
Preface

Algebraic Geometry

As the name suggests, algebraic geometry is the linking of algebra to geometry. For example, the unit circle, a geometric object, can be

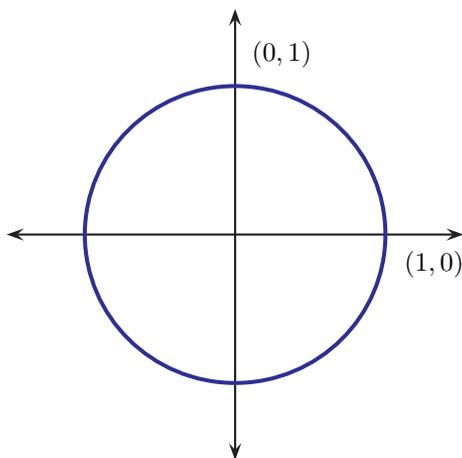


Figure 1. The unit circle centered at the origin.

described as the points (x, y) in the plane satisfying the polynomial

equation

$$x^2 + y^2 - 1 = 0,$$

an algebraic object. Algebraic geometry is thus often described as the study of those geometric objects that can be defined by polynomials. Ideally, we want a complete correspondence between the geometry and the algebra, allowing intuitions from one to shape and influence the other.

The building up of this correspondence has been at the heart of much of mathematics for the last few hundred years. It touches on area after area of mathematics. By now, despite the humble beginnings of the circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 = 0\},$$

algebraic geometry is not an easy area to break into.

Hence this book.

Overview

Algebraic geometry is amazingly useful, yet much of its development has been guided by aesthetic considerations. Some of the key historical developments in the subject were the result of an impulse to achieve a strong internal sense of beauty.

One way of doing mathematics is to ask bold questions about concepts you are interested in studying. Usually this leads to fairly complicated answers having many special cases. An important advantage of this approach is that the questions are natural and easy to understand. A disadvantage is that the proofs are hard to follow and often involve clever tricks, the origins of which are very hard to see.

A second approach is to spend time carefully defining the basic terms, with the aim that the eventual theorems and their proofs are straightforward. Here the difficulty is in understanding how the definitions, which often initially seem somewhat arbitrary, ever came to be. The payoff is that the deep theorems are more natural, their insights more accessible, and the theory is more aesthetically pleasing. It is this second approach that has prevailed in much of the development of algebraic geometry.

This second approach is linked to solving *equivalence problems*. By an equivalence problem, we mean the problem of determining, within a certain mathematical context, when two mathematical objects are *the same*. What is meant by *the same* differs from one mathematical context to another. In fact, one way to classify different branches of mathematics is to identify their equivalence problems.

Solving an equivalence problem, or at least setting up the language for a solution, frequently involves understanding the functions defined on an object. Since we will be concerned with the algebra behind geometric objects, we will spend time on correctly defining natural classes of functions on these objects. This in turn will allow us to correctly describe what we will mean by equivalence.

Now for a bit of an overview of this text. In Chapter 1 our motivation will be to find the natural context for being able to state that all nonsingular conics are the same. The key will be the development of the complex projective plane \mathbb{P}^2 . We will say that two curves in this new space \mathbb{P}^2 are *the same* (we will use the term “isomorphic”) if one curve can be transformed into the other by a projective change of coordinates (which we will define). We will also see that our conic “curves” can actually be thought of as spheres.

Chapter 2 will look at when two cubic curves are the same in \mathbb{P}^2 , meaning again that one curve can be transformed into the other by a projective change of coordinates. Here we will see that there are many different cubics. We will further see that the points on a cubic have incredible structure; technically they form an abelian group. Finally, we will see that cubic curves are actually one-holed surfaces (tori).

Chapter 3 turns to higher degree curves. From our earlier work, we still think of these curves as “living” in the space \mathbb{P}^2 . The first goal of this chapter is to see that these “curves” are actually surfaces. Next we will prove Bézout’s Theorem. If we stick to curves in the real plane \mathbb{R}^2 , which would be the naive first place to work, we can prove that a curve that is the zero locus of a polynomial of degree d will intersect another curve of degree e in at most de points. In our claimed more natural space of \mathbb{P}^2 , we will see that these two curves will intersect in exactly de points, with the additional subtlety of needing to give the correct definition for intersection multiplicity.

The other major goal of Chapter 3 is the Riemann-Roch Theorem, which connects the geometry and topology of a curve to its function theory. We will also define on a curve its natural class of functions, which will be called the curve's *ring of regular functions*.

In Chapter 4 we look at the geometry of more general objects than curves. We will be treating the zero loci of collections of polynomials in many variables, and hence looking at geometric objects in \mathbb{C}^n and in fact in k^n , where k is any algebraically closed field. Here the function theory plays an increasingly important role and the exercises work out how to bring much more of the full force of ring theory to bear on geometry. With this language we will see that there are actually two different but natural equivalence problems: isomorphism and birationality.

Chapter 5 develops the true natural ambient space, projective n -space \mathbb{P}^n , and the corresponding ring theory.

Chapter 6 increases the level of mathematics, providing an introduction to the more abstract, and more powerful, developments in algebraic geometry from the 1950s and 1960s.

Problem Book

This is a book of problems. We envision three possible audiences.

The first audience consists of students who have taken courses in multivariable calculus and linear algebra. The first three chapters are appropriate for a semester-long course for these students. If you are in this audience, here is some advice. You are at the stage of your mathematical career where you are shifting from merely solving homework exercises to proving theorems. While working the problems ask yourself what the big picture is. After working a few problems, close the book and try to think of what is going on. Ideally you would try to write down in your own words the material that you just covered. Most likely the first few times you try this, you will be at a loss for words. This is normal. Use this as an indication that you are not yet mastering this section. Repeat this process until you can describe the mathematics with confidence and feel ready to lecture to your friends.

The second audience consists of students who have had a course in abstract algebra. Then the whole book is fair game. You are at the stage where you know that much of mathematics is the attempt to prove theorems. The next stage of your mathematical development involves coming up with your own theorems, with the ultimate goal to become creative mathematicians. This is a long process. We suggest that you follow the advice given in the previous paragraph, and also occasionally ask yourself some of your own questions.

The third audience is what the authors refer to as “mathematicians on an airplane.” Many professional mathematicians would like to know some algebraic geometry, but jumping into an algebraic geometry text can be difficult. We can imagine these professionals taking this book along on a long flight, and finding most of the problems just hard enough to be interesting but not so hard so that distractions on the flight will interfere with thinking. It must be emphasized that we do not think of these problems as being easy for student readers.

History of the Book

This book, with its many authors, had its start in the summer of 2008 at the Park City Mathematics Institute’s Undergraduate Faculty Program on Algebraic and Analytic Geometry. Tom Garrity led a group of mathematicians on the basics of algebraic geometry, with the goal being for the participants to be able to teach algebraic geometry to undergraduates at their own college or university.

Everyone knows that you cannot learn math by just listening to someone lecture. The only way to learn is by thinking through the math on your own. Thus we decided to write a new beginning text on algebraic geometry, based on the reader solving many exercises. This book is the result.

Other Texts

There are a number of excellent introductions to algebraic geometry, at both the undergraduate and graduate levels. The following is a brief list, taken from the first few pages of Chapter 8 of [Fowler04].

Undergraduate texts. Bix's *Conics and Cubics: A Concrete Introduction to Algebraic Geometry* [Bix98] concentrates on the zero loci of second degree (conics) and third degree (cubics) two-variable polynomials. This is a true undergraduate text. Bix shows the classical fact, as we will see, that smooth conics (i.e., ellipses, hyperbolas, and parabolas) are all equivalent under a projective change of coordinates. He then turns to cubics, which are much more difficult, and shows in particular how the points on a cubic form an abelian group. For even more leisurely introductions to second degree curves, see Akopyan and Zaslavsky's *Geometry of Conics* [AZ07] and Kendig's *Conics* [Ken].

Reid's *Undergraduate Algebraic Geometry* [Rei88] is another good text, though the undergraduate in the title refers to British undergraduates, who start to concentrate in mathematics at an earlier age than their U.S. counterparts. Reid starts with plane curves, shows why the natural ambient space for these curves is projective space, and then develops some of the basic tools needed for higher dimensional varieties. His brief history of algebraic geometry is also fun to read.

Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra by Cox, Little, and O'Shea [CLO07] is almost universally admired. This book is excellent at explaining Groebner bases, which is the main tool for producing algorithms in algebraic geometry and has been a major theme in recent research. It might not be the best place for the rank beginner, who might wonder why these algorithms are necessary and interesting.

An Invitation to Algebraic Geometry by K. Smith, L. Kahanpaa, P. Kekaelaenen, and W. N. Traves [SKKT00] is a wonderfully intuitive book, stressing the general ideas. It would be a good place to start for any student who has completed a first course in algebra that included ring theory.

Gibson's *Elementary Geometry of Algebraic Curves: An Undergraduate Introduction* [Gib98] is also a good place to begin.

There is also Hulek's *Elementary Algebraic Geometry* [Hul03], though this text might be more appropriate for German undergraduates (for whom it was written) than U.S. undergraduates.

The most recent of these books is Hassett's *Introduction to Algebraic Geometry*, [Has07], which is a good introductory text for students who have taken an abstract algebra course.

Graduate texts. There are a number, though the first two on the list have dominated the market for the last 35 years.

Hartshorne's *Algebraic Geometry* [Har77] relies on a heavy amount of commutative algebra. Its first chapter is an overview of algebraic geometry, while chapters four and five deal with curves and surfaces, respectively. It is in chapters two and three that the heavy abstract machinery that makes much of algebraic geometry so intimidating is presented. These chapters are not easy going but vital to get a handle on the Grothendieck revolution in mathematics. This should not be the first source for learning algebraic geometry; it should be the second or third source. Certainly young budding algebraic geometers should spend time doing all of the homework exercises in Hartshorne; this is the profession's version of paying your dues.

Principles of Algebraic Geometry by Griffiths and Harris [GH94] takes a quite different tack from Hartshorne. The authors concentrate on the several complex variables approach. Chapter zero in fact is an excellent overview of the basic theory of several complex variables. In this book analytic tools are freely used, but an impressive amount of geometric insight is presented throughout.

Shafarevich's *Basic Algebraic Geometry* is another standard, long-time favorite, now split into two volumes, [Sha94a] and [Sha94b]. The first volume concentrates on the relatively concrete case of subvarieties in complex projective space, which is the natural ambient space for much of algebraic geometry. Volume II turns to schemes, the key idea introduced by Grothendieck that helped change the very language of algebraic geometry.

Mumford's *Algebraic Geometry I: Complex Projective Varieties* [Mum95] is a good place for a graduate student to get started. One of the strengths of this book is how Mumford will give a number of

definitions, one right after another, of the same object, forcing the reader to see the different reasonable ways the same object can be viewed.

Mumford's *The Red Book of Varieties and Schemes* [Mum99] was for many years only available in mimeograph form from Harvard's Mathematics Department, bound in red (hence its title "The Red Book"), though it is now actually yellow. It was prized for its clear explanation of schemes. It is an ideal second or third source for learning about schemes. This new edition includes Mumford's delightful book *Curves and their Jacobians*, which is a wonderful place for inspiration.

Fulton's *Algebraic Curves* [Ful69] is a good brief introduction. When it was written in the late 1960s, it was the only reasonable introduction to modern algebraic geometry.

Miranda's *Algebraic Curves and Riemann Surfaces* [Mir95] is a popular book, emphasizing the analytic side of algebraic geometry.

Harris's *Algebraic Geometry: A First Course* [Har95] is chock-full of examples. In a forest versus trees comparison, it is a book of trees. This makes it difficult as a first source, but ideal as a reference for examples.

Ueno's two volumes, *Algebraic Geometry 1: From Algebraic Varieties to Schemes* [Uen99] and *Algebraic Geometry 2: Sheaves and Cohomology* [Uen01], will lead the reader to the needed machinery for much of modern algebraic geometry.

Bump's *Algebraic Geometry* [Bum98], Fischer's *Plane Algebraic Curves* [Fis01] and Perrin's *Algebraic Geometry: An Introduction* [Per08] are all good introductions for graduate students.

Another good place for a graduate student to get started, a source that we used more than once for this book, is Kirwan's *Complex Algebraic Curves* [Kir92].

Kunz's *Introduction to Plane Algebraic Curves* [Kun05] is another good beginning text; as an added benefit, it was translated into English from the original German by one of the authors of this book (Richard Belshoff).

Holme's *A Royal Road to Algebraic Geometry* [Hol12] is a quite good recent beginning graduate text, with the second part a serious introduction to schemes.

An Aside on Notation

Good notation in mathematics is important but can be tricky. It is often the case that the same mathematical object is best described using different notations depending on context. For example, in this book we will sometimes denote a curve by the symbol C , while at other times denote the curve by the symbol $V(P)$ when the curve is the zero locus of the polynomial $P(x, y)$. Both notations are natural and both will be used.

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writing this book, though the section on duality in Chapter 1 was in part inspired by his insights. We are grateful for having had the chance to work with him, if ever so briefly.