Preface

By covering a carefully selected subset of topics, offering detailed explanations and examples, and with the occasional assistance of technology, this book aims to introduce undergraduate students to a subject normally only encountered by graduate students and researchers. Because of its interdisciplinary nature (bringing together different branches of mathematics as well as having connections to science and engineering), it is hoped that this book would be ideal for a one semester special topics class, "capstone" or reading course.

About Soliton Theory

There are many different phenomena in the real world which we describe as "waves". For example, consider not only water waves but also electromagnetic waves and sound waves. Because of tsunamis, microwave ovens, lasers, musical instruments, acoustic considerations in auditoriums, ship design, the collapse of bridges due to vibration, solar energy, etc., this is clearly an important subject to study and understand. Generally, studying waves involves deriving and solving some *differential equations*. Since these involve derivatives of functions, they are a part of the branch of mathematics known to professors as *analysis* and to students as *calculus*. But, in general, the differential equations involved are so difficult to work with that one needs advanced techniques to even get approximate information about their solutions.

It was therefore a big surprise in the late 20th century when it was realized for the first time that *some* of these equations are much easier than they first appeared. These equations that are not as difficult as people might have thought are called "soliton equations"

because among their solutions are some very interesting ones that we call "solitons". The original interest in solitons was just because they behave a lot more like *particles* than we would have imagined. But shortly after that, it became clear that there was something about these soliton equations that made them not only interesting, but also ridiculously *easy* as compared with most other wave equations.

As we will see, in some ways it is like a magic trick. When you are impressed to see a magician pull a rabbit out of a hat or saw an assistant in half it is because you imagine these things to be impossible. You may later learn that these apparent miracles were really the result of the use of mirrors or a jacket with hidden pockets.

In soliton theory, the role of the "mirrors" and "hidden pockets" is played by a surprising combination of algebra and geometry. Just like the magician's secrets, these things are not obvious to a casual observer, and so we can understand why it might have taken mathematicians so long to realize that they were hiding behind some of these wave equations. Now that the tricks have been revealed to us, however, we can do amazing things with soliton equations. In particular, we can find and work with their solutions much more easily than we can for your average differential equation.

Just as solitons have revealed to us secrets about the nature of waves that we did not know before (and have therefore benefited science and engineering), the study of these "tricks" of soliton theory has revealed hidden connections between different branches of mathematics that also were hidden before. All of these things fall under the category of "soliton theory", but it is the connections between analysis, algebra and geometry (more than the physical significance of solitons) that will be the primary focus of this book. Speaking personally, I find the interaction of these seemingly different mathematical disciplines as the underlying structure of soliton theory to be unbelievably beautiful. I know that some people prefer to work with the more general – and more difficult – problems of analysis associated with more general wave phenomena, but I hope that you will be able to appreciate the very specialized structure which is unique to the mathematics of solitons.

About This Book

Because it is such an active area of research, because it has deep connections to science and engineering, and because it combines many

different areas of mathematics, soliton theory is generally only encountered by specialists with advanced training. So, most of the books on the subject are written for researchers with doctorates in math or physics (and experience with both). And even the handful of books on soliton theory intended for an undergraduate audience tend to have expectations of prerequisites that will exclude many potential readers.

However, it is precisely this interdisciplinary nature of soliton theory – the way it brings together material that students would have learned in different math courses and its connections to science and engineering – that make this subject an ideal topic for a single semester special topics class, "capstone" experience or reading course.

This textbook was written with that purpose in mind. It assumes a minimum of mathematical prerequisites (essentially only a calculus sequence and a course in linear algebra) and aims to present that material at a level that would be accessible to any undergraduate math major.

Correspondingly, it is not expected that this book alone will prepare the reader for actually working in this field of research as would many of the more advanced textbooks on this subject. Rather, the goal is only to provide a "glimpse" of some of the many facets of the mathematical gem that is soliton theory. Experts in the field are likely to note that many truly important topics have been excluded. For example, symmetries of soliton equations, the Hamiltonian formulation, applications to science and engineering, higher genus algebro-geometric solutions, infinite dimensional Grassmannian manifolds, and the method of inverse scattering are barely mentioned at all. Unfortunately, I could not see a way to include these topics without increasing the prerequisite assumptions and the length of the book to the point that it could no longer serve its intended purpose. Suggestions of additional reading are included in footnotes and at the end of most chapters for those readers who wish to go beyond the mere introduction to this subject that is provided here.

On the Use of Technology

This textbook assumes that the reader has access to the computer program *Mathematica*. For your convenience, an appendix to the book is provided which explains the basic use of this software and offers "troubleshooting" advice. In addition, at the time of this writ-

ing, a file containing the code for many of the commands and examples in the textbook can be downloaded from the publisher's website: `www.ams.org/bookpages/stml-54`.

It is partly through this computer assistance that we are able to make the subject of soliton theory accessible to undergraduates. It serves three different roles:

➢ The solutions we find to nonlinear PDEs are to be thought of as being waves which change in time. Although it is hoped that readers will develop the ability to understand some of the simplest examples without computer assistance, *Mathematica*'s ability to produce animations illustrating the dynamics of these waves allows us to visualize and "understand" solutions with complicated formulae.

➢ We rely on *Mathematica* to perform some messy (but otherwise straightforward) computations. This simplifies exposition in the book. (For example, in the proof of Theorem 10.6 it is much easier to have *Mathematica* demonstrate without explanation that a certain combination of derivatives of four functions is equal to the Wronskian of those four functions rather than to offer a more traditional proof of this fact.) In addition, some homework problems would be extremely tedious to answer correctly if the computations had to be computed by hand.

➢ Instead of providing a definition of the elliptic function $\wp(z; k_1, k_2)$ that is used in Chapter 4 and deriving its properties, we merely note that *Mathematica* knows the definition of this function, calling it `WeierstrassP[]`, and can therefore graph or differentiate it for us. Although it would certainly be preferable to be able to provide the rigorous mathematical definition of these functions and to be able to prove that it has properties (such as being doubly periodic), doing so would involve too much advanced analysis and/or algebraic geometry to be compatible with the goals of this textbook.

Of course, there are other mathematical software packages available. If *Mathematica* is no longer available or if the reader would prefer to use a different program for any reason, it is likely that everything could be equally achieved by the other program merely by appropriately "translating" the code. Moreover, by thinking of the *Mathematica* code provided as merely being an unusual mathematical notation, patiently doing all computations by hand, and referring to

the suggested supplemental readings on elliptic curves, it should be possible to fully benefit from reading this book without any computer assistance at all.

Book Overview

Chapters 1 and 2 introduce the concepts of and summarize some of the key differences between linear and nonlinear differential equations. For those who have encountered differential equations before, some of this may appear extremely simple. However, it should be noted that the approach is slightly different than what one would encounter in a typical differential equations class. The representation of linear differential equations in terms of differential operators is emphasized, as these will turn out to be important objects in understanding the special nonlinear equations that are the main object of study in later chapters. The equivalence of differential equations under a certain simple type of change of variables is also emphasized. The computer program *Mathematica* is used in these chapters to show animations of exact solutions to differential equations as well as numerical approximations to those which cannot be solved exactly. Those requiring a more detailed introduction to the use of this software may wish to consult Appendix A.

The *story* of solitons is then presented in Chapter 3, beginning with the observation of a solitary wave on a canal in Scotland by John Scott Russell in 1834 and proceeding through to the modern use of solitons in optical fibers for telecommunications. In addition, this chapter poses the questions which will motivate the rest of the book: What makes the KdV Equation (which was derived to explain Russell's observation) so different than most nonlinear PDEs, what other equations have these properties, and what can we do with that information?

The connection between solitary waves and algebraic geometry is introduced in Chapter 4, where the contribution of Korteweg and de Vries is reviewed. They showed that under a simple assumption about the behavior of its solutions, the wave equation bearing their name transforms into a familiar form and hence can be solved using knowledge of elliptic curves and functions. The computer program *Mathematica* here is used to introduce the Weierstrass $\wp$-function and its properties without requiring the background in complex analysis which would be necessary to work with this object unassisted.

(Readers who have never worked with complex numbers before may wish to consult Appendix B for an overview of the basic concepts.)

The n-soliton solutions of the KdV Equation are generalizations of the solitary wave solutions discovered by Korteweg and de Vries based on Russell's observations. At first glance, they appear to be linear combinations of those solitary wave solutions, although the nonlinearity of the equation and closer inspection reveal this not to be the case. These solutions are introduced and studied in Chapter 5.

Although differential operators were introduced in Chapter 1 only in the context of *linear* differential equations, it turns out that their algebraic structure is useful in understanding the KdV equation and other nonlinear equations like it. Rules for multiplying and factoring differential operators are provided in Chapter 6.

Chapter 7 presents a method for making an $n \times n$ matrix M depending on a variable t with two interesting properties: its eigenvalues do not depend on t (the matrix is *isospectral*) and its derivative with respect to t is equal to $AM - MA$ for a certain matrix A (so it satisfies a differential equation). This digression into linear algebra is connected to the main subject of the book in Chapter 8. There we rediscover the important observation of Peter Lax that the KdV Equation can be produced by using the "trick" from Chapter 7 applied not to matrices but to a differential operator (like those in Chapter 6) of order two. This observation is of fundamental importance not only because it provides an algebraic method for solving the KdV Equation, but also because it can be used to produce and recognize *other* soliton equations. By applying the same idea to other types of operators, we briefly encounter a few other examples of nonlinear partial differential equations which, though different in other ways, share the KdV Equation's remarkable properties of being exactly solvable and supporting soliton solutions.

Chapter 9 introduces the KP Equation, which is a generalization of the KdV Equation involving one additional spatial dimension (so that it can model shallow water waves on the surface of the ocean rather than just waves in a canal). In addition, the Hirota Bilinear version of the KP Equation and techniques for solving it are presented. Like the discovery of the Lax form for the KdV Equation, the introduction of the Bilinear KP Equation is more important than it may at first appear. It is not simply a method for producing solutions to this one equation, but a key step towards understanding the geometric structure of the solution space of soliton equations.

The wedge product of a pair of vectors in a 4-dimensional space is introduced in Chapter 10 and used to motivate the definition of the Grassmann Cone $\Gamma_{2,4}$. Like elliptic curves, this is an object that was studied by algebraic geometers before the connection to soliton theory was known. This chapter proves a finite dimensional version of the theorem discovered by Mikio Sato who showed that the solution set to the Bilinear KP Equation has the structure of an infinite dimensional Grassmannian. This is used to argue that the KP Equation (and soliton equations in general) can be understood as algebro-geometric equations which are merely *disguised* as differential equations.

Some readers may choose to stop at Chapter 10, as the connection between the Bilinear KP Equation and the Plücker relation for $\Gamma_{2,4}$ makes a suitable "finale", and because the material covered in the last two chapters necessarily involves a higher level of abstraction.

Extending the algebra of differential operators to pseudo-differential operators and the KP Equation to the entire KP Hierarchy, as is done in Chapter 11, is only possible if the reader is comfortable with the infinite. Pseudo-differential operators are infinite series and the KP Hierarchy involves infinitely many variables. Yet, the reader who persists is rewarded in Chapter 12 by the power and beauty of Sato's theory which demonstrates a complete equivalence between the soliton equations of the KP Hierarchy and the infinitely many algebraic equations characterizing all possible Grassmann Cones.

A concluding chapter reviews what we have covered, which is only a small portion of what is known so far about soliton theory, and also hints at what more there is to discover. The appendices which follow it are a *Mathematica* tutorial, supplementary information on complex numbers, a list of suggestions for independent projects which can be assigned after reading the book, the bibliography, a Glossary of Symbols and an Index.

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Air Corps whose public use policy allowed me to reproduce their photo as Figure 9.1-4.