Introduction

This book is an introduction to some topological, analytical, and algebro-geometric aspects of the theory of completely integrable Hamiltonian systems.

Hamiltonian systems are differential equations that describe the movements of an object whose energy is conserved (conservative mechanics). Certain of these systems can be integrated "by quadratures", these are the completely integrable Hamiltonian systems. One example of such a system, which all children know well, is that of the motion of a top. The motion of a free particle on a surface of revolution or an ellipsoid are other simple examples of completely integrable Hamiltonian systems.

What Is in This Book. The theory of integrable systems finds methods, followers, and applications in most of the fields of mathematics. It has as a result been the subject of an immense amount of work. No book—even one ten times thicker than this booklet—could address all these aspects.

Integrable systems belong to the field of (conservative) Hamiltonian mechanics, which today means the field of symplectic geometry. They have many conserved quantities traditionally called "first integrals". It is this definition which Chapter I addresses.

The heart of the material presented here is the Arnold–Liouville theorem, which describes the properties of solutions for which the values of these conserved quantities are generic. There are certain coordinates in which these systems take a very simple form. We have said that these systems are "integrable by quadratures". Geometrically, this means that their trajectories lie on tori and, furthermore, are *linear* on these tori (Figure 1). This is the Arnold–Liouville theorem and Chapter II is devoted to its proof and to some examples.

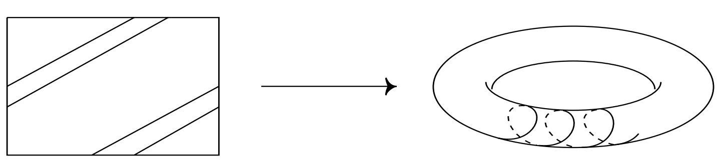

FIGURE 1

Next the material branches into two independent directions:

- In Chapter III, we turn to the algebraic theory of differential equations. The Arnold–Liouville theorem asserts that the solutions near a solution that is general enough have the same nature as their neighbor (again, see Figure 1). It is then tempting to look at what happens in general for the solutions that are (infinitesimally) near a trajectory of a Hamiltonian vector field. The word “infinitesimally” indicates that the differential equation has been linearized. With suitable hypotheses of analyticity, we can then make use of the Picard–Vessiot theory (differential Galois theory). The most interesting result in this area is a theorem of Morales and Ramis [**62**], which asserts that the differential Galois group of the linearized equation cannot be too complicated when the system is integrable (its identity component must be an Abelian group). This theorem can be used to show that certain Hamiltonian systems are *not* completely integrable.

- In Chapter IV, we turn to algebraic geometry, which in some cases is able to give us a more precise, even effective, version of the Arnold–Liouville theorem. When the differential system can be written as a “Lax equation”, there is associated with it an algebraic curve (or family of algebraic curves) which will lead to a (family of) Jacobian(s), which are tori ready to play the role of Liouville tori. In particular, we will see how this machinery gives us tools to the answer to concrete, naive questions such as determining whether a value of the first integrals is generic.

Finally, two appendices summarize some useful definitions and properties from differential Galois theory and the theory of algebraic curves, respectively.

While the methods of algebraic geometry have been the object of a great many works, including many books, the same cannot be said for the “Galois-theoretic” methods. In writing this book, one of my goals was to understand and popularize the beautiful Morales–Ramis theorem by presenting it to a less specialized audience than the original texts were written for. Although the criterion for non-integrability it provides belongs to the world of differential Galois theory, it certainly still is about objects belonging to the world of geometry!

This is why I tried

- to use the most classical and especially the most elementary examples possible. The main example used here to illustrate the Galois methods (a special case of the Hénon–Heiles system leading to the Airy equation) is, probably, the simplest possible one although it does not appear to be in any of the original articles;
- to clarify the situation a bit by separating
  - what is drawn from the general theory of differential equations (the variational equation, initial forms, first integrals, Galois groups)
  - from what is actually drawn from symplectic geometry.

  Similarly, I have tried to give the essential ingredients of the proofs in order to get straight to the desired goal;
- to maintain an acceptable level; in fact, these notes were developed from a graduate course (cours de DEA) given to real students (who contributed much to the simplicity of this book).

I am a firm believer in the virtue of examples, and so there are numerous examples (classical ones, as I have said, but they are the best) in Chapters I and II. While the mathematics used in Chapter III are no more difficult than in the first two chapters, the *applications* of the Morales–Ramis theorem are more elaborate. Therefore, unfortunately, Chapter III contains mainly academic examples. Chapter IV, on the other hand, is almost solely composed of examples—the main one being that of geodesics on quadrics.

The Exercises. Each chapter ends with a section of exercises. Most of these are direct applications of the material presented in that chapter. Against tradition, none of the important proofs have been left as an exercise.

The Index. I made a particular effort to make the index as complete and redundant as possible, both essential properties for its usefulness.

Notation. The notation is as standard as possible. There is no reason to let $\mathbb{P}^n(\mathbb{C})$ be the stabilizer of an element x under the action of a group G, or to let G_x be a symplectic form, or to let ω to be complex projective space, so I have simply conformed to tradition. I will use *slanted* characters to designate a notion that I am in the process of defining. I will use *italics* to emphasize something.

What Is Not in This Book. As I have said, there are many things, including

- the use of algebro-geometric machinery for counting Liouville tori, which would only be a repetition of [**8**];
- examples of infinite-dimensional integrable systems: no Korteweg–de Vries equation, no inverse scattering, all of which one can be found in Moser's little book [**66**] and Segal's article in [**42**];
- questions of perturbation theory, also found in [**66**].

Bibliographic Note. I will utilize the basics of differential calculus on manifolds (vector fields, differential forms, Lie derivative) which the readers can find, for example, in [**54**] or in the first volume of [**75**]. I will also use the definition of Lie groups and Lie algebras (and not much more)[1] and the basic concepts of group actions on manifolds (which the readers can find in [**21**] and also [**15**]). Apart from this, the book is "self-contained" in the sense that all the definitions of objects utilized in the text and all of the proofs, except for

- the uniqueness of Picard–Vessiot extensions and the fact that the Galois group is algebraic (see [**58**]);
- the triviality of holomorphic vector bundles on open Riemann surfaces, which I use very marginally and which can be found in [**29**];
- some results for Riemann surfaces (completion, Riemann–Roch,...), used in Chapter IV and listed explicitly in Appendix B, which can be found in [**72**].

In 1999, I gave three courses on integrable systems, where I presented some material contained in this book:

[1] Again, see [**54**].

- in Lisbon in June, where I particularly addressed algebro-geometric methods (Chapter IV);
- at the Centre International de Recherches en Mathématiques (Marseilles) in July, where I briefly discussed Galois-theoretic methods (Chapter III in this book), but not the algebro-geometric part, at the summer school "Symétries et application moment", organized by Patrick Iglesias and Elisa Prato;
- in Strasbourg in the fall, for a graduate course (Cours de DEA) where I presented all the material found in this book, except Chapter IV and, in particular, the material on Galois methods in a fair amount of detail.

This book is a development of the notes I wrote for the "students" of all these courses.

Acknowledgements. I would like to thank all the participants of the various courses I gave, and especially the warm audience of the Lisbon course. The tenacity and fidelity of those subjected to me twice a day, every day for two weeks, touched me greatly.

A special thanks goes to Ana Cannas da Silva, who not only organized the course in Lisbon[2] but also actively followed the course in Marseilles, providing remarks, questions, and numerous corrections and improvements.

The material on Galois methods was put together during the working group on the Morales–Ramis theorem that Claudine Mitschi and myself organized in Strasbourg during the spring of 1999. Most of what I understand about differential Galois theory comes from her presentations and explanations. She is also owed thanks for reading a draft of part of this book and suggestion several improvements.

I would also like to thank Leonor Godinho, Christelle Holtzmann, Marie-Laure Kostyra, and Julien Stiker for reading all or parts of various preliminary versions of this text and for the comments and corrections they provided.

A good-natured anonymous referee expressed a few wishes that I was happy to fulfill. I would like to thank him for the resulting improvements.

Finally, I thank Raymond Seroul, who designed the most beautiful figures in this book, Donald Babbitt, who carefully edited the English translation, and Claude Sabbah, who twice patiently edited the LaTeX files. It goes without saying that I, the author, am the only responsible for the remaining errors.

[2]Um mês de junho muito agradável em Lisboa... É um grande prazer agradecer o convite.