Introduction

In the first place, what are the properties of space properly so called? ...1st, it is continuous; 2nd, it is infinite; 3rd, it is of three dimensions; ...

HENRI POINCARÉ, 1905

So will the final theory be in 10, 11 *or* 12 *dimensions?*

MICHIO KAKU, 1994

As a separate branch of mathematics, topology is relatively young. It was isolated as a collection of methods and problems by HENRI POINCARÉ (1854–1912) in his pioneering paper *Analysis situs* of 1895. The subsequent development of the subject was dramatic and topology was deeply influential in shaping the mathematics of the twentieth century and today.

So what is topology? In the popular understanding, objects like the Möbius band, the Klein bottle, and knots and links are the first to be mentioned (or maybe the second after the misunderstanding about topography is cleared up). Some folks can cite the joke that topologists are mathematicians who cannot tell their donut from their coffee cups. When I taught my first undergraduate courses in topology, I found I spent too much time developing a hierarchy of definitions and

too little time on the objects, tools, and intuitions that are central to the subject. I wanted to teach a course that would follow a path more directly to the heart of topology. I wanted to tell a story that is coherent, motivating, and significant enough to form the basis for future study.

To get an idea of what is studied by topology, let's examine its prehistory, that is, the vague notions that led Poincaré to identify its foundations. GOTTFRIED W. LEIBNIZ (1646–1716), in a letter to CHRISTIAAN HUYGENS (1629–1695) in the 1670's, described a concept that has become a goal of the study of topology:

> *I believe that we need another analysis properly geometric or linear, which treats PLACE directly the way that algebra treats MAGNITUDE.*

Leibniz envisioned a calculus of figures in which one might combine figures with the ease of numbers, operate on them as one might with polynomials, and produce new and rigorous geometric results. This science of *PLACE* was to be called *Analysis situs* ([**68**]).

We don't know what Leibniz had in mind. It was LEONHARD EULER (1701–1783) who made the first contributions to the infant subject, which he preferred to call *geometria situs*. His solution to the Bridges of Königsberg problem and the celebrated *Euler formula*, $V - E + F = 2$ (Chapter 11), were results that depended on the relative positions of geometric figures and not on their magnitudes ([**68**], [**46**]).

In the nineteenth century, CARL-FRIEDRICH GAUSS (1777–1855) became interested in *geometria situs* when he studied knots and links as generalizations of the orbits of planets ([**25**]). By labeling figures of knots and links Gauss developed a rudimentary calculus that distinguished certain knots from each other by combinatorial means. Students who studied with Gauss and went on to develop some of the threads associated with *geometria situs* were JOHANN LISTING (1808–1882), AUGUSTUS MÖBIUS (1790–1868), and BERNHARD RIEMANN (1826–1866). Listing extended Gauss's informal census of knots and links and he coined the term *topology* (from the Greek $\tau o \pi o \upsilon \ \lambda o \gamma o \varsigma$, which in Latin is *analysis situs*). Möbius extended Euler's formula to

surfaces and polyhedra in three-space. Riemann identified the methods of the emerging *analysis situs* as fundamental in the study of complex functions.

During the nineteenth century analysis was developed into a deep and subtle science. The notions of continuity of functions and the convergence of sequences were studied in increasingly general situations, especially in the work of Georg Cantor (1845–1918) and finalized in the twentieth century by Felix Hausdorff (1869–1942) who proposed the general notion of a *topological space* in 1914 ([**33**]).

The central concept in topology is continuity, defined for functions between sets equipped with a notion of nearness (topological spaces) which is preserved by a continuous function. Topology is a kind of geometry in which the important properties of a figure are those that are preserved under continuous motions (homeomorphisms, Chapter 2). The popular image of topology as *rubber sheet geometry* is captured in this characterization. Topology provides a language of continuity that is general enough to include a vast array of phenomena while being precise enough to be developed in new ways.

A motivating problem from the earliest struggles with the notion of continuity is the problem of dimension. In modern physics, higher dimensional manifolds play a fundamental role in describing theories with properties that combine the large and the small. Already in Poincaré's time the question of the physicality of dimension was on philosophers' minds, including Poincaré. Cantor had noticed in 1877 that, as sets, finite-dimensional Euclidean spaces were indistinguishable (Chapter 1). If these identifications were possible in a continuous manner, a requirement of physical phenomena, then the role of dimension would need a critical reappraisal. The problem of dimension was important to the development of certain topological notions, including a strictly topological definition of dimension introduced by Henri Lebesgue (1875–1941) [**47**]. The solution to the problem of dimension was found by L. E. J. Brouwer (1881–1966) and published in 1910 [**10**]. The methods introduced by Brouwer reshaped the subject.

The story I want to tell in this book is based on the problem of dimension. This fundamental question from the early years of the

subject organizes the exposition and provides the motivation for the
choices of mathematical tools to develop. I have not chosen to follow
the path of Lebesgue into dimension theory (see the classic text [**38**])
but the further ranging path of Poincaré and Brouwer. The funda-
mental group (Chapters 7 and 8) and simplicial methods (Chapters
10 and 11) provide tools that establish an approach to topological
questions that has proven to be deep and is still developing. It is this
approach that best fits Leibniz's wish.

In what follows, we will cut a swath through the varied and beau-
tiful landscape that is the field of topology with the goal of solving the
problem of invariance of dimension. Along the way we will acquire
the necessary vocabulary to make our way easily from one landmark
to the next (without staying too long anywhere to pick up an accent).
The first chapter reviews the set theory with which the problem of
dimension can be posed. The next five chapters treat the basic point-
set notions of topology; these ideas are closest to analysis, including
connectedness and compactness. The next two chapters treat the
fundamental group of a space, an idea introduced by Poincaré to as-
sociate a group to a space in such a way that equivalent spaces lead to
isomorphic groups. The next chapter treats the Jordan Curve Theo-
rem, first stated by Jordan in 1882, and given a complete proof in 1905
by OSWALD VEBLEN (1880–1960). The method of proof here mixes
the point-set and the combinatorial to develop approximations and
comparisons. The last two chapters take up the combinatorial theme
and focus on simplicial complexes. To these conveniently constructed
spaces we associate their homology, a sequence of vector spaces, which
turn out to be isomorphic for equivalent complexes. This leads to a
proof of the topological invariance of dimension using homology.

Though the motivation for this book is historical, I have not
followed the history in the choice of methods or proofs. First proofs
of significant results can be difficult. However, I have tried to imitate
the mix of point-set and combinatorial ideas that was topology before
1935, what I call classical topology. Some beautiful results of this time
are included, such as the Borsuk-Ulam theorem (see [**9**] and [**56**]).

How to use this book

I have tried to keep the prerequisites for this book at a minimum. Most students meeting topology for the first time are old hands at linear algebra, multivariable calculus, and real analysis. Although I introduce the fundamental group in Chapters 7 and 8, the assumptions I make about experience with groups are few and may be provided by the instructor or picked up easily from any book on modern algebra. Ideally, a familiarity with groups makes the reading easier, but it is not a hard and fast prerequisite.

A one-semester course in topology with the goal of proving Invariance of Dimension can be built on Chapters 1–8, 10, and 11. A stiff pace is needed for most undergraduate classes to get to the end. A short cut is possible by skipping Chapters 7 and 8 and focusing the end of the semester on Chapters 10 and 11. Alternatively, one could cover Chapters 1–8 and simply explain the argument of Chapter 11 by analogy with the case discussed in Chapter 8. Another suggestion is to make Chapter 1 a reading assignment for advanced students with a lot of experience with basic set theory. Chapter 9 is a classical result whose proof offers a bridge between the methods of Chapters 1–8 and the combinatorial emphasis of Chapters 10 and 11. This can be made into another nice reading assignment without altering the flow of the exposition.

For the undergraduate reader with the right background, this book offers a glimpse into the standard topics of a first course in topology, motivated by historically important results. It might make a good read in those summer months before graduate school.

Finally, for any gentle reader, I have tried to make this course both efficient in exposition and motivated throughout. Though some of the arguments require developing many interesting propositions, keep on the trail and I promise a rich introduction to the landscape of topology.

Acknowledgements

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