$$\frac{k-2}{3}\pi < \arg\lambda < \frac{\pi k}{3}, \quad \lambda \to \infty,$$

$$k = 1, \dots, 7,$$

where

$$x = ia - 2uv, \quad \nu = vw - uz.$$

The monodromy data $\mathfrak{m}$ are given by the parameter ν and the set of *Stokes matrices* S_k defined as

$$\mathsf{S}_k = \Psi_k^{-1}\Psi_{k+1}, \qquad k = 1, \dots, 6.$$

These matrices have a triangular structure,

$$\mathsf{S}_{2l} = \begin{pmatrix} 1 & s_{2l} \\ 0 & 1 \end{pmatrix}, \quad \mathsf{S}_{2l+1} = \begin{pmatrix} 1 & 0 \\ s_{2l+1} & 1 \end{pmatrix},$$

and satisfy the cyclic relation

$$\mathsf{S}_1\mathsf{S}_2 \dots \mathsf{S}_6 = e^{-2\pi i\nu\sigma_3}.$$

Therefore,

$$\dim\mathcal{M} = 4 = \dim\mathcal{A} - 1,$$

and again we expect nontrivial *isomonodromy deformations.* In fact (cf. [**FN1**], [**JMU**]), if one takes the quantity x as a parameter for the isomonodromy curve, so that u, v, z and w become functions of x, then the condition $\mathfrak{m} \equiv$ const. is equivalent to the system

$$\text{(7)} \qquad \begin{cases} w = u_x, & u_{xx} - xu - 2u^2v = 0\ , \\ z = v_x, & v_{xx} - xv - 2v^2u = 0\ . \end{cases}$$

Similarly to the Fuchsian case considered earlier, the proof of (7) is based on the fact that the x-independence of the Stokes matrices $\mathsf{S}_1, \dots, \mathsf{S}_6$, implies that the function $\Psi \equiv \Psi_1$ satisfies the Lax pair (details in Chapter 4),

$$\text{(8)} \qquad \begin{cases} \dfrac{\partial\Psi}{\partial\lambda} = \mathsf{A}(\lambda)\Psi\ , \\ \dfrac{\partial\Psi}{\partial x} = \mathsf{U}(\lambda)\Psi\ , \end{cases}$$

where the matrix $\mathsf{A}(\lambda)$ is the one from (6), whereas the matrix $\mathsf{U}(\lambda)$ is given by the equation,

$$\mathsf{U}(\lambda) = -i\lambda\sigma_3 + i\lambda \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}. \tag{9}$$

This time it is straightforward to show that the associated zero-curvature relation,

$$\mathsf{A}_x(\lambda) - \mathsf{U}_\lambda(\lambda) + [\mathsf{A}(\lambda), \mathsf{U}(\lambda)] = 0 \quad \text{(identically in } \lambda\text{)}, \tag{10}$$

is equivalent to system (7).

Under the reduction,

$$u = v,$$

system (7) becomes a particular case ($\alpha = 0$) of the second Painlevé equation. The reduction $u = v$ implies the restrictions

$$s_{k+3} = -s_k, \quad \nu = 0,$$

so that

$$\mathcal{M} = \{(s_1, s_2, s_3) : s_2 - s_1 - s_3 - s_1 s_2 s_3 = 0\} \tag{11}$$

and hence

$$\dim \mathcal{M} = 2.$$

This means that for generic (s_1, s_2, s_3), one can take s_1 and s_3 as independent coordinates on $\mathcal{M}$.

In order to obtain the general Painlevé II equation it is necessary (cf. [**FN1**]) to add in (6) (under the reduction $u = v$, $w = z$) a regular singularity at $\lambda = 0$ of the form

$$-\frac{\alpha}{\lambda} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Also, there exists an elementary transformation which maps the general u–v system (7) into the general Painlevé II equation, whose parameter α is related to the first integral $\nu \equiv v u_x - u v_x$ of system (7); we will describe this map in Section 3.5 of Chapter 4.

Summarizing the above discussion, we can say that *the Painlevé transcendents mark the border beyond which both the inverse and the direct monodromy problems cannot be solved in terms of elementary functions and their contour integrals.* At the same time, this is the border beyond which the monodromy map ceases to be one-to-one.

The above relation was used actually by R. Garnier and L. Schlesinger for the solution of the inverse monodromy problem. Their scheme for the constructive solution of this inverse monodromy problem can be formulated as follows: Given the monodromy data, the coefficients of the corresponding linear system can be expressed in terms of the solution of the associated nonlinear equations of the isomonodromy deformations, i.e., in the first nontrivial cases — the Painlevé equations.

The principal question which remained open was: What are the initial data for the isomonodromy equations (in our examples for the PVI and PII equations)?

What apparently was missed in the classical works, was precisely the opposite point of view. Namely, the fact that the initial linear system and the associated RH problem can be used for solving the Painlevé equations themselves. The latter is the modern achievement, which was initiated in the works [**FN1**], [**JMU**] mentioned above, and which constitutes what is now called the *isomonodromy* or *Riemann-Hilbert method*. As an important by-product of this development, it is now possible to answer the above question in the classical scheme, namely it is possible to choose the correct initial data of the isomonodromy equations (see paper [**IN2**] and Theorem 2 below).

By the word "solving" in the previous paragraph, we mean the ability to perform a comprehensive *global* analysis of the Painlevé transcendents, i.e., to derive explicitly all relevant connection formulae. It also should be mentioned that, throughout the book, we accept the terminology, that an equation, a formula, an expression, etc. is called *explicit* if it is given in terms of a finite algebraic combination of elementary and elliptic functions, and a finite number of contour integrals (including quadratures) of these functions. In particular, all classical "linear" special functions admit explicit representations. The Painlevé transcendents do not. At the same time, the Painlevé transcendents, as we know now, admit explicit connection formulae. This fact, in our opinion, makes them as efficient in applications as the linear special functions.

In the second part of the book, two general approaches for solving the connection problem for the Painlevé equations will be presented. Both of them use the relation of the Painlevé transcendents to the RH problem. Due to J. Plemelj [**Pl**], the latter problem can be reformulated as a factorization problem (homogeneous Hilbert problem) in the theory of sectionally analytic matrix valued functions (see below). Using geometric language, this means that the Painlevé functions (and their natural generalizations), which can be called *nonlinear* special functions, can be analyzed within the theory of holomorphic vector bundles of rank > 1 over compact Riemann surfaces (of genus ≤ 1). In the same spirit, the classical *linear* special functions belong to the same theory, but for rank $= 1$ (the scalar factorization problem).

To explain in more detail the essence of the isomonodromy method let us consider the particular case, $\alpha = 0$, of the PII equation

$$u_{xx} - xu - 2u^3 = 0. \tag{12}$$

Reversing the arguments used earlier relating this equation to the isomonodromy deformations of the system (6), we can now say that the monodromy data,

$$s_1 = s_1(x, u, u_x), \qquad s_3 = s_3(x, u, u_x)$$

are the *first integrals* of equation (12). Formally this means that we have solved equation (12)! Indeed, we have found a complete set of independent first integrals. However, since these integrals are *not* explicit, the important question is: Can we make this fact an efficient tool for the study of the PII equation? It turns out that the answer is positive. Of course, one does not expect an explicit expression for the functions $s_1(x, u, w)$ and $s_3(x, u, w)$, since this would imply the possibility of expressing the second Painlevé transcendent in terms of classical special functions.

Nevertheless, one can calculate the functions s_1 and s_3 asymptotically. Assuming a certain kind of behavior of $u(x)$ as $x \to \pm\infty$, and applying an *extended version* of the classical WKB-technique to the λ-equation (2), it is possible to evaluate s_1 and s_3 asymptotically as $x \to \pm\infty$. However, since these functions are independent of x, this implies the connection formulae between the asymptotic parameters of the solution $u(x)$ at "$+\infty$" and "$-\infty$". Following this scheme, which will be explained in detail later, many global asymptotic results of the Painlevé equations have been obtained during the last twenty years (A. P. Bassom, P. A. Clarkson, M. Jimbo, A. R. Its, A. A. Kapaev, A. V. Kitaev, C. K. Law, B. M. McCoy, J. B. McLeod, V. Yu .Novokshenov, B. I. Suleimanov, Sh. Tang). Among these results are:

(a) The parametrization of all solutions by their asymptotic behaviour near critical points and the construction of the corresponding connection formulae.
(b) The global description of the asymptotic behaviour for complex x-nonlinear Stokes phenomenon.
(c) The determination of the distributions of zeros and poles.

The above scheme is based on the asymptotic analysis of the *direct monodromy problem* associated with a given Painlevé equation. It has the disadvantage that it requires the use of a priori information about the solution. An alternative approach, which remedies the above disadvantage is based on the asymptotic solution of the *inverse monodromy problem*. The starting point of this approach is, following ideas of Birkhoff [**Bir**], to consider the basic monodromy relation, $\mathrm{S}_k = {\Psi_k}^{-1}\Psi_{k+1}$, as a jump condition for the sectionally analytic function $Y(\lambda)$:

$$Y(\lambda) = \Psi_k(\lambda)e^{\theta(\lambda)\sigma_3} \equiv Y_k(\lambda), \qquad \theta(\lambda) = \frac{4i}{3}\lambda^3 + ix\lambda, \tag{13}$$
$$\frac{\pi}{3}(k-2) + \frac{\pi}{6} \le \arg\lambda \le \frac{\pi}{3}k - \frac{\pi}{6}, \quad k = 1, \dots, 6,$$

on the anti-Stokes rays, $\arg\lambda = \pi k/3 - \pi/6$, $k = 1, \dots, 6$. This gives rise to a special type of oscillatory Riemann-Hilbert problem, which is now understood as a *factorization or jump problem*:

$$Y_{k+1}(\lambda) = Y_k(\lambda)e^{-\theta(\lambda)\sigma_3}\mathrm{S}_k e^{\theta(\lambda)\sigma_3}, \tag{14}$$
$$\arg\lambda = \frac{\pi k}{3} - \frac{\pi}{6}, \qquad k = 1, \dots, 6,$$
$$\lim_{\lambda\to\infty} Y(\lambda) = I.$$

(The last normalization equation is actually the reason for the introduction of the function $Y(\lambda)$.)

The problem (14) is depicted in Figure I. The corresponding solution $u(x, s_1, s_3)$ of the second Painlevé equation (12) can be determined from $Y(\lambda)$ by the equation

$$u(x; s_1, s_3) = 2\lim_{\lambda\to\infty}(\lambda Y_{12}(\lambda)). \tag{15}$$

The problem (14), as well as the Fuchsian inverse monodromy problem (which also can be reformulated as a factorization problem — see Chapter 3), are particular cases of a *Riemann-Hilbert factorization problem.* This problem involves finding an analytic (matrix valued) function having prescribed jumps across a given contour. Following the tradition developed in mathematical physics, it is such more general

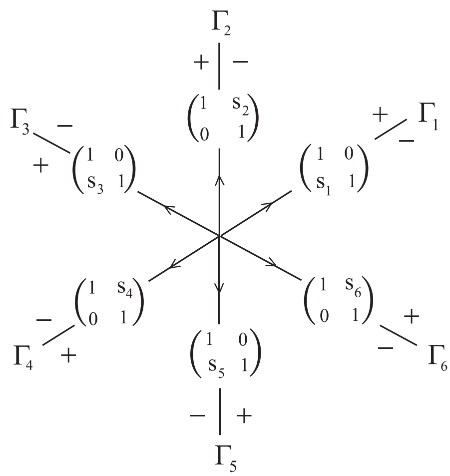

FIGURE I. The contour and jump matrices for the RH problem (14)

factorization problems (and not just the inverse monodromy Fuchsian and/or the Birkhoff problems) that we will call *Riemann-Hilbert problems.*[2]

The investigation of the Cauchy problem for the Painlevé II equation based on the analysis of the RH problem (14), as well as the investigation of its discontinuous generalization for $\alpha \neq 0$ (complete Painlevé II equation) was initiated by A. S. Fokas and M. J. Ablowitz in [**FoA3**], [**FoA1**]. The rigorous theory was developed by A. S. Fokas and X. Zhou in [**FoZ**] for PII and PIV, and was later applied to PI, PIII and PV in [**FoMA**]. In the works of Fokas, Mugan, Ablowitz and Zhou (see the review paper [**FoI1**]) the first rigorous proofs were given (based on the isomonodromy method) that all solutions of PI – PV possess the Painlevé-property. Furthermore, it was shown that if the monodromy data satisfy explicit constraints, then the solutions are free from poles (on the real line).

The last basic step was taken in the works of P. Deift and X. Zhou [**DeZ1**], [**DeZ2**], who introduced an elegant scheme of the asymptotic analysis of this type of oscillatory factorization problems; their method can be thought of as a *nonlinear steepest descent method*. The asymptotic method of Deift and Zhou, in complete analogy with the classical method, exploiting the analytic structure of the relevant jump matrices deforms the original contours of the RH problem to contours where the relevant oscillatory factors become exponentially small as $x \to \infty$. In this way, the original RH problem reduces to a collection of local RH problems associated with the relevant saddle points. However, the noncommutative nature of the RH problem requires the development of several totally new and rather sophisticated technical ideas, which, in particular, allow us to solve the local RH problems in closed form. The remarkable fact is that the final result of the analysis is as efficient as the asymptotic evaluation of the classical oscillatory integrals.

It is worth mentioning that the first step towards the direct asymptotic analysis of oscillatory RH problems was made in 1973 by S. V. Manakov [**Mn**] in connection

[2] It should be mentioned that in the theory of boundary value problems for analytic functions, the problem of reconstructing a function from its jumps across a curve is sometimes called the “Hilbert boundary-value problem”.

with the theory of integrable PDEs. Manakov's approach was further developed in [**I2**], [**I3**], where certain aspects of the nonlinear steepest descent method (e.g. the analysis and the explicit solution of some basic local RH problems) already existed.

Asymptotic results

In order to give a flavor of the type of results which are contained in this book, we first present the complete asymptotic description of the purely imaginary solutions of the PII equation, x real, $\alpha = 0$, as $x \to \pm\infty$.[3]

THEOREM 1. *An arbitrary purely imaginary solution $u(x)$ of the second Painlevé equation (12), has the following asymptotic behavior as $x \to -\infty$:*

$$u(x) = i(-x)^{-1/4} d \sin\left\{\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \varphi\right\} + o((-x)^{-1/4}). \tag{16}$$

Here the constants $d > 0$ and $\varphi \in \mathbb{R}$ (mod 2π) can be any real numbers; they determine the solution $u(x)$ uniquely and hence form a set $\mathbf{a}_- \equiv (d, \varphi)$ of asymptotic parameters at $-\infty$.

Let

$$\Delta(d,\varphi) \equiv \frac{3}{2}d^2\log 2 - \frac{\pi}{4} - \arg\Gamma\left(i\frac{d^2}{2}\right) - \varphi,$$

where $\Gamma(z)$ denotes Euler's gamma function. The behavior of $u(x) \equiv u(x; d, \varphi)$ as $x \to +\infty$ depends on the value of $\Delta(d,\varphi)$. If (generic case)

$$\Delta(d,\varphi) \neq 0 \quad (\text{mod } \pi), \tag{17}$$

then

$$u(x) = \sigma i\sqrt{\frac{x}{2}} + \sigma i(2x)^{-1/4}\rho\cos\left\{\frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2\log x + \theta\right\} + o(x^{-1/4}), \tag{18}$$
$$\text{as } x \to +\infty,$$

where $\sigma = \pm$, $\rho > 0$, and $\theta \in \mathbb{R}$ (mod 2π). If instead,

$$\Delta(d,\varphi) = 0 \quad (\text{mod } \pi), \tag{19}$$

then

$$u(x) = \sigma i\frac{\rho}{2\sqrt{\pi}}x^{-1/4}e^{-(2/3)x^{3/2}}(1 + o(1)), \quad x \to +\infty, \tag{20}$$

where $\sigma = \pm$ and $\rho > 0$. The set $\mathbf{a}_+$ of asymptotic parameters at $+\infty$ is the triple (ρ, θ, σ) in the generic situation and the pair (ρ, σ) in the special case.

The following explicit connection formulae characterize the map $\mathbf{a}_- \mapsto \mathbf{a}_+$:

$$\begin{aligned}
\rho^2 &= d^2 - \frac{1}{\pi}\log 2(e^{\pi d^2} - 1)^{1/2}|\sin\Delta(d,\varphi)|, \\
\theta &= -\frac{3\pi}{4} - \frac{7}{2}\rho^2\log 2 + \arg\Gamma(i\rho^2) + \arg\left(1 + (e^{\pi d^2} - 1)e^{2i\Delta(d,\varphi)}\right), \\
\sigma &= -\text{sign}\,(\sin\Delta(d,\varphi)),
\end{aligned} \tag{21}$$

if $\Delta(d,\varphi) \neq 0$ (mod π), and

[3] By letting $u \to iu$ in (12), the results stated below are transformed to the complete asymptotic description of the real solutions of the equation $u_{xx} - xu + 2u^3 = 0$, x real, as $x \to \pm\infty$.

$$\rho = (e^{\pi d^2} - 1)^{1/2}, \quad \sigma = (-1)^n \tag{22}$$

if $\Delta(d, \varphi) = n\pi$.

Here are some remarks about Theorem 1.

- A detailed proof of the theorem is given in Part II of the book.
- Connection formulae for the one-parameter family of solutions (19) were first obtained in 1977 in [**AS**].
- Connection formulae for the generic case were first obtained in 1986 in [**IK1**]; some technical gaps in the original proof were filled later in the paper [**IFoK**]. Another proof of the theorem based on the nonlinear steepest descent method was obtained in [**DeZ2**].
- Some parts of the theorem can be obtained without using the Riemann-Hilbert formalism. This is true for the existence, for a given pair (d, φ), of a solution $u(x)$ with the asymptotics (16) (see [**Abd**]), or with the asymptotics (18) and (20). Since these are *local* statements, they do not reflect the integrability of the second Painlevé equation. The *global* fact that formulae (16) and (18)–(20) describe *all* possible types of asymptotic behavior of the purely imaginary solutions of the second Painlevé equation (12) as $x \to \pm\infty$, can also be proved without appealing to the Riemann-Hilbert representation (see [**JoKr1**]), but one has to make use of the integrability of equation (12); indeed, the arguments of [**JoKr1**] exploit in an essential way the Painlevé property.

  Without using the Riemann-Hilbert formalism, it does not seem feasible to obtain the parts of the theorem concerning the bifurcation condition (17) and the connection formulae (21)–(22).
- In the limit of small d the Ablowitz-Segur connection formulae (19), (22) become the classical Airy connection formulae.

We would like to emphasize that it is precisely the connection formulae that make the Airy functions so useful in applications. Theorem 1 shows that in the nonlinear case of PII, the associated connection formulae, though more complicated, are still explicit. In the Tables 1 and 2 below, we present similar results for PIII (the relevant linear counterparts are the Bessel functions).

Because u is imaginary, there exists only one independent monodromy datum $s_1 = -\overline{s}_3 = s$. As we will see in Chapter 9, the proof of Theorem 1, also provides the explicit representations of the monodromy parameter s in terms of the asymptotic parameters d, φ and ρ, θ associated with the "nonlinear" Stokes rays, $\arg x = \pi, 0$. For instance, the formula $s = s(d, \varphi)$ reads as follows,

$$s = (e^{\pi d^2} - 1)^{1/2} \exp\{\Delta(d, \varphi)\}. \tag{23}$$

It is important to emphasize that it is also possible to obtain similar results for the remaining Stokes lines, i.e., for the rays $\arg x = \pm\pi/3$ and $\arg x = \pm 2\pi/3$ (see paper [**K5**] by A. A. Kapaev). The formulae presented in Theorem 1 and the formulae of Kapaev show that the asymptotic behavior of PII on the Stokes lines is characterized by elementary (trigonometric) functions.

TABLE 1. Connection formulas (oscillating sector)

Bessel functions	Third Painlevé functions
$u_{xx} + \frac{1}{x}u_x + u = 0$	$u_{xx} + \frac{1}{x}u_x + \sin u = 0$
$u(x;s):\quad (sJ_0(x))$ $u(0;s) = s$ $u_x(0;s) = 0$	$u(x;s):$ $u(0;s) = s$ $u_x(0;s) = 0$
$x \to +\infty$ $u(\vert s) \cong \alpha x^{-1/2}\cos(x+\varphi)$ $\alpha = s\sqrt{\frac{2}{\pi}}$ $\varphi = -\frac{\pi}{4}$	$x \to +\infty$ (V.Yu. Novokshenov [**N2**]) $u(\vert s) \cong 2\pi k + \alpha x^{-1/2}\cos(x - \frac{\alpha^2}{16}\log x + \varphi)$ $\alpha^2 = -\frac{16}{\pi}\log\vert\cos\frac{s}{2}\vert$ $\varphi = -\frac{\alpha^2}{8}\log 2 - \arg\Gamma(-\frac{i\alpha^2}{16}) + \frac{\pi}{4}$ $k = \textit{ entier }\{\frac{1}{2\pi}[\pi - s]\}$

TABLE 2. Connection formulas (decreasing sector)

Bessel functions	Third Painlevé functions
$u_{xx} + \frac{1}{x}u_x - u = 0$	$u_{xx} + \frac{1}{x}u_x - \sinh u = 0$
$u(x\vert\alpha):\quad (\alpha K_0(x))$ $x \to +\infty$ $u(0\vert\alpha) \cong \alpha\sqrt{\pi}2x^{-1/2}e^{-x}$	$u(x\vert\alpha):$ $x \to +\infty$ $u(0\vert\alpha) \cong \alpha\sqrt{\pi}2x^{-1/2}e^{-x}$
$x \to 0$ $u(0\vert\alpha) \cong r\log x + s$ $r = -\alpha$ $s = -r(\log 2 + \gamma)$	$x \to 0$ (B.McCoy, T.Wu, C.A.Tracy [**McTW**]) $u(0\vert\alpha) \cong r\log x + s$ $\alpha = -\frac{4}{\pi}\sin\frac{\pi r}{4}$ $e^s = 2^{-3r}\Gamma^2(\frac{1}{2} - \frac{r}{4})\Gamma^{-2}(\frac{1}{2} + \frac{r}{4})$

It turns out that if x is <u>not</u> on the Stokes lines, then the asymptotic behavior is more complicated and is characterized by elliptic functions. It is shown by V. Yu. Novokshenov [**N1**] and A. A. Kapaev [**K5**] (see also Chapters 7 and 8 of the book) that for $u(x;s_1,s_3)$ the asymptotics at infinity in the sectors $\frac{\pi}{3}j < \arg x < \frac{\pi}{3}(j+1)$,

$j = 0, 1, \ldots, 5$, is given by the following formula:

$$(24)\ u(x) = e^{\frac{2\pi i}{3}n}|x|^{1/2}(\xi_1 - \xi_2) \times \mathrm{sn}\left\{(\xi_1+\xi_2)\left(e^{i\varphi + \frac{2\pi i}{3}n}\frac{2}{3}|x|^{3/2} + \chi_{n,k}\right)\Big|\frac{\xi_1-\xi_2}{\xi_1+\xi_2}\right\} + O(|x|^{-1}),$$

where

$$\frac{\pi}{6}(7-k) - \frac{2\pi}{3}n < \varphi < \frac{\pi}{6}(9-k) - \frac{2\pi}{3}n, \qquad |x| \to \infty, \qquad x \in \{\arg x = \varphi\} \cap D_\varepsilon,$$
$$n = 0, \pm 1, \ldots, \qquad k = 1, 3,$$

and D_ε is the complement to the union of the ε-neighborhoods of all the poles of the elliptic function appearing in (24). The module parameters, $\xi_{1,2} \equiv \xi_{1,2}(\varphi)$, of the elliptic sine are given by the equations (Boutroux equations)

$$(25)\quad {\xi_1}^2 + {\xi_2}^2 = -\frac{1}{2}e^{i\varphi + \frac{2\pi i}{3}n}, \qquad \mathrm{Im}\left(\int_{\xi_2}^{\xi_1}, \int_{-\xi_1}^{\xi_1}\right)\sqrt{(\xi^2-\xi_1^2)(\xi^2-\xi_2^2)}d\xi = 0\,.$$

The connection formulae between the asymptotic phases $\chi_{n,k}$, and the monodromy data s, are given by

$$(26)\qquad \chi_{n,k} = \chi_{n,k}(s_1, s_3) = \frac{\omega_2}{2\pi i}(-1)^{\frac{k+1}{2}}\log s_{k+2n} + \frac{\omega_1}{2\pi i}\log(1 - s_{1+2n}s_{3+2n}),$$

$$\omega_{1,2} = \left(i\int_{-\xi_1}^{\xi_1}, 2i\int_{\xi_2}^{\xi_1}\right)\frac{d\xi}{\sqrt{(\xi^2-\xi_1^2)(\xi^2-\xi_2^2)}},$$

where the monodromy data satisfy

$$s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, \qquad -\overline{s_3} = s_1 \equiv s.$$

The elliptic asymptotics of this type for the Painlevé I equation, was first obtained by P. Boutroux in 1913 [**Bou**], but without the phase connection formulae.

Formulae (24-26) are actually valid for the general complex valued solution of the Painlevé II equation (without the restriction that u is purely imaginary). Also, in [**K5**] the general case of the second Painlevé equation is considered (without the restriction $\alpha = 0$). The complete connection formulae for the elliptic asymptotics of Painlevé I equation are given in papers [**Ki7**], [**KK2**].

The connection formulae valid for $\arg x = 0, \pm\pi/3, \pm 2\pi/3, \pi$, together with the formulae (26) for the complex phases $\chi_{n,k}$, provide an explicit description of the *nonlinear Stokes phenomenon* for the second Painlevé equation. Indeed, knowing the asymptotic behaviour of the solution $u(x)$ on one of the rays, one can calculate the corresponding monodromy parameter s, and hence obtain complete information about the asymptotic behaviour of $u(x)$ everywhere on the complex x-plane.

The connection formulae can be used to fill the gap of the classical approach to the inverse monodromy problem mentioned earlier, namely these formulae specify the proper initial conditions for the isomonodromy equations [**IN2**].

THEOREM 2. *Consider the RH problem, i.e., the inverse monodromy problem (IMP) for the system (6),*

$$\{S_k, \nu, x\} \to \{u, v, w, z, x\}$$

and assume that

$$s_{k+3} = -s_k, \quad \nu = 0, \tag{27}$$

$$-\overline{s_3} = s_1 \equiv s. \tag{28}$$

Then, for all real x the IMP is uniquely solvable and the solution is given by

$$v = u = u(x,s),\ y = w = u_x(x,s),$$

where $u(x,s)$ is the solution of the second Painlevé equation,

$$u_{xx} = xu + 2u^3$$

specified by the following asymptotic condition ("initial" data):

$$\begin{aligned} u(x) \quad &= i(-x)^{-1/4} d \sin\left\{\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \varphi\right\} \\ &+ o((-x)^{-1/4}), \quad x \to -\infty, \end{aligned} \tag{29}$$

where (cf. (23))

$$d^2 = \frac{1}{\pi}\log(1+|s|^2), \quad d > 0, \tag{30}$$

$$\varphi = \frac{3}{2}d^2\log 2 - \frac{\pi}{4} - \arg\Gamma(i\frac{d^2}{2}) - \arg s. \tag{31}$$

The analogous result for the Fuchsian system (1), i.e., an explicit expression of the initial data for the sixth Painlevé equation in terms of the given monodromy data, was obtained for the generic case by M. Jimbo [**J1**]. Important special cases, not covered by Jimbo's results, were obtained in the recent works of B. Dubrovin and M. Mazzocco [**DM**] and D. Guzzetti [**Guz1**].

If instead of (28), the restriction

$$\overline{s_3} = s_1 \equiv s \tag{32}$$

is imposed, then results similar to the statement of Theorem 2 are valid but only for $x < 0$, $|x|$ sufficiently large, and $|s| < 1$.

If in addition to the restriction (32), the conditions

$$|s| > 1, \quad \mathrm{Re} s \neq 0, \tag{33}$$

hold, then the Painlevé transcendent $u(x,s)$ has sequences $\{x_n{}^{\pm}\}$ of real poles which accumulate at $x = \pm\infty$. The asymptotic behaviour of $x_n{}^{\pm}$ as $n \to \infty$ can also be described explicitly in terms of s [**K1**], [**KN**]:

$$\begin{aligned} (x_n^+)^{3/2} = \frac{3}{\sqrt{2}}\pi n - \frac{3}{2\sqrt{2}}\mu^+\log\frac{3}{2\sqrt{2}}\pi n - \frac{21}{4\sqrt{2}}\mu^+\log 2 - \frac{3}{2\sqrt{2}}\arg\Gamma(1/2 - i\mu^+) \\ -\frac{3}{2\sqrt{2}}\arg\frac{1+s^2}{1-|s|} + o(1), \end{aligned} \tag{34}$$

$$n \to \infty,\ \mu^+ = \frac{1}{\pi}\log\frac{2|\mathrm{Re} s|}{|s|^2 - 1}$$

and

$$(-x_n^-)^{3/2} = 3\pi n - \frac{3}{2}\mu^-\log 3\pi n - \frac{9}{2}\mu^-\log 2 + \frac{3}{2}\arg\Gamma(1/2 + i\mu^-)$$

$$-\frac{3\pi}{4}(1-(-1)^{n+n_0})+\frac{3}{2}\arg s+o(1), \tag{35}$$

$$n\to\infty,\ \mu^-=\frac{1}{2}\pi\log(|s|^2-1).$$

It is shown in [**IN2**] that the poles of $u(x,s)$ give the values of x for which the RH problem for the system (6) is not solvable. Equations (34) and (35) give the asymptotic distribution of these values on the real axis in the case that the monodromy data satisfy the restrictions (32) and (33). At the same time, these equations give, in closed form, the asymptotic distribution of the real values of x for which the holomorphic bundle over $\mathbb{C}\mathrm{P}^1$ defined by the associated Stokes cocycle (see paper [**Si1**]),

$$S_k(\lambda)=\mathrm{e}^{\left\{-i\frac{4}{3}\lambda^3-ix\lambda\right\}\sigma_3}S_k\mathrm{e}^{\left\{i\frac{4}{3}\lambda^3+ix\lambda\right\}\sigma_3}$$

is not trivial.

When (33) is replaced by,

$$|s|=1,\quad \mathrm{Re}s=0, \tag{36}$$

i.e., when $s=\pm i$, then for real x the solution becomes smooth. It decays exponentially as $x\to+\infty$, and it grows as $\mp\sqrt{-x}$ as $x\to-\infty$. This particular (but still transcendental) solution, which was first singled out and studied by S. P. Hastings and J. B. McLeod [**HasM**], plays now an important role in the theory of random matrices (it determines the so-called Tracy-Widom distributions; see the applications to random matrices discussed later in the introduction).

Regarding asymptotic results we also mention the following:

(a) The asymptotics of particular families of solutions of Painlevé equations can be investigated rigorously by linear integral equations of the Gel'fand-Levitan-Marchenko type (P. A. Clarkson, J. B. McLeod, [**ClM**]).
(b) Important facts (but not explicit connection formulae) about the complex asymptotics of the Painlevé functions can be obtained without using the isomonodromy method (N. Joshi and M. D. Kruskal [**JoKr1**]).
(c) An improved version of the direct monodromy asymptotic approach has been developed by Bassom, Clarkson, Law and McLeod in [**BCLM**].
(d) The direct monodromy asymptotic approach outlined above can be thought of as the ODE version of the Zakharov-Manakov scheme [**ZM**], which was developed for integrable PDEs.
(e) An alternative approach to the study of the elliptic asymptotics for the Painlevé equations, which uses a combination of the isomonodromy, algebrogeometric, and Bogolyubov-Whitham techniques, has been developed by S. P. Novikov and P. Grinevich in [**GN**].
(f) The asymptotics of some important special families of the third and fifth Painlevé equations (and their higher order generalizations) have been studied via the Fredholm operator techniques by C. Tracy and H. Widom [**TW3**], [**TW5**], and by E. Basor [**Bas**]. This approach, in particular, allows one to evaluate exactly the important constants of integration appearing in the applications of the Painlevé transcendents.
(g) A study of the various connection formulae for the first, third, fourth and fifth Painlevé equations via the direct isomonodromy approach has been carried out by the following authors: for Painlevé I, by A. A. Kapaev and

A. V. Kitaev [**K2**], [**KK2**], [**Ki7**]; for Painlevé III, by V. Yu. Novokshenov [**N2**], [**N3**], A. V. Kitaev [**Ki3**], [**Ki4**], A. I. Bobenko and A. R. Its [**BobI**], V. Yu. Novokshenov and A. G. Shagalov [**N5**], [**N6**], and by A. V. Kitaev and A. H. Vartanian [**KiV**]; for Painlevé IV, by A. V. Kitaev [**Ki2**], [**Ki5**], A. A. Kapaev [**K6**], and by A. R. Its and A. A. Kapaev [**IK2**]; and for Painlevé V, by B. M. McCoy and Sh. Tang [**McT1**], [**McT2**], and by F. V. Andreev and A. V. Kitaev [**AnK2**].

(h) The asymptotics and connection formulae for the sixth Painlevé equation were evaluated via the isomonodromy approach for the generic case by M. Jimbo [**J1**], and for important special cases, not covered by Jimbo's results, by B. Dubrovin and M. Mazzocco [**DM**] and D. Guzzetti [**Guz1**].

(i) The first connection formulae for specific families of the Painlevé transcendents were obtained in [**AS**] (the one-parameter family (19) of the solutions of the second Painlevé equation presented in Theorem 1) and in [**McTW**] (a one-parameter family of solutions of the third Painlevé equation arising in the 2D Ising model and presented in Table 2). The authors of [**AS**] used the results of [**ZM**] and the fact (first discovered by the authors of [**AS**]) that the second Painlevé equation is a self-similar reduction of the KdV equation. The Ablowitz-Segur connection formulae were rigorously justified later in the works [**Su2**] and [**ClM**]. The work of B. M. McCoy, C. A. Tracy and T. T. Wu [**McTW**] is actually the first rigorous work on the Painlevé connection formulae. Remarkably, it was done before the discovery of the Riemann-Hilbert formalism for Painlevé equations. However, there is an important specific technical point used by the above authors for the one-parameter family considered in [**McTW**]; this family admits a Fredholm determinant representation, which, in a sense, is a "shadow" of the Riemann-Hilbert formalism.

(j) Among the pioneering studies of the connection formulae for Painlevé equations, the works of J. W. Miles [**Mil1**], [**Mil2**] must be mentioned. These are the works where the view on the Painlevé transcendents as the *nonlinear special functions* was stressed, perhaps, for the first time.

Applications

We conclude our brief review of the modern theory of Painlevé functions, by mentioning some of their typical physical applications. The first group of these applications is related to the fact that all reductions of integrable nonlinear PDEs, and also some ODE reductions of some nonintegrable PDEs, are Painlevé equations. For example, ODE reductions of Korteweg-de Vries equation yield PI and PII, of nonlinear Schrödinger equation — PIV, of Ernst equation – PVI, of sine-Gordon equation — PIII, etc. Some of these ODEs play a crucial role in the asymptotic analysis of the corresponding PDEs. For example, it was shown by A. S. Fokas, D. Kaup and C. Menyuk [**FoM**], [**KaM**] that a certain important asymptotic limit of the PDEs describing stimulated Raman scattering is provided by a particular class of solutions of PIII.

Another impressive field of applications is exactly solvable models in statistical physics and in quantum field theory. Among field-theoretical problems which can be solved in terms of the Painlevé transcendents we mention the following:

1. The two-point correlation function for the two-dimensional Ising model (E.Barouch, B.M. McCoy, C.A. Tracy, T.T. Wu [**Bar**], [**WTMTB**], [**McTW**]).
2. The two-point correlation functions at zero temperature for the one-dimensional impenetrable Bose-gas (M. Jimbo, T. Miwa, Y. Mari, M. Sato [**JMMS**]).
3. The two-point correlation functions at zero temperature for the one-dimensional isotropic XY-model (B. M. McCoy, J.H.H. Perk, R.E. Shrock, H.G. Vaidya, C.A. Tracy [**McPS**], [**VT**]).
4. The correlation functions in some topological field theories (C. Vafa, S. Cecotti [**CeVa1**], B. Dubrovin [**D**]).
5. The partition function of 2D quantum gravity (E. Brézin, V. Kazakov [**BreKaz**], M. Douglas, S. Shenker [**DoS**], D. Gross, A. Migdal [**GrMi**]).

A very interesting (as well as “visual”) application of the modern theory of Painlevé equations is the differential geometry of surfaces (see the monograph of A. I. Bobenko and U. Eitner [**BobEi2**]). More recent applications include:

(1) Orthogonal polynomials and random matrices.
(2) Random permutations, growth processes, last passage percolation.
(3) Number theory (the distribution of zeros of zeta-function).
(4) String Theory.

An intriguing appearance of the second Painlevé function occurs in recent studies of the classical Hele-Shaw problem (A. S. Fokas and S. Tanveer [**FoT**]), where it seems that for the first time the properties of the Painlevé function on the *complex plane* were used. Another exciting example, which is also related to the Hele-Shaw problem, is the appearance of the Painlevé functions in the analysis of the normal matrix ensembles (see the series of works of A. Zabrodin, P. Wiegmann and their collaborators [**TBAZ**] and references therein).

In what follows we give a brief outline of some specific applications.

Three-dimensional wave collapse and PII.

The so-called strong wave collapse in the three-dimensional nonlinear Schrödinger equation,

$$iU_t + \frac{1}{2} \triangle U + |U|^2 U = 0, \quad x \in \mathbb{R}^3 \tag{37}$$

(condition for the existing of collapse: $H \equiv \int_{R^3} (|\bigtriangledown U|^2 - |U|^4) dx < 0$) was first analyzed in detail by V.E. Zakharov, E.A. Kuznetsov, and S.L. Musher [**ZK**]. In particular, they discovered that the corresponding transition mode can be described in terms of the second Painlevé transcendent. The analysis of [**ZK**] had been completed by V. Yu. Novokshenov who calculated the uniform asymptotics of the corresponding solution of equation (37) using the modern theory of PII. Below we present this result following Chapter 15 of the monograph [**IN1**].

Suppose that collapse occurs at the origin $x = 0$, at $t = t_0 > 0$. Suppose also, that $U = U(r,t)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The aim is to find an asymptotic expansion for $U(r,t)$ uniform in r as $t \to t_0$. As usual, we have an inner, an outer and a transition regions.

The inner expansion

$$U_{in} = \exp\left\{i\tau^{-1/5}\Phi(\xi)\right\} \cdot \left[\tau^{-3/5} f(\xi) + \tau^{-1/2}\alpha(\xi)\cos\theta(\xi,\tau) + \dots\right],$$
$$\tau \to 0,$$
$$\xi = \frac{r}{l\tau^{2/5}},\ \tau = t_0 - t > 0,\ \Phi(\xi) = \frac{3}{5}l^2\xi_c - \frac{1}{5}l^2\xi,$$
$$f(\xi) = \frac{l}{5}\sqrt{3(\xi_c^2 - \xi^2)}, \quad \xi < \xi_c,$$
$$\Theta(\xi,\tau) = \tau^{-1/5}(\xi_c - \xi)^{3/2}\left(-\frac{4l^2}{5\sqrt{3}}\sqrt{2\xi_c}\right)$$
$$-\frac{C^2}{\sqrt{6\xi_c}}\log\tau + \frac{5C^2}{4\sqrt{\xi_C}}\sqrt{6}\log(\xi_c - \xi) + \beta_0 + \dots$$
$$\alpha(\xi) = \frac{C}{2}(\xi_c - \xi)^{-1/4} + \dots, \quad \xi \to \xi_c - 0.$$

The above expansion implies the parameters : $(l,\ \xi_c,\ C,\ \beta_0)$.

The outer expansion

$$U_{out} = \exp\left\{i\tau^{-1/5}\Phi(\xi)\right\} \cdot \left[\tau^{-1/2}\tilde{\alpha}(\xi)\sin\tilde{\theta}(\xi,\tau) + \dots\right],$$
$$\tau \to 0, \quad \tau > 0, \quad \xi > \xi_c,$$
$$\tilde{\Theta}(\xi,\tau) = \frac{4l^2}{5\sqrt{3}}\sqrt{\xi_c}(\xi - \xi_c)^{3/2}$$
$$-\frac{\tilde{C}^2}{4\sqrt{3\xi_c}}\log\tau + \frac{5\tilde{C}^2}{8\sqrt{\xi_c}}\sqrt{3}\log(\xi_c - \xi) + \tilde{\beta}_0 + \dots$$
$$\alpha(\xi) = \frac{\tilde{C}}{2}(\xi - \xi_c)^{-1/4} + \dots, \quad \xi \to \xi_c + 0.$$

The above expansion implies the parameters : $(l,\ \xi_c,\ \tilde{C},\ \tilde{\beta}_0)$.

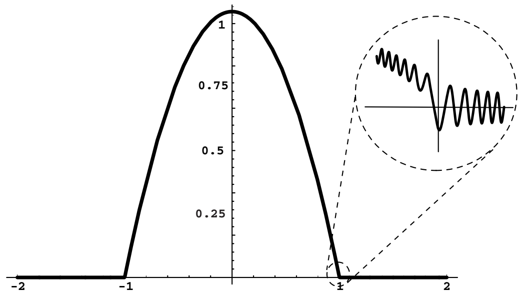

Figure II. The matching of the core parabolic profile with the zero tail by the transition mode expressed in terms of a solution of PII

Transition formulae

$$U_{trans} \cong \tau^{-8/15}u(\zeta)\exp\left\{i\tau^{-1/5}\Phi(\xi)\right\},$$

(38) $$\frac{6}{25}\xi_c l^2\zeta u + \frac{1}{2l^2}u_{\zeta\zeta} + u^3 = 0,$$

$$\xi = \xi_c + \tau^{2/15}\zeta, \quad \text{transition region}: \quad \xi_c - \xi = O(\tau^{2/15-\epsilon}).$$

Introducing the new variables,

$$\zeta = -\left(\frac{12l^4}{25}\xi_c\right)\eta^{-1/3}, \;\; u = \left(\frac{12l}{25}\xi_c\right)^{1/3} v(\zeta),$$

we can rewrite (37) in the canonical form of the second Painlevé equation,

$$v_{\eta\eta} = \eta v - 2v^3.$$

The known connection formulae for PII (use Theorem 1 as well as the transformation $u \to v = iu$, which changes the sign before the nonlinearity) allow us to connect the inner and outer parameters,

$$(C,\ \beta_0) \leftrightarrow (\tilde{C},\ \tilde{\beta}_0). \tag{39}$$

For example, denoting

$$b \doteq C\sqrt{\frac{5}{2}}(3\xi_c)^{-1/4}, \quad a \doteq \tilde{C}\sqrt{\frac{5}{2}}(\xi_c)^{-1/4},$$

we find

$$b^2 = \frac{3a^2}{4\sqrt{2}} - \frac{1}{2\pi\sqrt{2}}\log 2\sinh\frac{\pi a^2}{2} -$$

$$\frac{1}{\pi\sqrt{2}}\log\left|2\sin\left\{\frac{3}{2}a^2\log 2 + \frac{\pi}{4} - \arg\Gamma(\frac{ia^2}{2}) - \tilde{\beta}_0 + \frac{a^2}{4}\log(\frac{12l^4}{25}\xi_c)\right\}\right|.$$

Finally, using the conservation laws,

$$E = \int_{R^3}|U|^2dx, \quad H = \int_{R^3}(|\bigtriangledown U|^2 - |U|^4)dx,$$

one can determine the remaining parameters $l, \xi_c, \tilde{C}, \tilde{\beta}_0$, through E and H, i.e., through the initial data $U_0(x)$.

Bound states of the elliptic sine-Gordon equation and PIII

Singular solutions of the 2D elliptic sine-Gordon equation

$$u_{xx} + u_{yy} = \sin(u) - f, \quad f = \text{const}, \tag{40}$$

which admit point-like singularities of the type

$$u \approx \rho\ \log r + \sigma + o(1), \qquad r^2 = x^2 + y^2 \to 0, \tag{41}$$

have important applications in condensed-matter physics, as models of defects. Such solutions are associated with supercurrent inhomogeneities near the corners of large area Josephson junctions [**Sha1**], as well as with patterns generated by dust particles in liquid crystals located in a rotating magnetic field [**Mi, Sha2**]. Moreover, equation (40) is an approximation [**Fri**] of the anisotropic Ginzburg-Landau equation, thus it represents a simple model for stationary phase defects in this equation.

The physical solutions should have *finite energy* E,

$$E = \int\left[\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \cos(u) + \cos(u_0) - f(u-u_0)\right]dxdy, \tag{42}$$

in analogy with the bound states in the scalar field theories. Here, u_0 represents a homogeneous asymptotic state $u \to u_0$, $r \to \infty$. However, it is well-known that in a

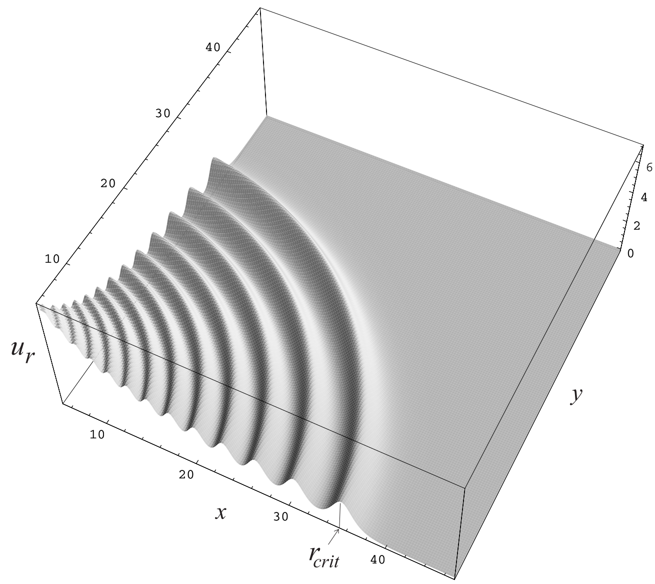

FIGURE III. The Josephson current intensity at the corner of a large rectangular junction.

class of smooth solutions, equation (40) has no bound states in $\mathbb{R}^2$. Nevertheless, if we admit solutions with point-like singularity satisfying (41), then global solutions with finite energy, i.e., bound states, do exist. But now, we have to exclude a small disk with radius $a \ll 1$ around the singular point, where a is the "cut-off" radius defined by the physical applicability of the continual model (40) [**L**].

In the force free case $f = 0$, bound state solutions with one singular point were found by computer simulation in [**Sha1, Sha2**]. They are radially symmetric, and they vanish exponentially at infinity $r \to \infty$. In addition, they have oscillatory radial structure up to a certain critical distance $r = r_{crit}$, where the exponentially decreasing mode comes to effect (see Figure).

This structure was interpreted in [**Sha1, Mi**] as a system of ring-like solitons (quasi-plane kinks of equation (40) at large distances), with common center at the singular point. Qualitatively, the picture does not depend on the driving force f up to a certain critical value f_{crit}. It was also found numerically [**Sha1, Sha2**] that for $f > f_{crit}$, the stationary radial-symmetric solutions of (40) become unstable and give rise to new dynamical patterns which satisfy the nonstationary equation

$$-u_t + u_{xx} + u_{yy} = \sin(u) - f.$$

The detailed treatment of the above radially symmetric structure was given by V. Novokshenov and A. Shagalov in [**N6**]. In the symmetric case the sine-Gordon equation (40) takes the form

$$u_{rr} + \frac{1}{r}u_r = \sin(u) - f. \tag{43}$$

For the case $f = 0$, the equation (43) reduces to a special case of the PIII equation. The solution of PIII describing the bound-state solutions can be characterized by the isomonodromy method as a limit case of the monodromy data manifold (see

Chapter 14 for details). At the same time, this solution can be described by its asymptotics both at the origin and at the infinity

$$u(r) = \begin{cases} \rho \ln x + \sigma + o(r), & r \to 0, \\ 2\pi \left(k + \frac{1}{2}\right) - \gamma \sqrt{\frac{2}{\pi}} r^{-\frac{1}{2}} \mathrm{e}^{-r} \left(1 + o(1)\right), & r \to +\infty, \end{cases} \tag{44}$$

where the constants γ, ρ and σ are related by the constraints

$$\gamma = 2 \sinh \frac{\pi \rho}{4}, \quad \sigma = -3\rho \log 2 - 4 \arg \Gamma \left(\frac{1}{2} - \frac{i\rho}{4} \right),$$

and the integer k is defined by

$$k = \left[\frac{1}{2\pi} \left\{ \pi - \sigma + 3\rho \ln 2 - 4 \arg \Gamma \left(\frac{1}{2} + \frac{i\rho}{4} \right) \right\} \right].$$

Due to the explicit connection formula (44), one can estimate the bound state radius r_{crit}, which is, roughly speaking, the coordinate of the most distant annular soliton of the target-like radial structure

$$r_{crit} = \frac{\pi}{4}\rho - \frac{1}{2} \log \rho + O(1), \quad \rho \to \infty.$$

This above analysis provides a guideline for the study of bound-state solutions to the nonintegrable equation (43). Namely, for large ρ and $|f| \leq 1$, it is possible to derive a critical value $f_c(\rho) \approx \mathrm{const} \cdot \rho^{-1}$, which governs the existence and stability of the bound-state solutions. For large area Josephson junctions and for nematic liquid crystals in a rotating magnetic field, these critical values were found in [**N6**]; they are in a good correlation with computer simulations and experimental data [**Sha1, Sha2**].

Regarding the sinh-Gordon equation, i.e., equation (43) with sine replaced by sinh, we note that bound-state solutions vanishing exponentially at infinity were studied in detail in [**McTW**]. However, these solutions correspond to the infinite-energy solutions of equation (43) under the map $u \to iu$, transforming sinh-Gordon into sine-Gordon equation. On the other hand, the connection formula for the amplitude $\gamma = \gamma(\rho)$ in (44) is correctly reproduced under this map.

Another important application of PIII (and actually somewhat similar to the one considered above) is the occurrence of the sinh-Painlevé III, as well as of some of its generalizations, in the theory of polyelectrolytes [**TW4**]. In this application, the principal role is played by the connection formulae derived in [**McTW**], [**Ki4**], and in [**TW3**].

Random matrices and random permutations

One of the basic random matrix models is the so-called *Unitary Ensemble* which is defined (see [**Meh**]) as the set of $N \times N$ Hermitian matrices $M = \{M_{ij}\}$, equipped with the unitary invariant probability measure given by the equation

$$\mu_N(dM) = Z_N^{-1} e^{-NTrV(M)} dM, \tag{45}$$

where

$$V(z) = \sum_{j=1}^{2m} t_j z^j, \quad t_{2m} > 0,$$

$$dM = \prod_{j<k}(d\,\mathrm{Re}\; M_{jk} d\,\mathrm{Im}\; M_{jk}) \prod_j dM_{jj},$$

and Z_N is the normalization constant, i.e.,

$$Z_N = \int e^{-NTrV(M)} dM. \tag{46}$$

An important particular case, the *Gaussian Unitary Ensemble GUE*, is characterized by the choice

$$V(z) = z^2.$$

The central question of random matrix theory is the analysis of the statistical properties of the ordered eigenvalues,

$$z_1(M) < z_2(M) < \cdots < z_N(M),$$

as $N \to \infty$. For instance, an important question is the computation of the probability $E_N(J)$ of finding no eigenvalues in the interval J. Another interesting question is whether there exist constants a and δ such that

$$(z_N - a)N^\delta$$

converges in distribution to a nontrivial distribution function $F(x)$, i.e.

$$\lim_{N\to\infty} \text{Prob}\left(z_N \le a + \frac{x}{N^\delta}\right) = F(x); \tag{47}$$

if such constants exist, what is this limiting distribution?

Consider the GUE case and choose the scaling $z \to \gamma_N z$, so that the expected number of eigenvalues per unit interval is 1. Then one can show (see [**Meh**], [**TW1**] or [**De1**] for more detail) that the limit

$$E(J) = \lim_{N\to\infty} E_N(\gamma_N J),$$

exists for every Borel subset J of $\mathbb{R}$. Furthermore, this limit is given by the formula

$$E(J) = \det(1 - K_{\text{sine}}), \tag{48}$$

where K_{sine} is the (trace class) operator with kernel,

$$K_{\text{sine}}(z, z') = \frac{\sin \pi(z - z')}{\pi(z - z')}$$

acting on $L^2(J, dz)$. In 1980 M. Jimbo, T. Miwa, Y. Môri, and M. Sato [**JMMS**] expressed this Fredholm determinant in terms of the solution of a system of integrable differential equations describing the isomonodromy deformations of a certain linear matrix ODE with rational coefficients. In the simplest case of a single interval, $J = (0, x)$, the differential equation is a particular case of the fifth Painlevé equation. In this case, the Fredholm determinant (48) is given by

$$\det(1 - K_{\sin}) = \exp\left(\int_0^{\pi x} \frac{\sigma(t)}{t} dt\right), \tag{49}$$

where $\sigma(x)$ is the solution of the so-called sigma (or Hirota) form of PV (see, e.g. [**JMMS**]),

$$(x\sigma'')^2 + 4(\sigma - x\sigma' - (\sigma')^2)(\sigma - x\sigma') = 0, \tag{50}$$

which is characterized by the initial condition

$$\sigma(x) \sim -\frac{x}{\pi}, \quad x \to 0. \tag{51}$$

The Riemann-Hilbert integrability of equation (50), suggests the possibility of the evaluation of the large x asymptotics of the probability $E(J)$. Indeed, the following asymptotic expansion is valid:

$$\log E(J) \sim -\frac{\pi^2 x^2}{8} - \frac{1}{4}\log\frac{\pi x}{2} + c + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots, \tag{52}$$

where

$$c = \frac{1}{12}\log 2 + 3\zeta'(-1), \tag{53}$$

and the constants c_1, c_2, $\dots$, can be computed, in principle, in terms of integrals of Bessel functions. The first two terms of this expansion were found in 1973 by des Cloizeaux and Mehta [**dCM**], whereas the full expansion, including equation (53) for the constant term, were derived in 1976 by Dyson [**Dy2**]. The rigorous proof of the asymptotic series (52) was obtained (using the Riemann-Hilbert approach) in [**DeIZ**], but without the explicit evaluation of the constant term c. The analysis of [**DeIZ**] was preceded by [**Wi1**], where a rigorous derivation of the first term of the asymptotics was suggested, and by [**Su1**] where the leading term of the asymptotics of the corresponding fifth Painlevé transcendent was rigorously evaluated. The proof of Dyson's formula (53) for the constant c was obtained in 2003 by I. V. Krasovsky [**Kra**] and, independently, by T. Ehrhardt [**Eh**]. The asymptotic representations (51) and (52), constitute an example of connection formulae for a Painlevé equation. The ability of deriving such formulas makes equation (49) an efficient tool for the calculation of the gap probability $E(J)$.

A Fredholm determinant representation, similar to (49), also exists for the limiting distribution function $F(x)$ defined by relation (47). More precisely, one can show (see [**TW2**] and references therein) that in the case of GUE, $a = \sqrt{2}$, $\delta = 2/3$, and

$$\mathrm{Prob}\left(z_N \le \sqrt{2} + \frac{\sqrt{2}x}{N^{2/3}}\right) \to F(x) = \det(1 - K_{\mathrm{airy}}), \tag{54}$$

where K_{airy} is the (trace class) operator with kernel,

$$K_{\mathrm{airy}}(z, z') = \frac{\mathrm{Ai}(z)\mathrm{Ai}'(z') - \mathrm{Ai}'(z)\mathrm{Ai}(z')}{z - z'} \tag{55}$$

acting on $L^2((x,\infty), dz)$. The notations $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ are used for the classical Airy function and its derivative, respectively. It was established in [**TW2**] by Tracy and Widom that the Airy kernel determinant (55) can also be evaluated in terms of a Painlevé transcendence - this time Painlevé II:

$$\begin{aligned} F(x) &\equiv \det(1 - K_{\mathrm{airy}}) \\ &= \exp\left(\int_x^\infty (x - y)u^2(y)dy\right), \end{aligned} \tag{56}$$

where $u(x)$ is the solution of the second Painlevé equation,

$$u_{xx} = xu + 2u^3,$$

uniquely specified by the boundary condition,

$$u(x) = \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left\{-\frac{2}{3}x^{3/2}\right\}(1 + o(1)), \quad x \to +\infty. \tag{57}$$

This particular solution of the second Painlevé equation was first singled out and studied by S. P. Hastings and J. B. McLeod in [**HasM**], where it is shown that the solution $u(x)$ is smooth for all real x, and its behavior as $x \to -\infty$ is given be the equation,

$$u(x) = \sqrt{-\frac{x}{2}}\left(1 + O\left(\frac{1}{x^2}\right)\right), \quad x \to -\infty. \tag{58}$$

This is again an example of an explicit connection formula for a Painlevé equation which again makes the representation (56) an efficient tool for evaluating the distribution function $F(x)$. The solution $u(x)$ was studied extensively in the later works [**Su2**], [**K5**], [**DeZ2**], and [**K10**]. A detailed analysis of this solution, which includes its behavior for complex x, and the generalization of this solution to the case that the parameter α in PII is nonzero, is presented in Chapter 11 of this book.

The importance of the results presented above is based on the *universality* of the distribution functions $E(J)$ and $F(x)$. Namely, the fact is that equations (48) and (54) are valid for an arbitrary generic choice of the polynomial $V(z)$ in the measure (45) which defines the random matrix model (provided that the number $\sqrt{2}$ in (54) is replaced by a proper value of the constant a, which, of course, depends on $V(z)$). This statement, which constitutes the *Dyson conjecture*, has been proven for different classes of the potential $V(z)$, (see [**PS**], [**BlI1**], [**DeKMVZ**]; see also the review article [**TW7**], and the monograph [**De1**] for more details and for the history of the subject). We note that the Riemann-Hilbert approach has played a crucial role in the rigorous analysis of the random matrix universality. We also mention that, as it was shown by A. Soshnikov [**So**], the Tracy-Widom edge scaling limit (47), is also valid for the so-called Wigner ensembles (where the matrix elements on and above the diagonal are i.i.d. random variables). This result is particularly important because the corresponding measures are not unitary invariant, and therefore the "integrable" techniques, such us Fredholm determinant and Riemann-Hilbert methods, are not applicable.

The Hastings-McLeod Painlevé solution and the related hierarchy of integrable ODEs, appear in yet another universality issue. Namely, these functions describe the so-called *double scaling* limits of the random matrix distribution functions at the neighborhoods of certain critical values of the coefficients $t_1, t_2, \dots$ of the polynomial $V(z)$ (see [**BlI2**], [**BlE**], and [**CK**]; see also [**BlI2**] and [**BlE**] for the history of this problem and basic references).

All above results are based on the *orthogonal polynomial method* in the random matrix theory, i.e., on the fact that the eigenvalue correlations for finite N can be expressed in terms of orthogonal polynomials on the line with respect to the weight $\mathrm{e}^{-NV(z)}$, where the function $V(z)$ is the same as in (45) (see [**Dy1**]; see also [**Meh**], [**TW1**] or [**De1**]). This implies the equivalence of the asymptotic analysis of random matrices with the asymptotic investigation of orthogonal polynomials. All the results mentioned above have their counterparts in the theory of orthogonal polynomials, and they provide solutions to some of the most basic long-standing questions of this theory. In fact, the majority of the works cited above deals with the relevant problems of the theory of orthogonal polynomials. The random matrix statements follow as simple corollaries. In other words, *the theory of orthogonal polynomials, provides yet another important field of applications of modern Painlevé theory.*

Painlevé equations do not only appear in the large N limit and in the investigation of universality, but they also occur before the large N limit is taken. Actually, *all* basic distribution functions can be expressed in terms of solutions of the various Painlevé equations. However, for finite N, these are *classical solutions*, unlike the "genuine" Painlevé transcendents which appear at $N = \infty$. For instance, as it was first shown in [**FoIK3**] (see also [**Ma**]), in the case of a quartic potential, $V(z) = t_2z^2 + t_4z^4$, the partition function Z_N, as a function of the ratio $t_2/\sqrt{t_4}$, is given in terms of a special solution of the fourth Painlevé equation which can be expressed in the parabolic cylinder functions. We refer the reader to the works of M. Adler and P. van Moerbeke [**AvM**], P. Forrester and N. Witte [**ForW1**], [**ForW2**], and G. Tracy and H. Widom [**TW6**] for more details on this aspect of the random matrix theory (in these papers, the universal Painlevé functions appearing at $N = \infty$ limit are also discussed).

As it is becoming increasingly evident, the limiting distributions of random matrix theory, and hence the related Painlevé transcendents, represent new limit laws in probability theory, which have numerous applications in number theory, combinatorics, statistical physics, growth processes, solid state physics, and statistics. We will now describe one of these applications, which concerns random permutations.

Let S_n be the symmetric group of degree n, i.e., the group of permutations σ of the numbers 1, 2, ..., n. If $1 \leq m_1 < m_2 < \cdots < m_k \leq n$, then $\sigma(m_1), \sigma(m_2), \dots, \sigma(m_k)$ is an increasing subsequence of σ of length k provided that

$$\sigma(m_1) < \sigma(m_2) < \cdots < \sigma(m_k).$$

Let $l_n(\sigma)$ be the length of the longest increasing subsequence in the permutation σ. Then,

$$\lim_{N\to\infty} \text{Prob}\left(l_n \leq 2\sqrt{N} + xN^{1/6}\right) = F(x), \tag{59}$$

where $F(x)$ is the Tracy-Widom distribution function of the random matrix theory (54), (56). This remarkable fact, proven in the celebrated paper [**BDJ**], led to an avalanche of activity in random permutations and related areas of combinatorics, representation theory and the theory of percolations and growing processes. The interested reader can find more about these exciting developments, in the review articles [**TW7**] and [**De2**]. For most recent results and references - see [**BKMM**].

2D quantum gravity and PI

Consider the partition function of the random matrix model (cf. (46)),

$$Z_N(t) = \int e^{-NTrV(M)}dM, \tag{60}$$

corresponding to the quartic potential,

$$V(z) \equiv V(z;t) = \frac{1}{2}z^2 + tz^4. \tag{61}$$

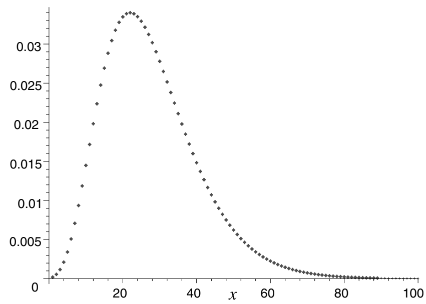

FIGURE IV. The density of the Tracy - Widom distribution

It was shown in the seminal paper of D. Bessis, C. Itzykson, and J. B. Zuber [**BIZ**], that the integral (60) (which is well defined for all $t \geq 0$) admits the following formal expansion[4] over N^{-2},

$$\log \frac{Z_N(t)}{Z_N(0)} \sim \sum_{g=0}^{\infty} N^{2-2g} E_g(t), \tag{62}$$

where

$$E_g(t) = \sum_{n \geq 1} (-t)^n \frac{\kappa_g(n)}{n!}. \tag{63}$$

The coefficients κ_n of the series (62) have a profound topological meaning, namely

$$\kappa_g(n) = \#\{\text{connected maps of genus } g \text{ with } n \text{ 4-valent vertices}\}. \tag{64}$$

See [**EM**] and also the survey [**FGZ**] for details.

The series (62) is a convergent series; moreover, by analyzing the explicit formulae that exist for $E_g(t)$ for small values of the genus g, one can suggest (see e.g. [**FGZ**]) that the radius of convergence of series (62) is equal to $1/48$ for all g, and that the point

$$t = -\frac{1}{48},$$

is a singular point for all the functions $E_g(t)$. The basic idea of E. Brézin and V. Kazakov [**BreKaz**], M. Douglas and S. Shenker [**DoS**], and D. Gross and A. Migdal [**GrMi**], is to associate the free energy functional $\mathcal{F}(x)$ of two-dimensional quantum gravity (for definitions see [**FGZ**]), with the second term in the following double-scaling expansion:

$$\log \frac{Z_N(t)}{Z_N(0)} = N^2 \mathcal{F}^0 + \mathcal{F}(x) + \dots, \tag{65}$$

$$N \to \infty, \quad t = -\frac{1}{48} - c_o N^{-4/5} x.$$

To be more precise, one has to perform an analytic continuation of the function $Z_N(t)$ to the negative values of t (which exists) and then take the limit indicated. The parameter $-x$ has the meaning of a renormalized string coupling constant. A

[4]The rigorous proof of the fact that the formal series in the right-hand side of equation (62) represents the large N asymptotics of its left-hand side was obtained only in 2003 by N. Ercolani and K. McLaughlin [**EM**].

detailed explanation of this terminology and for the reason that this particular limit should be taken as a definition of 2D quantum gravity, can be found in [**FGZ**]. Here, we only note that the double scaling limit (65) is supposed to make all the terms of series (62) of the same order. This fact, the assumption that the asymptotics (62) is valid for $-1/48 < t < 0$, and the topological meaning of the numbers $\kappa_g(n)$, imply that the limit (65) can be thought of as a "Riemann sum approximation" to the formal functional integral of the string theory.

The remarkable fact, discovered in [**BreKaz**], [**DoS**] and [**GrMi**], is that the limiting function $\mathcal{F}(x)$ satisfies (with the proper choice of the numerical coefficient $c_0 > 0$) the relation

$$\mathcal{F}''(x) = -u(x), \tag{66}$$

where $u(x)$ is the solution of the first Painlevé equation,

$$u_{xx} = 6u^2 + x, \tag{67}$$

uniquely specified by the asymptotic condition,

$$u(x) \sim \sqrt{\frac{-x}{6}} + \sum_{l=1}^{\infty} c_l (-x)^{1/2 - 5l/2}, \tag{68}$$

$$x \to \infty, \quad -\frac{\pi}{5} < \arg x < \frac{7\pi}{5}. \tag{69}$$

Strictly speaking, it was shown in [**BreKaz**], [**DoS**] and [**GrMi**] only that $u(x)$ belongs to a one-parameter family of solutions of PI with the asymptotics (68) as $x \to -\infty$. The characterization (68 - 69) was obtained in the works [**Da**] and [**FoIK1**] (see also [**FoIK2**], [**FoIK3**], [**FoIK4**], [**Mo**], and [**Mor**]). In [**FoIK1**], an alternative characterization of $u(x)$ was also found, using the earlier results of the global asymptotic analysis of PI obtained by Kapaev [**K2**] together with the Riemann-Hilbert approach: On the boundaries of the sector (69), the solution $u(x)$ exhibits the following oscillatory behavior,

$$\begin{aligned} u(x) = {} & \sqrt{\frac{-x}{6}} + \sum_{l=1}^{n} c_l (-x)^{1/2 - 5l/2} + O(x^{-2-5n/2}) \\ & -is(-x)^{-1/8} \exp\left\{-\frac{8}{5}\left(\frac{3}{2}\right)^{\frac{1}{4}} (-x)^{5/4}\right\} \Big(1 + o(1)\Big), \end{aligned} \tag{70}$$

as

$$x \to \infty, \quad \arg x = \arg(-x) + \pi = \frac{7\pi}{5},$$

and

$$\begin{aligned} u(x) = {} & \sqrt{\frac{-x}{6}} + \sum_{l=1}^{n} c_l (-x)^{1/2 - 5l/2} + O(x^{-2-5n/2}) \\ & +s(-x)^{-1/8} \exp\left\{\frac{8}{5}\left(\frac{3}{2}\right)^{\frac{1}{4}} (-x)^{5/4}\right\} \Big(1 + o(1)\Big), \end{aligned} \tag{71}$$

as

$$x \to \infty, \quad \arg x = \arg(-x) + \pi = -\frac{\pi}{5},$$

where

$$s = \frac{1}{\sqrt{8\pi}} \left(\frac{2}{3}\right)^{1/8}. \tag{72}$$

Each of the two asymptotics (70) and (71) characterizes $u(x)$ uniquely.

It is also worth noticing that the Riemann-Hilbert method in the theory of matrix models and orthogonal polynomials, mentioned in the previous subsection, was first introduced in [**FoIK1**]–[**FoIK4**].

A formal evaluation of the parameter s, as the coefficient of the exponentially decaying term in the asymptotics of $u(x)$ as $x \to -\infty$, was also performed in [**SY**].

The solution $u(x)$ is actually the classical *tri-tronquée* solution of the first Painlevé equation, which was already singled out and studied by P. Boutroux in [**Bou**]. Equations (70) - (72) give an *explicit* evaluation of the *Stokes phenomenon* for this solution, and they constitute a modern achievement ([**K2**], [**KK2**], [**GN**], [**K11**]), based on the isomonodromy method. In Chapter 11 we will discuss the issue of Stokes phenomenon in great detail for the case of PII.

Equation (66) has also an interesting combinatorial application. In fact, (66) suggests that the singularities of the functions $E_g(t)$ in (63) at $t = -1/48$ are of the following power form:

$$E_g(t) \sim e_g \left(t + \frac{1}{48}\right)^{5/2(1-g)} + \text{constant}. \tag{73}$$

Indeed, if we assume (73) and the validity of (62) for $-1/48 < t < 0$ (the absence of the Stokes phenomenon for the partition function $Z_N(t)$!) then by a direct formal calculations we find (cf. [**FGZ**])

$$\log \frac{Z_N(t)}{Z_N(0)} - N^2 \,\text{const} \sim \sum_{g=0}^{\infty} N^{2-2g} e_g \left(t + \frac{1}{48}\right)^{5/2(1-g)}$$

$$= \sum_{g=0}^{\infty} N^{2-2g} e_g \left(-c_0 x N^{-4/5}\right)^{5/2(1-g)} = \sum_{g=0}^{\infty} \hat{c}_g (-x)^{5/2-5g/2}.$$

The second derivative of the last series with respect to x is of the form of the series (68). This, in view of (66), can be considered as a formal confirmation of the estimate (73) (see the survey [**FGZ**] and references therein for more details). In turn, the estimate (73), following a standard reasoning, implies the following estimate for the combinatorial numbers (64):

$$\kappa_g(n) \sim C n! \frac{48^n}{n^{1+5/2(1-g)}}. \tag{74}$$

The above arguments are, of course, formal. The status of estimate (73), and hence of (74), remains that of a conjecture. Nevertheless, the current level of development of the Riemann-Hilbert techniques, and the experience with other combinatorial problems e.g. in random permutations [**BDJ**], suggest that all the gaps in the above construction will be soon filled. It should be mentioned, that the estimate (73) is also predicted by the continuum string theory (Liouville gravity, see [**FGZ**]).

A guidance to the book and to the related literature

The book consists of three parts (with a continuous numeration of chapters) and of two appendices. Each part has a short introduction, which provides a brief outline of the material discussed in this part. In what follows, we present a more extended overview of the content of the book.

Part 1 consists of Chapters 1 – 6, and it has two goals. The first goal is to develop the general theory of linear systems of ODEs. This is carried out in the first four chapters, which are essentially based on the monographs of W. Balser [**Bal**], E. A. Coddington and N. Levinson [**CodL**], Y. Sibuya [**Si2**], and W. Wasow [**W1**], as well as on the papers of M. Jimbo, T. Miwa, and K. Ueno [**JMU**], [**JM**]. The second goal is to introduce (following [**I1**] and [**I4**] as the "blueprint" outlined earlier) the both Painlevé functions and the "linear" special functions of hypergeometric type, as an intrinsic element of the monodromy theory of linear ODEs. This allows us to place the theory of Painlevé equations, from the very beginning, in the context of the Riemann-Hilbert method. We believe that this point of view on the theory of special functions has not been presented in the existing textbook literature; at least, it has not been put forward to the extent we do. The closest to our methodology (though still rather different), are the recent monographs [**IwKSY**] and [**SL**], as well as the isomonodromy approach to special functions initiated in [**IKK**] and further developed in [**Ki8**]. At the same time, in order to present the Riemann-Hilbert method as fully as possible we were obliged to limit our presentation of more conventional approaches to Painlevé equations. In particular, the classification problem mentioned at the beginning of the introduction and several important issues concerning the general properties of Painlevé functions are not addressed in this book. These issues are well covered in the existing literature, and we refer the interested reader to the classical book of Ince [**In**], to the recent monographs [**GroLS**], [**Cos**], and to the original articles [**JoK**] and [**JoKr2**]. We also recommend the proceedings volume [**Con**], as a good source of the classical as well as modern results concerning the Painlevé equations together with many interesting applications.

Chapters 3, 4, and 5 are the central chapters of this part of the book. This is where the general Riemann-Hilbert formalism is introduced (Chapter 3), the general theory of isomonodromy deformations (following mostly [**JMU**]) is presented (Chapter 4), the Painlevé equations are introduced and the specific Riemann-Hilbert formalism for integrating each of the six Painlevé transcendents (following mostly [**FoZ**] and [**FoMZ**]) is developed (Chapters 4 and 5).

As we have already indicated, we have attempted to make the book self-contained; therefore, in Chapter 3 we present and discuss (following very closely [**Zh**], [**DeZ4**], and [**DeZ5**]) the general theory of Riemann-Hilbert factorization and the related singular integral operators. The main analytical statement of Chapter 3 is meromorphic solvability of the Riemann-Hilbert problem related to the second Painlevé equation. The relevant proof is presented in **Appendix A** and it is based on the classical Birkhoff-Grothendieck theorem with a parameter, whose detailed proof is presented in **Appendix B**. These two appendices are based on the work [**BolIK**], and they relate parts of the book to the theory of holomorphic vector bundles. However, we do not develop further this aspect of modern theory of isomonodromy deformations. The interested reader can find more on this and other important geometrical aspects of the monodromy theory in the works of P. Boalch [**Bo3**], B. Malgrange [**Mal**], T. Miwa [**Miw**], P. Palmer [**Pa**], L. Mason, M. A. Singer, and N. Woodhouse [**Mas**]. It should be emphasized that the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems. Bearing this in mind, we believe that, although we do not use vector bundles directly in our study (in the case of Painlevé equations it is sufficient to use the "elementary" language of classical complex analysis), Appendices A and B prepare the interested reader to the more advanced texts on the general Riemann-Hilbert-Birkhoff problems.

An alternative approach to the inverse monodromy problems is based on the analytic Fredholm theory of the relevant singular integral operators. This theory is outlined in Chapter 3. It is used then in Chapter 5 (where we follow [**FoZ**] and [**FoMZ**]) to analyze the global solvability of the Riemann-Hilbert problems associated to the Painlevé I, II, III, IV, and V equations, and hence to establish the global solvability of these equations (see also [**BS**] where a similar technique is applied to the isomonodromy deformations of general linear systems). Painlevé VI is rather different than the other five Painlevé equations (although the relevant Riemann-Hilbert formalism for this equation was developed [**Mu**]). In the analysis of the sixth Painlevé equation a very important role is played by certain nontrivial algebraic properties of the space of its monodromy data. For the modern theory and for applications of the sixth Painlevé equation, we refer the reader to the pioneering work of M. Jimbo [**J1**] and to the more recent works of P. Boalch [**Bo4**], B. Dubrovin and M. Mazzocco [**DM**], D. Guzzetti [**Guz1**] - [**Guz2**], and A. Kitaev [**Ki9**] - [**Ki10**]. We will indicate later (see the description of Part III below) some of the specific features of the asymptotic analysis of the sixth Painlevé equation, which distinguish this equation from the other Painlevé functions.

In the last chapter of the first part we introduce the notion of *Bäcklund transformation* which maps solutions of a Painlevé equation into solutions of the same equation but with different parameters. These transformations play an important role in the theory of integrable systems. A large class of Bäcklund transformations for the Painlevé equations, is generated by the so-called Schlesinger transformations of the associated Riemann-Hilbert problems (see [**FoA2**], [**FoY**], [**Ki6**], [**MuFo**]). In Chapter 6 we construct sets of Bäcklund transformations for Painlevé II–IV equations (Painlevé I does not have a parameter). An important application of the Bäcklund transformations is that they generate the classical solutions of the Painlevé equations.

The theory of Bäcklund transformations has beautiful algebraic and algebrogeometrical ramifications, which we do not discuss in this book. We refer the reader to the works of A. P. Bassom, P. A. Clarkson and A. C. Hicks [**BCH**], P. Clarkson, N. Joshi and A. Pickering [**ClJP**], H. Sakai [**S**], to the monograph of M. Noumi [**No2**], and to the references in Chapter 6. Closely related to the algebrogeometrical approach to Painlevé equations are their Hamiltonian theory and the modern proofs of the irreducibility of the Painlevé equations. The main references on these aspects of the theory are the works of P. Boalch [**Bo1**]–[**Bo2**], H. Flaschka and A. Newell [**FN2**], J. Harnad [**Ha1**], J. Harnad and M. Wisse [**Ha2**], D. Korotkin and

H. Samtleben [**Ko3**], I. Krichever [**Kri1**] , K. Okamoto [**Ok1**]–[**Ok6**], M. Ugaglia [**U**], H. Umemura and H. Watanabe [**Um1**]–[**UmW**], [**Wa1**], [**Wa2**].

Bäcklund transformations are also closely related to the most recent direction in the theory of Painlevé equations, namely to the theory of the *discrete Painlevé equations* (for which, unfortunately, we have no room in our book). In fact, "half" of the discrete Painlevé equations are just the chains of Bäcklund transformations of the usual continuous Painlevé equations. For instance, one of the first discrete Painlevé equations, which was identified as the discrete Painlevé I equation, is the Bäcklund chain of the Painlevé IV equation (see [**FoIK3**], [**FoIK2**]). We also note that the PI and PII double scaling limits of the random matrix theory and the PII double scaling limit of random permutations, which were discussed above, are so far the only nontrivial examples for which the global asymptotic analysis (via a proper version of the Riemann-Hilbert method) of discrete Painlevé functions was carried out (d-PI in the case of [**FoIK3**] and [**BlI2**], and d-PII in the case of [**BDJ**]).

The other "half" of the discrete Painlevé equations consists of the so-called q-discrete equations. Their theory is not part of the continuous theory. They too are the isomonodromy (Schlesinger) transformations, but of *difference* linear equations, i.e., *both* equations of their Lax pairs are discrete. This is an extremely interesting area, which is in the very beginning of its development. We refer the reader to the works of M. Jimbo and H. Sakai [**JS**], A. Borodin [**Br**] and I. Krichever [**Kri2**], for more details on the q-Lax pairs and the setting of the q-monodromy problem.

For more details on the discrete Painlevé equations see the survey [**GrNR**]. We also refer the reader to the beginning of Chapter 6 where we briefly review some of the algebraic aspects of the Painlevé theory.

Part 2 which consists of Chapters 7–11, is the main part of the book. Taking the second Painlevé equation as a case study, we present in full detail the two major approaches to the asymptotic analysis of Painlevé equations via the Riemann-Hilbert method. The first scheme is based on the asymptotic solution of the *direct monodromy problem* for the associated linear λ-equation and uses the fact that the monodromy data form a complete set of first integrals of the Painlevé equations. This approach is presented in Chapter 7, Sections 1-3 of Chapter 9 and in Chapter 10. The basic technical tool of this approach is an extended version of the complex WKB method which was first suggested in [**IPe**], [**I3**], [**N3**], and [**IN1**] and which was further used and substantially developed in the works of A. Kapaev [**K1**]–[**K9**], A. Kitaev [**Ki2**]–[**Ki7**], F. Andreev and A. Kitaev [**AnK2**], and A. Kitaev and A. Vartanian [**KiV**]. A similar approach was also used by B. McCoy and Sh. Tang [**McT1**]–[**McT2**]. The key element of the technique is the introduction of a set of complex WKB solutions of the λ-equation. These solutions can be glued with the canonical solutions at the irregular singular points, and, at the same time, can be connected to each other through the relevant complex turning points (with the help of explicitly solvable model systems arising at the turning points). This allows an asymptotic evaluation of the corresponding Stokes matrices. Since the latter are *integrals of motion*, their asymptotic evaluation yields in fact the parametrization of the asymptotics of the solution of the Painlevé equation in different directions and near different critical points by the *same* parameters — the constant values of the entries of the Stokes matrices. This, in turn, provides the connection formulae. In its simplest form, this scheme is presented in Sections 1-3 of Chapter 9 (where we mostly follow the original method of [**IN1**]). Chapters 7 and 10 use a more advanced version of the WKB scheme which was developed by A. Kapaev in [**K7**]. Somewhat similar to this version, although still different in several important points, is the WKB method of A. P. Bassom, P.A. Clarkson, C. K. Law and J. B. McLeod [**BCLM**].

A crucial methodological element of the above direct monodromy approach, is the necessity of (at least some) a priori information about the local asymptotic behavior of the solution. Hence the connection formulae obtained by this approach must be rigorously justified. This issue can be resolved with the help of an elegant method based on the Brouwer fixed point theorem suggested by A. Kitaev [**Ki1**] (see Subsection 6.1 of Chapter 7; see also the discussion of the justification problem at the end of Section 1 of Chapter 9).

Chapters 8, Sections 4 and 5 of Chapter 9 and Chapter 11 present the second scheme of the asymptotic analysis of Painlevé equations. This scheme is based on the asymptotic solution of the *inverse monodromy problem* for the associated linear λ-equation, and it uses the nonlinear steepest descent method of P. Deift and X. Zhou for oscillatory Riemann-Hilbert problems [**DeZ1**]. This method was outlined earlier in the introduction; its important advantage is that it does not rely on any sort of a priori information, and therefore the rigorous proofs of the asymptotics and connection formulae are obtained in a much more straightforward way than in the case of the first scheme. At the same time, in many situations, and especially when one is confident about the asymptotic form of the solution, the formal derivation of the relevant connection formulae (with its a posteriori justification), is often easier via the first scheme. In particular, this is the case when one deals with the *generic* types of asymptotic behavior, such as the ones considered in Chapters 7 - 8 and Chapters 9 - 10. In order to help the reader to appreciate both schemes, in Chapter 8 we present the steepest descent derivation of the complex asymptotics, which was already obtained via the WKB technique in Chapter 7, while in Sections 4, 5 of Chapter 9 we rederive the WKB results of Sections 1–3 of the same chapter. However, in the case of the *special separatrix families* of Painlevé transcendents, the Deift-Zhou method is often better than the WKB scheme. We demonstrate this in Chapter 11 where, with the help of the nonlinear steepest descent approach, we describe rigorously and in full detail the quasi-linear Stokes phenomenon associated with the so-called *tronquée* solutions of the second Painlevé equation (this is methodologically similar with the contour integral techniques of the linear theory). One can also observe that the steepest descent approach, although technically more involved in the generic complex case, reveals in a most elegant way all the nontrivial elements of the asymptotics as the result of the proper deformation of the jump contours of the relevant Riemann-Hilbert problems.

The results obtained in the second part of the book give a complete description of the leading terms of the asymptotic behavior for complex $x \to \infty$ and the relevant connection formulae for all the generic families of solutions in the special case of the second Painlevé equation with the parameter $\alpha = 0$ (Chapters 7 - 10). For the general case of the second Painlevé equation with $\alpha \neq 0$, we only describe the quasi-linear Stokes phenomenon exhibited by the separatrix families of solutions. These families play a significant role in random matrix and string theory applications [**CKV**], [**Se**].

Part 3 consists of Chapters 12–16 where we study the important particular case of the third Painlevé equation which appears as a self-similar reduction of the sine-Gordon equation. From the methodological point of view, the analysis of the third Painlevé equation adds an important new component to the scheme developed in the preceding part. Indeed, unlike the second Painlevé equation, the third Painlevé equation has two critical point, $x = 0$ and $x = \infty$. This means that the connection problem for this equation, in addition to the nonlinear Stokes phenomenon near $x = \infty$, includes the so-called central connection problem, i.e., the evaluation of the relation between the asymptotic parameters at $x = 0$ and the asymptotic parameters at $x = \infty$. In fact, it is the central connection problem for the SG-PIII, or, in other words, the asymptotic behavior of the solution of the *Cauchy problem*, which is the question we consider in this part of the book.

Unlike the limit $x \to \infty$, whose analysis is very similar to the one performed in the case of the second Painlevé equation, the evaluation of the PIII monodromy data at the limit $x \to 0$ does not require the WKB arguments or the steepest descent constructions: The key observation is that, as $x \to 0$, the corresponding λ-equation and the Riemann-Hilbert problem can be, with the help of a proper rescaling, solved explicitly (see Chapter 14 for more details). This is a dramatic difference, between the investigation of the Riemann-Hilbert problem at a critical point which is an essential singularity of the Painlevé equation (the point $x = \infty$ in the case of PII and PIII), and the investigation of the Riemann-Hilbert problem at the point which is a branch singularity of the equation (the point $x = 0$ in the case of PIII). The evaluation of the relevant monodromy data in terms of the Cauchy data at the branch points was first performed, in the cases of the sixth and fifth Painlevé equations, by M. Jimbo in [**J1**]. A similar technique (together with the WKB analysis at the limit $x \to \infty$) was then used in the papers [**N2**]–[**N3**], [**Ki3**]–[**Ki4**], and [**KiV**] and in [**McT1**]–[**McT2**], [**AnK2**], for the third and fifth Painlevé equations, respectively. It should be emphasized that the papers [**N2**]–[**N3**] were the first works where the isomonodromy approach was applied to a Painlevé equation which is not the sixth Painlevé equation (which was done in [**J1**]). Also, the works [**McT1**]–[**McT2**] and [**AnK2**] were the works where the analysis of [**J1**], for the fifth Painlevé equation was completed by evaluating the monodromy data at $x = \infty$ and hence producing the Painlevé V connection formulae. An extended version of the method of [**J1**] has also been used in the recent studies of the sixth Painlevé equation in the papers [**DM**], [**Guz1**]–[**Guz2**], and [**Bo4**].

A detailed exposition of the asymptotic solution of the generalized Cauchy problem for the SG-PIII equation is presented in Chapters 13 - 16. In these chapters we discuss the smooth (for real x) two-parameter families of solutions (Chapter 14), the important separatrix solutions (Chapter 13 and Chapter 14), and the two-parameter singular (for real x) solutions (Chapter 15). The latter analysis adds yet another new component to the scheme of Part 2; namely it presents the evaluation (in terms of the Cauchy data) of the limiting distribution of the poles of the solutions of the Painlevé equations. In the last chapter of this part, the asymptotic behavior of the solution of the Cauchy problem in the complex plane is evaluated. The important new feature of this analysis is that unlike the PII case, the setting of the Riemann-Hilbert problem *depends on the value of* $\arg x$. It is important to notice that this feature is also shared with the fifth and sixth Painlevé equations and, in fact, with the isomonodromy deformations of the general linear systems.

In the third part of the book we also present (see Chapter 12) the sets of rational and algebraic solutions of the general PIII equation, presenting more details than those presented for the analogous case of the second Painlevé equation (Part 2 of the book).

In the third part of the book we mostly follow the original works of V. Novokshenov and A. Shagalov [**N2**]–[**N6**], and some of the parts of the monograph [**IN1**]. It should be mentioned that the third part of the book is mainly intended for the prospective “users” of the Painlevé functions. Therefore, we emphasize the qualitative properties and the physical meaning of the connection formulae derived, as well as the physical meaning of the third Painlevé equation itself. Also, for the same reason (and in contrast to Part 2), when we obtain the relevant asymptotic results, we focus on the formal derivation rather than on the details of the rigorous proofs. The proofs are often only sketched, with the understanding that the interested reader could complete them using the relevant material of the preceding parts.

The Parts 2 and 3, although methodologically related to the general Part 1, can be read independently of the latter by a reader who is already familiar with the general monodromy theory of linear systems.

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