

Introduction

Conformal infinity of hyperbolic spaces

The most obvious examples of Einstein metrics are the symmetric spaces, and the simplest are the symmetric spaces of rank one. In this book, we shall be interested in those of negative curvature, whence of noncompact type: these are the hyperbolic spaces $\mathbb{K}\mathbf{H}^m$ ($m \geq 2$), where \mathbb{K} is the field of the real numbers (\mathbb{R}), of the complex numbers (\mathbb{C}), or of the quaternions (\mathbb{H}), or the algebra of the octonions (\mathbb{O}); in this latter case, only the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$ exists. The real dimension of \mathbb{K} will be denoted by d (whence $d = 1, 2, 4$ or 8) and that of $\mathbb{K}\mathbf{H}^m$ by $n = md$.

The sphere at infinity \mathbb{S}^{n-1} of a hyperbolic space has an interesting natural geometric structure, namely a conformal Carnot–Carathéodory metric. Let us consider this explicitly for the simplest cases.

Real hyperbolic space. The real hyperbolic space can be realized as the unit ball \mathbb{B}^n of \mathbb{R}^n with the metric (normalized so that the sectional curvature is -1)

$$g = 4 \frac{\text{euc}}{(1 - \rho^2)^2},$$

where ‘euc’ is the flat metric on \mathbb{R}^n and ρ is the radius. This metric induces the metric

$$(0.1) \quad \gamma = \lim_{\rho \rightarrow 1} (1 - \rho^2)^2 g_{\mathbb{S}^n}$$

on the boundary of \mathbb{S}^{n-1} . The function $t = (1 - \rho^2)$ is a function which goes to zero exactly at order one on the boundary (t is said to define the boundary), and the metric γ only depends on the choice of such a function up to a conformal factor, so that the conformal class $[\gamma]$ is intrinsically defined. We shall call this conformal class the conformal infinity of g , an expression borrowed from LeBrun.

Complex hyperbolic space. As for the complex hyperbolic space, this can be represented as the unit ball of \mathbb{C}^m with the Bergmann metric

$$g = \frac{\text{euc}}{1 - \rho^2} + \frac{\rho^2(d\rho^2 + (Id\rho)^2)}{(1 - \rho^2)^2}.$$

Instead of equation (0.1), which would lead to a highly degenerate tensor on the boundary, we set

$$(0.2) \quad \gamma = \lim_{\rho \rightarrow 1} (1 - \rho^2) g_{\mathbb{S}_\rho}.$$

This metric is now infinite, except on the distribution

$$V = \ker \eta \subset T\mathbb{S},$$

where $\eta = Id\rho$ is a connection 1-form on the \mathbb{S}^1 -bundle $\mathbb{S}^{2m-1} \rightarrow \mathbb{C}\mathbf{P}^{m-1}$. Such a metric, defined on a contact distribution, is called a Carnot–Carathéodory metric. As in the real case, only the conformal class $[\gamma]$ is intrinsically defined, and we shall again call it the conformal infinity of g .

General hyperbolic space. Let us now turn to a more general description, which is valid for all hyperbolic spaces. Having fixed a base point, the distance to this point is called r and the sphere of radius r \mathbb{S}_r . The metric γ on the sphere at infinity \mathbb{S}^{n-1} of the hyperbolic space $\mathbb{K}\mathbf{H}^m$ is given by the formula

$$(0.3) \quad \gamma = \lim_{r \rightarrow \infty} e^{-2r} g_{\mathbb{S}_r}.$$

This metric takes infinite values, except on a distribution $V \subset T\mathbb{S}$, of codimension 1 in the complex case, 3 in the quaternionic case, and 7 in the octonionic case; in the real case, γ is a true metric and $V = T\mathbb{S}$. Since the brackets of elements of V generate all the vector fields of \mathbb{S} , the metric γ is again a *Carnot–Carathéodory metric*; the definition of equation (0.3) only depends on the choice of the base point through a conformal factor, thus this metric is again called the *conformal infinity of g* (see the more precise Definition B). The hyperbolic metric can be described in terms of information on the boundary: there is a contact form η , with values in $\text{Im}(\mathbb{K}) = \mathbb{R}, \mathbb{R}^3$ or \mathbb{R}^7 , with kernel V , such that the metric can be written as

$$(0.4) \quad g = dr^2 + \sinh^2(2r)\eta^2 + \sinh^2(r)\gamma,$$

where the sectional curvature is normalized between -4 and -1 . In the real case, there is no term in η^2 ; in the other cases, the formula presupposes the choice of a complement to V in $T\mathbb{S}$ to extend the metric γ on V to a quadratic form on $T\mathbb{S}$. This complement is provided by the fibres of the fibration

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \longrightarrow & \mathbb{S}^{n-1} \\ & & \downarrow \\ & & \mathbb{K}\mathbf{P}^{m-1} \end{array}$$

Dirichlet problem. These symmetric metrics are of course Einstein metrics, whence

$$\text{Ric}^g = -\lambda g, \quad \lambda = n - 1, n + 2, n + 8, 36$$

in the real, complex, quaternionic and octonionic cases, respectively. The aim of this book is to study the following problem: given a conformal

Carnot–Carathéodory metric $[\gamma]$ on the boundary, study the nonlinear Dirichlet problem

- (i) $\text{Ric}^g = -\lambda g$;
- (ii) the conformal infinity of g is $[\gamma]$.

The fact that, as we shall see, we can deform hyperbolic Einstein metrics by solving this problem contrasts sharply with the rigidity of their compact quotients: Koiso [Koi78] showed that the Einstein metrics of compact quotients of hyperbolic spaces cannot be deformed; in a few cases in dimension 4, there is even a known global rigidity result, namely that the hyperbolic metrics on quotients are the only Einstein metrics there up to the action of diffeomorphisms (this was proved by Besson, Courtois and Gallot [BCG95] in the real case, and by LeBrun [LeB95] in the complex case). Quotients with a finite volume appear to behave in a similar manner [Biq97].

Two types of approach to the problem (i)–(ii) can be distinguished:

- (1) global solution of equation (i) with condition at infinity (ii): this approach relies on techniques of global analysis;
- (2) local solution near infinity: this more algebraic approach involves a strengthening of the equations, since the problem (i)–(ii) is an underdetermined Cauchy problem at infinity.

In this book, we present two situations, using the two types of approach described above, in which the solution of the problem allows one to construct new Einstein metrics: the first solution, using analytic techniques, is the subject of Chapter I (see the following section); the second situation, using twistorial techniques, is the subject of chapters II and III (see the following section).

Short historical review

The idea of considering the problem (i)–(ii) in order to understand Einstein metrics in terms of Carnot–Carathéodory metrics on the boundary has its roots in earlier works, on the one hand in real geometry (where the conformal infinity is a true conformal metric) and on the other hand in complex geometry (for Kähler–Einstein metrics). Let us develop this historical review a little.

Complex geometry. The most complete known results concern complex geometry. On a complex manifold, condition (i) can be replaced by the stronger condition of finding a Kähler–Einstein metric: the equation does not involve the tensor g , but rather a function which must satisfy a complex nonlinear Monge–Ampère equation. The problem is solved in full by the theorem of Cheng and Yau [CY80] which says, in particular, that any strictly pseudo-convex domain in \mathbb{C}^m admits a unique complete Kähler–Einstein metric, asymptotic to the CR structure of the boundary in the same way as in equation (0.2). Fefferman [Fef76] had earlier constructed formal solutions,

giving a high degree of approximation to the Monge–Ampère equation. Lee and Melrose studied the regularity of the solution near the boundary [LM82].

Real geometry. In the real case, when the Carnot–Carathéodory metric reduces to a true conformal metric on the boundary, the first approaches initially used the second method described above, and realized local constructions near the boundary. In dimension 4, LeBrun [LeB82] solved a local problem near an arbitrary real analytic boundary S ; the additional condition needed for the problem to be well-posed is

(iii) g is self-dual,

that is to say that the Weyl tensor is a self-dual 2-form. He also proved that any metric γ on a real analytic S is the conformal infinity of a unique solution g of equations (i), (ii) and (iii); if t is a function which defines the boundary S , then t^2g is analytic up to the boundary (there is thus a very strong regularity at infinity, which is not shared, for example, by the Cheng–Yau metrics, whose expansions at infinity involve logarithmic terms).

In higher dimensions, condition (iii) no longer makes sense. Fefferman and Graham [FG85] replaced it by a less geometric condition: if t defines the boundary S^{n-1} in a coordinate system in which

$$g = t^{-2} \left(dt^2 + \sum_1^{n-1} g_{ij}(x, t) dx^i dx^j \right),$$

we require the condition

(iv) $g_{ij}(x, t)$ is an even function of t

which is independent of the coordinate system. Fefferman and Graham show that, if n is even, for any metric γ on the manifold S^{n-1} , equations (i), (ii) and (iv) have a unique formal solution which is convergent in a neighbourhood of S if γ is analytic; on the other hand, if n is odd there are metrics γ for which no formal solution exists. This result, which was proved with the aim of constructing conformal invariants of γ from Riemannian invariants of g , seems to be a rediscovery of a theorem due to Schouten and Haantjes [SH36a, SH36b].

More recently, Graham and Lee [GL91] have tackled the problem (i)–(ii) via analysis, thereby producing global Einstein metrics: they have shown that if γ is close to the standard round metric on the sphere \mathbb{S}^{n-1} then the problem (i)–(ii) has a global solution close to the hyperbolic metric.

Quaternionic geometry. In the quaternionic case, LeBrun [LeB91] has used a twistorial method to construct an infinite-dimensional family of (global) quaternionic-Kähler deformations of the quaternionic hyperbolic metric. He did not study the behaviour of his metrics at infinity in terms of the problem (i)–(ii), but we shall see that they provide very interesting examples of solutions.

Explicit examples. Very few of these metrics are known explicitly. The only instance is that of SU_2 -invariant solutions in the ball \mathbb{B}^4 of the system (i), (ii) and (iii), when γ is an invariant metric or Carnot-Carathéodory metric. Hitchin [Hit95] has used the twistorial construction to find formulae giving solutions in terms of elliptic functions. Pedersen [Ped86] had previously dealt with the case in which γ is the metric of a Berger sphere.

Description of the results

We now turn to the description of the results proved in this book. In order to get an idea of the techniques used, the reader is referred to the introductions to Chapters I, II and III.

Carnot–Carathéodory metrics compatible with a contact structure. On the boundary S^{n-1} of the hyperbolic space $\mathbb{K}H^m$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$ or \mathbb{O} , the Carnot-Carathéodory metric γ , the conformal infinity defined on a distribution $V = \ker \eta$, has a $U_{m-1}, Sp_{m-1}Sp_1$ or $Spin_7$ structure compatible with the differential $d\eta$, where η is the contact 1-form with values in \mathbb{R}, \mathbb{R}^3 or \mathbb{R}^7 . This can be seen in the description of these boundaries as the homogeneous spaces $U_m/U_{m-1}, Sp_mSp_1/Sp_{m-1}Sp_1$ and $Spin_9/Spin_7$. In the complex case, this means that γ is obtained from the symplectic form $d\eta|_V$ through an almost complex structure; in the quaternionic and octonionic case, the fundamental 4-form $\sum_i d\eta_i^2|_V$ defines an $Sp_{m-1}Sp_1$ or $Spin_7$ structure which determines γ . It is this type of compatibility which we shall need.

DEFINITION A. *Let $H = U_{m-1}, Sp_{m-1}Sp_1$ or $Spin_7$, corresponding to the complex, quaternionic or octonionic cases, respectively. Let S^{n-1} be a manifold with a contact 1-form η with values in \mathbb{R}, \mathbb{R}^3 or \mathbb{R}^7 , respectively, and let $V = \ker \eta$. A Carnot–Carathéodory H -metric compatible with $d\eta$ is defined to be a metric γ on V such that*

- *in the complex case, the restriction to V of $d\eta$ is a symplectic form compatible with g (that is, $d\eta(\cdot, \cdot) = \gamma(I\cdot, \cdot)$ where I is an almost complex structure on V);*
- *in the quaternionic case, the three 2-forms $(d\eta_1, d\eta_2, d\eta_3)$ on V provide a quaternionic structure compatible with γ (that is, $d\eta_i(\cdot, \cdot) = \gamma(I_i\cdot, \cdot)$ for almost complex structures I_i satisfying the quaternionic commutation relations);*
- *in the octonionic case, the seven 2-forms $(d\eta_1, \dots, d\eta_7)$ on V provide a $Spin_7$ structure compatible with γ (that is, $d\eta_i(\cdot, \cdot) = \gamma(I_i\cdot, \cdot)$ for almost complex structures I_i satisfying the octonionic commutation relations).*

In the quaternionic and octonionic cases, a distribution V of codimension 3 or 7 for which such a metric exists will be called a *quaternionic contact structure* or an *octonionic contact structure*.

For a more precise discussion of the algebra associated with the octonions, see Section II.8. The complex case on the one hand, and the quaternionic and octonionic cases on the other hand, are different: in the complex case, given a contact 1-form, there are many compatible metrics; in the quaternionic and octonionic cases, the contact 1-form determines the metric, whose existence becomes a strong condition on the contact structure, completely. Let us also note that given the metric, the contact form with values in \mathbb{R}^{d-1} is completely determined in the complex case, and determined up to the action of the group SO_3 in the quaternionic case and of the group SO_7 in the octonionic case.

A word about the terminology: the expression ‘contact structure’ denotes the distribution itself; in the quaternionic and octonionic cases it thus determines the Carnot–Carathéodory metric up to a conformal factor.

Equation (0.4) motivates the following definition.

DEFINITION B. *A complete metric g on a manifold M with boundary S is asymptotically symmetric if S has a Carnot–Carathéodory H -metric g_V , compatible with a contact form η , such that the behaviour of g at infinity is given by*

$$g \sim dr^2 + \frac{1}{4}e^{4r}\eta^2 + \frac{1}{4}e^{2r}g_V,$$

where r is the distance to a fixed point. The conformal metric $[g_V]$ is called the conformal infinity of g .

In the real case, there is no longer any contact structure and g_V is a true metric on S .

To impart a sense to this definition, one has to extend the metric g_V by 0, by choosing a complement to V in TS ; the asymptotic behaviour does not depend on the choice.

The definitive definition (Definition I.3.1) indicates the derivatives needed. The term ‘asymptotically symmetric’ is justified by the fact that such metrics have a curvature tensor which is asymptotic to the curvature tensor of the corresponding symmetric space near infinity (Proposition I.1.3).

Global construction of Einstein metrics. We have seen that the Kähler–Einstein metrics of Cheng and Yau [CY80] and the quaternionic–Kähler metrics of LeBrun [LeB91] provide deformations of the hyperbolic spaces $\mathbb{C}\mathbf{H}^m$ and $\mathbb{H}\mathbf{H}^m$; these deformations preserve the holonomies of these spaces, namely U_m and $Sp_m Sp_1$; in the octonionic case there is no deformation preserving the $Spin_9$ holonomy of $\mathbb{O}\mathbf{H}^2$, since any metric with $Spin_9$ holonomy is locally symmetric. Conversely, the following theorem constructs, in some sense, the most general deformations.

THEOREM A. *Let $(M, g_0) = \mathbb{K}\mathbf{H}^m$ be the complex hyperbolic, quaternionic or octonionic space, respectively, and \mathbb{S} its sphere at infinity. Let $V \subset TS$ be a distribution of codimension $\ell = 1, 3$ or 7 , respectively. Let η be a 1-form with values in \mathbb{R}^ℓ with kernel V , and γ a Carnot–Carathéodory metric*

on the distribution V , compatible with $d\eta$ (Definition A). If γ is sufficiently close (in the Hölder $C^{2,\alpha}$ norm) to the standard Carnot–Carathéodory metric on the boundary of $\mathbb{K}\mathbf{H}^m$, then there exists an Einstein metric g on M which is asymptotically symmetric with conformal infinity $[\gamma]$ (Definition B). The metric g is locally unique modulo the action of diffeomorphisms inducing the identity on \mathbb{S} .

This theorem will be proved in Chapter I (Theorems I.4.8 and I.4.14), where, in particular, the behaviour of the metric g at infinity will be given precisely. These theorems also indicate the conditions under which one can (similarly) find Einstein deformations starting from an asymptotically symmetric Einstein metric g_0 , which is more general than the hyperbolic metric, and possibly definable on a manifold M different from the hyperbolic space. In particular, the theorem remains valid for an asymptotically symmetric metric g_0 on a manifold M if g_0 has a negative sectional curvature.

Of course, the theorem holds for the real hyperbolic space; the contact structures disappear and we rediscover the result of Graham and Lee [GL91], which thus finds its natural extension to other hyperbolic spaces¹.

In the complex case, in dimension $2m - 1 \geq 5$, the integrability of the CR structure (sufficiently close to the standard CR structure) is equivalent to the fact that the solution g produced by Theorem A should be Kähler–Einstein. Indeed, an integrable deformation of the \mathbb{S}^{2m-1} CR structure can be realized as a strictly pseudo-convex hypersurface of \mathbb{C}^m (see theorem 9.4 of [Tan75]), and the theorem of Cheng and Yau then provides a Kähler–Einstein solution to the problem. Thus, if the CR structure is integrable, Theorem A is a consequence of the theorem of Cheng and Yau, while in the case in which the CR structure is not integrable, the Einstein metrics constructed are new.

As often happens, a dividing line appears between the real and complex case on the one hand, and the quaternionic and octonionic case on the other hand; in the former case a contact structure close to the standard structure is diffeomorphic to that and the different metrics γ are obtained by varying the almost complex structure compatible with the symplectic form, in such a way that all the metrics g obtained are mutually bounded (moreover, ignoring the question of the regularity on the boundary at infinity, they include all bounded Einstein deformations of the complex hyperbolic metric); in the latter case, on the other hand, there is at most one metric γ which is compatible with the contact structure, and the metrics g obtained are not quasi-isometric (it can be shown that there are no bounded Einstein deformations of the hyperbolic quaternionic or octonionic metric).

In particular cases, there are known global uniqueness results. These are results on rigidity which relate to the scalar curvature rather than to the

¹In dimension 4, there is a slight technical difference between my result and that of Graham and Lee, since the latter had to assume the metric γ with $C^{3,\alpha}$ and not $C^{2,\alpha}$ regularity; my proof no longer needs this hypothesis.

Ricci tensor, and are modelled on Witten's spinorial proof of the positive mass conjecture [Wit81]. For example, Min-Oo [MO89] has shown that a metric h which is strongly asymptotic to the real hyperbolic metric g , with a scalar curvature $s^h \geq s^g$, must be equal to g ; in this result, strongly asymptotic means that $h - g$ and a derivative are of order $O(e^{-(n+\varepsilon)r})$; other results on rigidity based on the theorem of Min-Oo are given in [Leu93]. A similar result on rigidity for a Kählerian metric h which is asymptotic to the complex hyperbolic metric was proved by Herzlich [Her98].

Among the open questions raised by this theorem (see, for instance, the survey [Biq99]) one of the most difficult things is clearly to go beyond a neighbourhood of the hyperbolic metric, and hence to determine the conformal infinities for which the Einstein problem can be solved.

To get an idea of the proof of the theorem, as previously mentioned, readers might refer to the introduction to Chapter I, or, for further details but nothing technical, to the survey [Biq99].

Construction of local quaternionic-Kähler metrics. have to study. The quaternionic contact structures which I have introduced are new objects which I believe are interesting to study. This study is performed in Chapter II.

In sub-Riemannian geometry (see, for example, [FGR97]), one does not, in general, have a canonical connection on the manifold similar to the Levi-Civita connection in Riemannian geometry. However, in CR geometry, once a contact form (or, which amounts to the same thing, a representative of the conformal class of metrics) is chosen one has the Tanaka-Webster connection [Web79]. Remarkably, the same phenomenon occurs for quaternionic and octonionic contact structures, for which, after choosing a conformal factor, one can define an appropriate canonical connection (see Theorem II.1.3 and Definition II.2.8 in the quaternionic case and II.8.1 and II.8.3 in the octonionic case).

THEOREM B. *Let V be a quaternionic (respectively, octonionic) contact structure on a manifold S of dimension $4m + 3$ with $m \geq 2$ (respectively, of dimension 15) and g_V a compatible Carnot-Carathéodory metric. Then there exist a complement W of V and a corresponding connection ∇ on S , characterized by the properties:*

- (1) ∇ preserves V , and its restriction to V preserves its $Sp_m Sp_1$ (respectively, $Spin_7$) structure;
- (2) for $X, Y \in V$, the torsion satisfies $T_{X,Y} = -[X, Y]_W$;
- (3) for $R \in W$, the endomorphism of V defined by $X \mapsto (T_{R,X})_V$ is in the subspace $(\mathfrak{sp}_m \oplus \mathfrak{sp}_1)^\perp \subset \text{End}(V)$ (respectively, \mathfrak{so}_7^\perp);
- (4) ∇ preserves W and $\nabla|_W$ coincides with the connection induced on the sub-bundle \mathbb{R}^3 (respectively, \mathbb{R}^7) of $\text{End}(V)$ provided by the almost complex structures.

The first two conditions determine W uniquely.

The octonionic case of Theorem B is here to show the analogy with the quaternionic case, but the study of local octonionic contact structures stops there: actually it follows from the work of Yamaguchi [Yam93] that any octonionic contact structure is locally diffeomorphic to the standard octonionic structure of the sphere \mathbb{S}^{15} (I thank R. Bryant for pointing out this reference to me).

Conversely, one can go much further in the quaternionic case. Let us first note that the case $m = 1$ is excluded from the above theorem, as it will be later: in the same way that dimension 4 is special in quaternionic-Kähler geometry, dimension 7 is special for quaternionic contact structures and I have not dealt with this case here, but it has been later understood by Duchemin, see Section II.7.

The remainder of Chapter II is devoted to developing the properties of the torsion and the curvature of the adapted connection, ending with the construction of a twistor space which depends on the quaternionic contact structure, and not on a choice of conformal factor (see Theorem II.5.1 for more details).

THEOREM C. *A quaternionic contact structure on a manifold S of dimension $4m + 3$ with $m \geq 2$, admits a twistor space*

$$\begin{array}{ccc} \mathbb{S}^2 & \longrightarrow & \mathcal{T} \\ & & \downarrow \\ & & S \end{array}$$

which is an integrable CR manifold.

Somehow, the study of quaternionic contact structures leads back to study of a particular class of integrable CR manifolds (which are never pseudo-convex). This construction is clearly reminiscent of the construction of the twistor space of a quaternionic-Kähler manifold [Sal82].

This study culminates in Chapter III with the local construction of Einstein metrics which are solutions of the problem (i)–(ii) with a quaternionic contact structure as conformal infinity. As we have seen, in such a construction, in order to guarantee a well-determined Cauchy problem, one needs an extra condition which, in our case, is the direct generalization of condition (ii) of LeBrun, since in dimension 4, the ‘Einstein self-dual’ condition is the condition which is substituted for ‘quaternionic-Kähler’ in higher dimensions.

THEOREM D. *Let S be a manifold of dimension $4m + 3$ ($m \geq 2$) with a real analytic quaternionic contact structure; then S is the conformal infinity of a unique asymptotically symmetric quaternionic-Kähler metric which is real analytic up to the boundary (with a pole of order 2), defined in a neighbourhood of S .*

In the case of dimension 3 ($m = 0$), there is no contact structure and what we are given is simply a conformal metric and an orientation on S ,

and we rediscover the statement of LeBrun's theorem [LeB82]. In the case of dimension 7, the theorem can essentially be said to extend to quaternionic contact structures for which there exists a twistor space of the type constructed by Theorem C; the condition is the integrability condition found by Duhemin, see Section II.7.

A Sasakian manifold (see the survey [BG99]) manifestly induces a quaternionic contact structure; the metric constructed by Theorem D is explicit in this case, being provided by the formula which gives the quaternionic hyperbolic space. The 3-Sasakian structures are infinitesimally rigid, and thus form a discrete moduli space; they are therefore very special points in the moduli space of quaternionic contact structures, which must be infinite dimensional.

Indeed, unlike in the octonionic case, I know, thanks to LeBrun [LeB91] that there exist large numbers of quaternionic contact structures; in fact, a subproduct of the proof of Theorem D indicates that the metrics of the (infinite dimensional) family of deformations of the quaternionic hyperbolic metric constructed by LeBrun, are induced by quaternionic contact structures on the boundary. We see here an interesting case in which the quaternionic-Kähler extension to the interior is global.

However, this property has no chance of being true for all quaternionic contact structures. For the Einstein metrics on the ball produced by Theorem A, the question is closely related to the 'Positive Frequency Conjecture' stated in dimension 4 by LeBrun at the end of [LeB91]. One can make an analogy with the Dirichlet problem for harmonic functions on the disc; the only holomorphic ones are those whose value on the boundary belongs to the functions with 'positive frequencies'. In our situation, the quaternionic contact structures on the sphere with 'positive frequencies' are the ones such that the Einstein metric on the ball constructed by Theorem A is quaternionic-Kähler (in dimension 4, one looks at conformal metrics on the 3-sphere such that the Einstein metric in the interior is self-dual). In [Biq02], positive frequency quaternionic contact structures are determined: in dimension 3, they fill a half-dimensional space of conformal structures on the 3-sphere; in dimension 7, the space of quaternionic contact structures satisfying Duhemin's integrability condition; in higher dimension, all quaternionic contact structures.