Introduction

1.1. This memoir[1] is devoted to the study of the category of schemes from a homotopic point of view. More precisely, our objective is to define, for all reasonable[2] schemes k, the homotopy category $h(\mathcal{E}_k)$ of smooth k-schemes and to show that this category plays the same role for smooth k-schemes as it does for the classical homotopic category for differentiable varieties.

Working independently, V. Voevodsky proposed in [**33, 35**], an alternative definition to ours. We can show that at least when k is of finite Krull dimension, the two approaches lead to equivalent homotopic categories. In a concise way, we can say that our approach is combinatorial while Voevodsky's approach is topological, based on the concept of sheaves in the Nisnevich topology [**26**]. This latter approach is developed in [**25**].

The next step will be the study of "cohomology theories on the category of smooth k-schemes", which is intimately connected to the *stable* homotopic category on k, whose objects are the spectrum in the sense of homotopy theory. This category is triangulated and comes with a "smash-product" for which the pointed projective line $\mathbf{P}_k^1$ is invertible. The classical examples of cohomology theories (étale cohomology with torsion coefficients prime to the underlying characteristic, Betti cohomology associated to the complex points of k, Beilinson's (rational) motivic cohomology, Suslin-Voevodsky motivic cohomology) are the "ordinary" cohomology theories in the sense that they are factored by the triangulated category of mixed motives on k defined by Voevodsky [**34**] (at least when k is a field of characteristic 0). Quillen's higher algebraic K-theory [**28**] is the first example (just as in standard algebraic topology) of a generalized cohomology theory that is not ordinary. One of the results of this present work is a particular description of the spectra of algebraic K-theory which is implicit in the periodicity Theorem 4.3.6.

We will show elsewhere, when k is a field in which -1 is a square, that the stable homotopy category once tensored with $\mathbf{Q}$ contains in a natural way as a full subcategory the category of rational motives over k as defined by Grothendieck using Chow correspondences between smooth projective k-varieties, which conforms to the principle, well-known in algebraic topology, that once tensored with $\mathbf{Q}$ the cohomology theories again all become "ordinary" (as this is illustrated by a result of Grothendieck asserting that K_0 of a smooth variety over a field, after tensoring with $\mathbf{Q}$, is identified with the direct sum of Chow groups tensored with $\mathbf{Q}$).

The category $\mathcal{E}_k$ of k-spaces is the category of functors of that opposite category, denoted by $\mathcal{C}_k$, of smooth affine k-schemes to the category of sets. By its definition, it has a certain analogy with the category of simplicial sets (which is the category of functors $\Delta^{op} \to \mathcal{S}ets$, Δ denoting the category of "standard simplices"): we replace the standard simplices by smooth affine k-schemes. It is this analogy that provides the inspiration to define in §2.2 the homotopy category associated to these k-spaces. Indeed, the essential obstruction to establishing homotopy theory in the category of k-spaces is that there is no obvious concept of weak equivalences.

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[1]This text is a review and correction of the l'Institute de Mathématiques de Jussieu prepublication entitled, "Théorie de 'l'homotopie et motifs I: propriétés géomètriques fondamentales" (1995).

[2]noetherian, separated, admitting an ample family, *cf.* appendix B.

To arrive at such a concept, we analyze the method of Gabriel and Zisman [16] and then Quillen [27] to define the homotopy category of simplicial sets: one reverses in S the *anodyne extensions.* The set of anodyne extensions is the smallest set of S-morphisms satisfying certain well-known properties, and includes those maps $\Lambda^{n,r} \to \Delta^n$, where $\Lambda^{n,r}$ denotes the union of all the faces of the nth standard simplex Δ^n except the rth (face).

Inspired by this technique, we describe in appendix A.2.3 a method allowing us to associate to a reasonable quadruplet

$$(\mathcal{E}, \Delta^{\bullet}, S, S_{an})$$

formed from the category $\mathcal{E}$, a cosimplicial object $\Delta^{\bullet}$ of $\mathcal{E}$, a set S of *elementary cofibrations* and from a set S_{an} of *elementary anodyne extensions,* a homotopy theory, and in particular a homotopic category $h(\mathcal{E}, \Delta^{\bullet}, S, S_{an})$. The set of elementary cofibrations is to some extent the set of "generators" of the homotopy theory and the set of elementary anodyne extensions, its "relations".

The elementary cofibrations of the category of k-spaces correspond to the finite transverse families of closed immersions in a smooth affine k-scheme. This choice appeared reasonable to us by a culmination of other various "good" reasons. We can already remark that the faces of the affine k-space of dimension n, when one interprets this as the geometric realization over k of the nth standard simplex, are transverse. Moreover, all closed immersions between affine smooth k-schemes are transverse according to a classical result [**15, Corollary 17.12.2 d**]. A closed immersion between affine algebraic varieties over $\mathbf{C}$ induces a proper embedding of differentiable varieties, which is "triangulizable" (contrary to the induced map of an open immersion for example) and is therefore a good candidate to be a "cofibration" (think of the concept of a relative CW-complex). We were also influenced by the following principle, accordingly due to a statement by Jouanolu for k a field [**18**] and generalized by Thomason for arbitrary k (see appendix B): a "reasonable k-scheme" has the weak homotopy type of an affine k-scheme; to some extent, we have solidified this principle by affirming that any homotopy type is "manufactured" from smooth affine k-schemes. Finally, the last reason, and not the least, that guided our choice of elementary cofibrations: the technical but fundamental result of §4.1.10 which allows us to establish that when k is (noetherian) affine and regular, the infinite projective space, the infinite Grassmannian, the infinite multiplicative group and linear group are fibrations (in the terminology of Quillen).

The list of elementary anodyne extensions that we address (§2.2.8) translate two types of properties:

invariance under homotopy: corresponding to the fundamental simplicial anodyne extensions (*cf.* 2.2.1), which essentially translate the zero sections: $X \to X \times \mathbf{A}^1$, are weak equivalences.

homotopic excision: corresponding to the fundamental geometric anodyne extensions (*cf.* 2.2.8), which essentially translate the fundamental squares of smooth k-schemes (2.2.7), are homotopically cocartesian (see Theorem 3.1.2).

The yoga therefore is that these properties constitute the fundamental bond which connects geometry to homotopy theory. It is this same yoga which was developed elsewere by Voevodsky in a different form. Indeed, the fundamental squares are precisely those which allow the definition of sheaves in the Nisnevich topology [**25**].

Variants of the category of k-spaces. We have sought to define the smallest category of k-spaces. One could also consider the "category" $\mathcal{E}_k''$ of functors $(\mathcal{S}c_k)^{op} \to \mathcal{S}ets$ ($\mathcal{S}c_k$) denoting the category of k-schemes. One can also consider the category of functors $(\mathcal{L}_k)^{op} \to \mathcal{S}ets$, $\mathcal{L}_k$ designating this time the category of smooth k-schemes. One can then show that these categories define the same homotopic category if one applies the same method (same set of elementary cofibrations, same set of fundamental anodyne extensions).

One can also consider the full subcategory of $\mathcal{E}_k'$ (or $\mathcal{E}_k''$) formed by sheaves in some Grothendieck topology (Nisnevich [**26, 25**], étale, cdh, h-topology of Voevodsky, etc.). This time one takes for the set of elementary cofibrations, monomorphisms for a set of generators, (namely, the cofibrations are now the monomorphisms) and for elementary anodyne extensions the "same" as above. The resulting homotopic category depends this time on the topology. In the Nisnevich topology, one is dealing primarily with the homotopic category defined by Voevodsky [**25**] and one can show that this one is equivalent to ours (at least when k is of finite Krull dimension). One such result is to compare with the fact that the homotopic category of simplicial sets is equivalent to the homotopic category of topological spaces.

By applying the same method, but for the moment working with differentiable varieties, the homotopic category that one obtains is now equivalent to that of CW-complexes and homotopy classes of continuous maps.

Except for the fundamental definitions and the homotopic excision theorem that one can regard as "forming part" of the axioms, the two principle results of this memoir are the homotopic purity Theorem 3.2.8 and Theorem 4.2.6. The homotopic purity theorem means that the Thom space of closed immersions between two smooth k-schemes depends only on the normal bundle of the immersion. The essential difficulty by comparison to the case of differentiable varieties is that here one does not have at one's disposal the use of tubular neighbourhoods. One replaces this technique by using deformations of the normal bundle. The homotopic purity theorem is at the foundation of the localization exact sequences for any oriented cohomology theory and likewise at the foundation of Poincaré duality in its most concise form: the S-dual of a smooth projective k-scheme is, up to suspension, the Thom space of the normal bundle of its k-scheme.

A consequence of Theorem 4.2.6 is the following result (that one could regard as a new definition of higher algebraic K-theory, at least for regular schemes): when k is regular, then for any smooth k-scheme X and any integer $n \geq 0$ the group $[\Sigma^n(X_+), \mathbf{Z} \times \mathrm{Gr}]$ of morphisms in the homotopic category of pointed k-spaces of the n-th suspension of the pointed k-space X_+ (sum of X and a base point) to the product of $\mathbf{Z}$ with the infinite Grassmannian is canonically identified with the Quillen n-th algebraic K-theory group $K_n(X)$ of the scheme X.

1.2. Notation, conventions and review

We adopt the terminology of [**23**]. We denote by $\mathcal{S}ets$ the category of sets and S the category of simplicial sets, that is to say, functors $\Delta^{op} \to \mathcal{S}ets$, where Δ denotes the category of ordered sets $\{0, \dots, n\}$, $n \in \mathbb{N}$, and with increasing maps between these ordered sets.

Unless stated to the contrary, all rings are assumed commutative with unity and all algebras over a ring are assumed commutative.

In this memoir, and with the exception of §§B.1 and B.2, the term *scheme* refers to a noetherian, separated scheme admitting an ample family (*cf.* [**3, II.2.2.4**] and appendix B), and k denotes a fixed scheme (noetherian, separated and admitting an ample family!). A smooth k-scheme will always be assumed of finite type. For any scheme X, $\mathcal{O}X$ denotes the structure sheaf of rings of X and $A(X)$ the ring of global sections of $\mathcal{O}X$. For any closed subset F of a scheme X, we will often denote by F, by abuse of notation, the corresponding closed subscheme to F provided with its induced reduced structure.

For any integer $n \geq 0$, we let $\mathbf{A}^n$ denote affine n-space (over $\mathrm{Spec}(\mathbf{Z})$) and for any scheme X we denote by $\mathbf{A}^n_X$ the product over $\mathbf{Z}$ of $\mathbf{A}^n$ and X. The latter is a smooth affine $\mathrm{Spec}(\mathbf{Z})$-scheme. The "affine line", $\mathbf{A}^1$, comes with a canonical structure of a ring scheme (since for any scheme X, the set of morphisms of schemes from X to $\mathbf{A}^1$ is identified with the ring $\mathbf{A}(X)$).

Vector bundles and locally free $\mathcal{O}X$-modules. Let X be a scheme. A vector bundle ξ over X is an X-scheme $p(\xi) : E(\xi) \to X$ provided with a structure of an X-module scheme over the X-ring scheme $\mathbf{A}^1_X$ such that there exists an open cover $\{U_i\}$ of X that trivializes ξ (i.e., for each i, the U_i-module scheme $\xi\big|_{U_i}$ over the U_i-ring scheme $\mathbf{A}^1_X$ is isomorphic to $\mathbf{A}^{n_i}_{U_i}$ for a certain integer n_i). For any locally free $\mathcal{O}X$-module $\mathcal{M}$ of finite rank, we denote by $V(\mathcal{M})$ the vector bundle over X associated to $\mathcal{M}$ [**14, §1.7**]. The contravariant functor $\mathcal{M} \to V(\mathcal{M})$ realizes an antiequivalence from the category of locally free $\mathcal{O}X$-modules of finite rank with that of vector bundles. The functor which associates to each vector bundle ξ the associated locally free $\mathcal{O}X$-module $\mathcal{M}(\xi)$ of finite rank, of germs of sections of ξ, is an equivalence of categories, for which the inverse associates to each locally free $\mathcal{O}X$-module $\mathcal{M}$ of finite rank the vector bundle $E(\mathcal{M}) := V(\mathcal{M}^\vee)$ ($\mathcal{M}^\vee$ denotes the locally free $\mathcal{O}X$-module of finite rank dual to $\mathcal{M}$). For any vector bundle ξ over X, we denote by $\xi^\vee$ the dual vector bundle of ξ. For any locally free $\mathcal{O}X$-module $\mathcal{M}$ of finite rank over X, $\mathbf{P}(\mathcal{M})$ denotes the projective X-scheme associated to $\mathcal{M}$. For any X-scheme Y, the set of morphisms of X-schemes from Y to $\mathbf{P}(\mathcal{M})$ can be identified with the set of locally free quotients of $\mathcal{M}\big|_Y$ of rank one (see [**14, §3.1**]). We denote by $\mathcal{O}(1)_{\mathbf{P}(\mathcal{M})}$ (and often just by $\mathcal{O}(1)$) the locally free rank one tautological $\mathcal{O}(\mathbf{P}(\mathcal{M}))$-module (quotient of $\mathcal{M}\big|_{\mathbf{P}(\mathcal{M})}$). For any vector bundle ξ over X we denote by $\mathbf{P}(\xi)$ the projective bundle over X associated to ξ, that is to say, $\mathbf{P}(\mathcal{M}(\xi)^\vee)$. We denote by $\lambda_{\mathbf{P}(\xi)}$ (or just λ_ξ) the vector bundle $V(\mathcal{O}(1)_{\mathbf{P}(\xi)})$ of rank one and we call this the *canonical* rank one vector bundle over $\mathbf{P}(\xi)$.

Linear groups. For any integer $n \geq 0$, we denote by $\mathbf{GL}_n$ the linear group scheme of rank n. For any scheme X, the group of morphisms of schemes from X to $\mathbf{GL}_n$ is identified with the group $\mathbf{GL}_n(A(X))$ of invertible square matrices of order n with coefficients in the ring $A(X)$. We denote by $i_n : \mathbf{GL}_n \to \mathbf{GL}_{n+1}$ the morphism that associates to any scheme X and any element M of $\mathbf{GL}_n(A(X))$ the matrix $i_n(X)(M) := \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathbf{GL}_{n+1}(A(X))$. This is a closed immersion.

Grassmannians. Let n and r be two integers ≥ 0. For any scheme X, the set of locally free $\mathcal{O}X$-module quotients of $(\mathcal{O}X)^{n+r}$ of rank n is naturally identified with the set of morphisms of X to a scheme denoted by $\mathbf{Gr}_{n,r}$ (the Grassmannian of n-planes in $\mathbf{A}^{n+r}$). When $n = 1$, $\mathbf{Gr}_{1,r}$ is identified with projective space of

dimension r. We denote by $\tau_{n,r} : \mathbf{Gr}_{n,r} \simeq \mathbf{Gr}_{r,n}$ the isomorphism of schemes such that for any scheme X, $\tau_{n,r}(X)$ associates to the element $\pi : (\mathcal{O}X)^{n+r} \to P$ of $\mathbf{Gr}_{n,r}(X)$ the canonical epimorphism $(\mathcal{O}X)^{r+n} \simeq ((\mathcal{O}X)^{n+r})^{\vee} \to (\ker \pi)^{\vee}$, the isomorphism $(\mathcal{O}X)^{r+n} \simeq ((\mathcal{O}X)^{n+r})^{\vee}$ being that which reverses the order of the coordinates. We denote by $l_{n,r} : \mathbf{Gr}_{n,r} \to \mathbf{Gr}_{n,r+1}$ the morphism that associates to any scheme X and any element $\pi : (\mathcal{O}X)^{n+r} \to P$ of $\mathbf{Gr}_{n,r}(X)$ the element $l_{n,r}(X)(\pi)$ composed of the projection $(\mathcal{O}X)^{n+r+1} \to (\mathcal{O}X)^{n+r}$ (which annihilates the last component) and with π. We denote by $\varphi_{n,r} : \mathbf{Gr}_{n,r} \to \mathbf{Gr}_{n+1,r}$ the morphism that associates to any scheme X and any element $\pi(\mathcal{O}X)^{n+r} \to P$ of $\mathbf{Gr}_{n,r}(X)$ the epimorphism $\mathrm{Id}_{\mathcal{O}X} \oplus \pi : (\mathcal{O}X)^{n+r+1} \to \mathcal{O}X \oplus P$. One can easily verify the equalities $\tau_{n,r+1} \circ l_{n,r} = \varphi_{r,n} \circ \tau_{n,r}$ and $\varphi_{n,r+1} \circ l_{n,r} = l_{n+1,r} \circ \varphi_{n,r}$. By construction, the identity morphism of $\mathbf{Gr}_{n,r}$ corresponds to an epimorphism of locally free $\mathcal{O}\mathbf{Gr}_{n,r}$-modules $\pi : (\mathcal{O}\mathbf{Gr}_{n,r})^{n+r} \to \mathcal{M}_{n,r}$, with $\mathcal{M}_{n,r}$ of rank n. We denote by $\gamma_{n,r}$ the vector bundle $V(\mathcal{M}_{n,r})$ over $\mathbf{Gr}_{n,r}$ and we call it the canonical vector bundle of rank n over $\mathbf{Gr}_{n,r}$. When n is one, we also note that λ_r is the vector bundle $\gamma_{1,r}$ of rank one over $\mathbf{P}^r$.

I dedicate this work to the memory of Robert Thomason. For his interest he showed in it, for his encouragement and for a certain number of discussions which enabled me to quickly arrive at Theorem 4.1.10.