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Function Theory in Several Complex Variables

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Preface

> *"When you ask around and persist fondly in the Way, you can never find the Truth. Maintain your belief. Devote yourself to the Way. It will turn out to be the mighty Truth."*
>
> from "the 42 teachings of Buddha"

One of the subjects in mathematical nature is created by unifying three notions: complex numbers, coordinate systems, and the related notions of differentiation and integration. The space $\mathbf{C}^n$ of n-tuples of complex numbers is ruled by the coordinate system $(z_1, \ldots, z_n)$. If a complex-valued function in a domain in $\mathbf{C}^n$ is differentiable in each variable, it can be represented locally as a convergent power series. It has a natural domain of existence, in which it behaves in its own characteristic way, i.e., it creates its own mathematical world. We call such a function an analytic function.

In the case of one complex variable, an analytic function has a distinguishing property. In this case its real and imaginary parts are harmonic functions which are conjugate to each other. Namely, if one is considered as a potential, then the other is the flow which the potential induces. A harmonic function is uniquely determined by its boundary values; we can construct a harmonic function with prescribed boundary values and construct locally its conjugate harmonic function, which is unique up to an additive constant. This makes it easy to construct analytic functions of one complex variable. The main properties of analytic functions of one complex variable can be explained from this observation.

When we want to describe concepts in nature by using analytic functions, it is not enough to use only those of one complex variable. The theory of analytic functions of several complex variables is quite difficult to treat, compared to the theory in one complex variable. One reason for this is the freedom of the form of domains in $\mathbf{C}^n$ due to the increase in the dimension. Another reason is that both the real and the imaginary parts of an analytic function are now pluriharmonic functions, which imposes a stronger restriction than being merely harmonic. For example, in some cases, a pluriharmonic function is uniquely determined by its boundary values on some proper subset of the boundary, and we cannot always construct a pluriharmonic function with prescribed boundary values on a given portion of the boundary. Therefore, it is difficult to construct analytic functions of several complex variables. Since function theory in one complex variable generally proceeds by constructing analytic functions, we cannot simply use the one-variable approach in the case of several complex variables.

The most particular phenomena in the study of analytic functions in several complex variables which does not appear in the case of one complex variable is the fact that the natural domain of an analytic function is not arbitrary, i.e., it is not true that any domain in $\mathbf{C}^n$ is a natural domain of existence of some analytic function. This fact is important. We call a domain in $\mathbf{C}^n$ which is the natural domain of existence of some analytic function a *domain of holomorphy.* The principal problem in function theory in several complex variables is to study which domains are domains of holomorphy, and to determine which objects we can construct in a domain of holomorphy.

This book is an attempt to explain results in the theory of functions of several complex variables which were mostly established from the late 19th century through the middle of the 20th century. The focus is to introduce the mathematical world which was created by my advisor, Kiyoshi Oka (1901-1978). I have attempted to remain as close as possible to Oka's original work.

Kiyoshi Oka, at the beginning of his research, regarded the collection of problems which he encountered in the study of domains of holomorphy as large mountains which separate today and tomorrow. Thus, he believed that there could be no essential progress in analysis without climbing over these mountains.

The work of Oka can be divided into two parts. The first is the study of analytic functions in univalent domains in $\mathbf{C}^n$. Here he proved that three concepts: domains of holomorphy, holomorphically convex domains, and pseudoconvex domains, are equivalent; and, moreover, that the Poincaré problem, the Cousin problems, and the Runge problem – when stated properly – can be solved in domains of holomorphy satisfying the appropriate conditions. The second part was to establish a method by which we can study analytic functions defined in a ramified domain over $\mathbf{C}^n$ in which the branch points are considered as interior points of the domain. He proceeded in this later work under the assumption that the results valid in univalent domains in $\mathbf{C}^n$ should similarly hold in a ramified domain over $\mathbf{C}^n$. However, the true situation was contrary to his intuition, i.e., a ramified domain of holomorphy is not always a holomorphically convex domain.

Oka's establishment of his method to treat analytic functions in a ramified domain has proved to be indispensable not only in analysis but also in other fields of mathematics.

This book consists of parts I and II, according to Oka's earlier and later work mentioned above. In part I we treat analytic functions in a univalent domain in $\mathbf{C}^n$. In part II we treat analytic functions in an analytic space; this is a slight generalization of a ramified domain over $\mathbf{C}^n$. The one exception to our adherence to Oka's program is that the fact that a pseudoconvex univalent domain is a domain of holomorphy will be proved in part II in a more general setting by modifying Oka's original ideas.

A mathematical object is abstract and is described by use of words and notation. We should note that the words and the notation themselves are not really mathematics. Mathematics can be realized as a flow of the consciousness which is really creating mathematical nature. After such a process, mathematical nature lives individually in the mind of each person who has studied it. He seems to hear a voice coming from the bottom of his mind, or to feel the glow of a living object within his mind. This process is essential when we study the established works of the pioneers of a field. If mathematical nature lives correctly within a person's mind, then when he encounters a certain problem, he may not recall the knowledge to solve it immediately, but he will be able to understand the problem itself in order to solve it.

The difficulty in studying mathematics is the procedure for giving life and meaning to the mathematics. The first step is to organize and expand upon the material written by use of words and notation in a concrete form, so that we can proceed with further steps.

I hope that this book is a worthwhile initial step for the reader in order to understand the mathematical world which was created by Kiyoshi Oka.

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