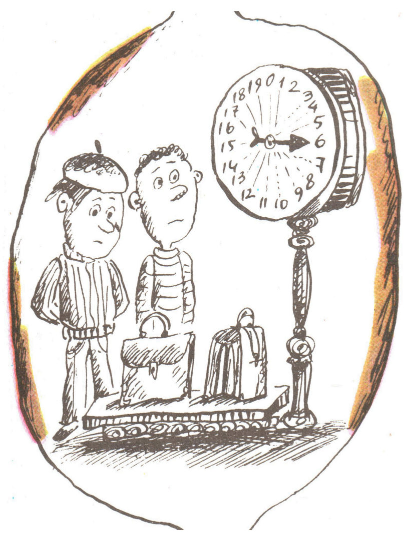
Problem Set 0

Problem 0.1. Grandma takes five minutes to climb from the first floor of a building to the fifth. If she climbs at the same speed, how long will it take her to get to the ninth floor from the first?

Problem 0.2. Juan and Candice are using a spring scale to weigh their book bags. When they are weighed separately, the scale shows 3 lbs and 2 lbs. When they weigh them together, the scale shows 6 lbs.



"That can't be right," said Candice. "Two plus three doesn't equal six!"

"Don't you see?" answered Juan. "The pointer on the scale is off."

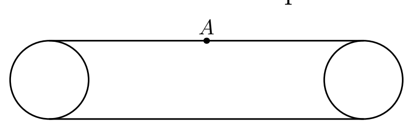
How much do the book bags actually weigh?

Problem 0.3. Use the fingers of one hand to count as follows: the thumb is first, the index finger is second, and so on to the pinkie which is fifth. Then reverse direction so that the ring finger is sixth, the middle finger is seventh, the index finger is eighth, and the thumb is ninth. Reverse direction again back toward the pinkie with the index finger tenth, and so on. If you continue to count back and forth along the fingers of one hand, which finger will be the 1000th one?

Problem 0.4. A dot is marked in a circle. (a) Cut the circle into at most three parts so that, by rearranging the parts, you get a circle with its center at the dot. (b) Is it possible to do so by cutting the circle into at most two parts?

Problem 0.5. A brother leaves his house 5 minutes after his sister. If he walks at 1.5 times her speed, how long will it take him to catch up?

Problem 0.6. The diagram shows the rolling track of a bulldozer, seen from the side. The bottom is in contact with the



ground. If the bulldozer moves forward 10 cm, how many centimeters does the point marked A move?

Problem 0.7. Together Winnie the Pooh, Owl, Rabbit, and Piglet ate 70 bananas. Each ate a whole number of bananas, and each ate at least one. Pooh ate more than each of the others; Owl and Rabbit together ate 45 bananas. How many bananas did Piglet eat?

Additional Problems

Problem 0.8. While walking in the park, Nicole and Valerie came to a large round clearing surrounded by a ring of cottonwood trees and decided to count the trees. Nicole walked around the clearing and counted all the trees. Valerie did the same, but started from a different tree. Nicole's 20th tree was Valerie's 7th, while Nicole's 7th tree was Valerie's 94th. How many trees were growing around the clearing?

Problem 0.9. A group from a summer camp leaves a forest where they have been gathering flowers. They walk in boy-girl pairs, and in each pair the boy has either three times as many or one third as many flowers as the girl has. Is it possible for the whole group to have 2006 flowers?

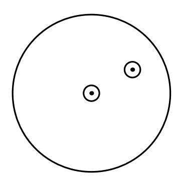
Solutions to Problem Set 0

Problem 0.1. It takes her 10 minutes. She climbs four flights of stairs to go from the first floor to the fifth floor, and climbs eight flights to go from the first floor to the ninth floor. This is exactly twice as many flights of stairs so it takes twice the time.

Problem 0.2. The fact that the pointer is shifted means the scale shifts (changes) every weight by the same quantity. Let's call this quantity x. We don't know the true weight of the first bag, but when we add x to it we get the reading 2; similarly x added to the true weight of the first bag gives the reading 3. The two readings add up to 5, which is $x + x$ different from the sum of the true weights. Weighing the bags together gives a reading of 6, which is x different from the sum of the true weights. Thus $x = -1$: the scale shows 1 lb less than the true weight. Hence the correct weights of the bags are 3 lbs and 4 lbs.

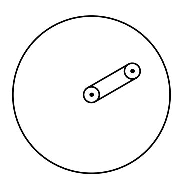
Problem 0.3. Let's watch the labels for the thumb. To begin with, it gets selected with 1, then we count four from the index finger to the pinkie and then four back again (from the ring finger to the thumb), so that the thumb gets selected on $1 + 8 = 9$. Next it gets selected on $9 + 8 = 17$, with each subsequent visit to the thumb eight more than the previous one. Since 1000 is divisible by 8, the thumb would be selected on $1 + 8 \times 125 = 1001$. Therefore, the index finger will be the 1000th one.

Problem 0.4. If the dot is at the center, we don't need to cut the circle at all. Suppose the dot is not in the center and not on the edge.



(a) Cut out two small, nonoverlapping circles of the same radius, centered at the dot and at the center of the original circle. The little circles should be completely within the given circle, as in the diagram. The original circle has been divided into three parts. Now interchange the two small circles.

(b) Solution 1: We can change the previous solution slightly. Draw two common tangents to the small circles, as shown below. Now cut out the oval-shaped figure thus obtained. We have divided the original circle into two parts; now just turn the cut-out figure through 180 degrees!



Solution 2: No matter where the dot is, draw a circle centered at the dot with the radius of the original circle, and cut along

the resulting arc that is inside the original circle. Except when the dot is at the original center, one gets a lens-shaped piece and a piece shaped like a crescent moon. Slide the crescent moon to the other side of the lens, at the same time turning it around. This yields a new circle with its center at the dot.

Problem 0.5. At the moment the brother leaves the house, his sister is a certain distance ahead of him; let's say she is x meters ahead. Then every subsequent 5 minutes the sister will walk the same distance of x meters while the brother will walk $1.5\,x$ meters. Therefore, every 5 minutes the distance between them will become $0.5x$ meters less. Since the distance between them was x meters to begin with, it would take 10 minutes for the brother to catch up.

Problem 0.6. Usually people think that if the bulldozer moved forward 10 cm then so did the point A. It's easy to see that this is wrong. If this were the case, then point A would be staying in the same place on the bulldozer, i.e., right at the middle, which would mean that the tread of the bulldozer does not move. What, then, is the distance that the point A advances?

Let's look at the point on the forward wheel of the bulldozer where it touches the ground after the bulldozer has moved. This point is 10 cm ahead of where the wheel touched the ground before the bulldozer moved. Therefore 10 cm of the tread has reached the ground as the bulldozer moved ahead 10 cm. Since point A has moved 10 cm with respect to the middle of the bulldozer, and the bulldozer has moved 10 cm forward, the point A has moved $10 + 10 = 20$ cm. This is the shift of the point A relative to the ground.

To see this a different way, let's imagine that we are by the side of the bulldozer and moving with it while recording it on a video camera. When we watch the video, we will see the bulldozer in a fixed position on the screen but with wheels rotating and the tread moving. During the screening of the video point A will move 10 cm on the screen. At the same time our camera, has gone 10 cm. Thus point A has indeed moved through 20 cm.

Problem 0.7. Since Owl and Rabbit together ate 45 bananas, one of them ate at least 23 bananas. Then Winnie the Pooh ate at least 24 bananas, and hence the three of them ate at least 69 bananas. Thus Piglet ate at most one banana. The problem states each ate at least one banana, so Piglet ate exactly one.

Problem 0.8. Label Nicole's seventh tree A and her 20th tree B. Following the direction of Nicole's count, there are 12 trees strictly between A and B. Now keep counting beyond B but use Valerie's count. There are 86 trees between B and A, since there are exactly 86 integers greater than 7 and smaller than 94. Remembering to include A and B in the final count, we see that the total number of trees is $12 + 86 + 2 = 100$.

Can you tell whether the girls walked in the same direction or in opposite directions while counting trees?

Problem 0.9. In a pair, if the boy has three times as many flowers as the girl, then this pair together has four times as many flowers as the girl. If, in a pair, the boy has one third as many flowers as the girl, then the girl has three times as many as the boy, so the total number of flowers for the pair is four times the number of flowers the boy has. In each case the total number of flowers in each pair is divisible by 4. Therefore, the total number of flowers must be divisible by 4. But 2006 is not divisible by 4 and so the whole group cannot have 2006 flowers.

Problem Set 9

Problem 9.1. Which is larger, $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024$ or 2048? By how much?

Problem 9.2. Each day at noon, a steamship leaves Savannah for Belfast while a steamship of the same line leaves Belfast for Savannah. Each ship spends exactly seven 24-hour days at sea, and all travel along the same route. How many Savannah to Belfast ships will a Belfast to Savannah ship meet while underway?

Problem 9.3. The length of one side of a triangle is 3.8 inches, and the length of another side is 0.6 inches. If the third side is a whole number of inches, find its length.

Problem 9.4. Simplify the fraction

$$\frac{1 \cdot 2 \cdot 3 + 2 \cdot 4 \cdot 6 + 4 \cdot 8 \cdot 12 + 7 \cdot 14 \cdot 21}{1 \cdot 3 \cdot 5 + 2 \cdot 6 \cdot 10 + 4 \cdot 12 \cdot 20 + 7 \cdot 21 \cdot 35}.$$

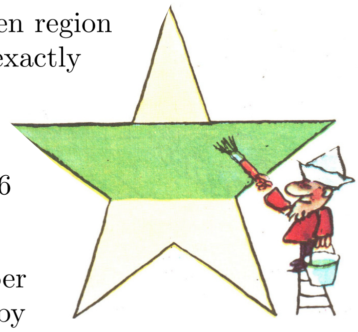
Problem 9.5. Using a pencil, an unmarked ruler, and a sheet of graph paper, how can you draw a square with area (a) double; (b) 5 times larger than the area of one square of the grid?

Problem 9.6. Which is greater, the sum of the lengths of the sides of a quadrilateral, or the sum of the lengths of its diagonals?

Problem 9.7. The menu in a school cafeteria always has the same 10 different items. To vary his meals, George decides to buy a different selection for every lunch. He can eat anywhere from 0 to 10 different items for lunch. (a) For how many days can he eat without repeating a selection? (b) What is the total number of items he will eat in that time?

Problem 9.8. Is it possible to write more than 50 different two-digit numbers on a blackboard without having two numbers on the board whose sum is 100?

Additional Problems



Problem 9.9. Show that the area of the green region of the regular pentagonal star in the picture is exactly half of the total area.

Problem 9.10. Find the sum

$$6 + 66 + 666 + 6666 + 66666 + \cdots + 66\ldots66$$

if the last string of 6's has 100 digits.

Problem 9.11. Is it possible to find a number of the form $11\ldots1100\ldots00$ that is divisible by 2003?

Problem 9.12. A straight bar of length 2 m is cut into five pieces with each piece at least 17 cm long. Prove that there are three of these pieces that can be put together to form a triangle.

Solutions to Problem Set 9

Problem 9.1. Let's add 1 to the sum. We are now looking at the number $1 + 1 + 2 + 4 + 8 + \cdots + 1024$. Let's replace $1 + 1$ with 2. The result is then $2 + 2 + 4 + 8 + \cdots + 1024$. Replace $2 + 2$ with 4 to get $4 + 4 + 8 + \cdots + 1024$. We can continue replacing the sum of two equal powers of 2 with the next power of 2. Eventually we reach $1024 + 1024 = 2048$. The original sum was 1 less than 2048.

Does this problem remind you of the goose problem from the previous lesson?

Problem 9.2. Solution 1: First we need to decide if we are going to count meetings in port. Let's not. Now analyze which ships will meet a ship that leaves Belfast. There are 6 ships that are already heading to Belfast. One left 6 days ago, the next 5 days ago, and so on with the last having left 24 hours ago. In the next 7 days another 7 ships will leave Savannah and head for Belfast. One will leave at the same time as our ship, another 24 hours later, and so on. The last one will leave port 24 hours before our ship arrives in Savannah. Therefore we get a total of 13 ships. If we also count ships as they meet in harbor, there will be 2 more. One will be arriving in Belfast as the one we are tracking is leaving Belfast, and the other will be leaving Savannah just as our ship is arriving there.

Solution 2: Divide the distance between Belfast and Savannah into 7 different segments because the ship covers each segment in 24 hours. Ships leave port every 24 hours and all are traveling the same speed so they approach each other at double that speed. Therefore ships will meet every 12 hours and they will meet at the endpoints and midpoints of the segments. If we don't count the ports, we get 6 ends and 7 midpoints of the segments giving a total of 13 places where ships meet.

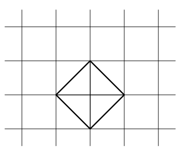
Solution 3: Let's say the ships that are heading to Belfast don't leave from Savannah but from a third city, let's say London, that is a 7 days' sail from Savannah. Think of Savannah as an intermediate point on the journey from London to Belfast. Because the ships leave London every 24 hours, when a ship leaves Belfast, there are 13 ships heading toward it not counting the one entering Belfast harbor or the one just leaving London. The Belfast to Savannah ship will meet these 13 ships on its voyage.

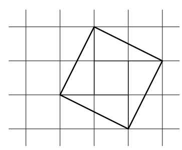
Problem 9.3. Here we can use the triangle inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third side. Let us examine triangle ABC, where $AB = 3.8$ inches, and $BC = 0.6$ inches. Let us find the length of AC knowing that it is a whole number of inches. AC can't be 3 inches or less because the sum of the lengths of AC and BC would be $3+0.6$ which is less than 3.8, the length of AB. AC can't be 5 inches or greater because the sum of the lengths of AB and BC would be $3.8 + 0.6$ which is less than 5, a number at least as big as the length of AC. Therefore, there is only one option left, which is 4 inches.

There is, in fact, such a triangle. It can be constructed with a ruler and compass as follows. First construct leg AC of length 4 inches. Then draw a circle about A with radius 3.8 inches, and a circle about C with radius 0.6 inches. These circles will intersect. Either of the two points of intersection can be taken as vertex B.

Problem 9.4. The numerator is equal to $1\cdot2\cdot3(1+2\cdot2\cdot2+4\cdot4\cdot4+7\cdot7\cdot7)$, and the denominator is equal to $1\cdot3\cdot5(1+2\cdot2\cdot2+4\cdot4\cdot4+7\cdot7\cdot7)$. Thus the fraction is equal to $\frac{2}{5}$.

Problem 9.5.(a) The square depicted in diagram on the left consists of four half-cells, so its area is equal to the area of two cells.





(b) The square depicted in diagram on the right consists of a central cell and four adjacent right triangles. Each triangle has area equal to one cell because it is half of a rectangle formed by two cells. Therefore the total area of the square is equal to the area of five cells.

Problem 9.6. If $ABCD$ is our quadrilateral, then AC and BD are its diagonals. Using the triangle inequality, we conclude that $AB + BC > AC$ and $CD + DA > AC$. Adding these inequalities gives us that the sum of the sides of the quadrilateral must be greater than twice AC. Similar reasoning, which you can complete, shows that the sum of the sides of the quadrilateral is greater than twice BD. Since twice the sum of the sides of the quadrilateral is greater than twice the sum of its diagonals, the sum of the sides is greater than the sum of the diagonals.

Problem 9.7. We are assuming that the order in which George eats items at a meal doesn't matter; only which items he eats matters.

(a) We have already seen several problems like this one. While putting together his lunch, George can either choose or not choose each item. Therefore, he has a total of 2^{10} options, from not eating at all to stuffing himself by buying everything. So George can eat lunch 1024 times without repeating a meal.

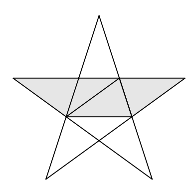
(b) Let's construct a table where the rows correspond to days, and the columns correspond to items George buys. Place a "+" at the intersection of day n and item m if George eats that item on that day, and place a "−" if he does not. Therefore row n will consist of 10 signs which will indicate which items George bought that day and which ones he did not buy. There are 1024 rows in the table because that is the number of options for a meal, and there are 10 columns representing the 10 different items. To count the number of items George eats, we need to count the number of plus signs in the table. Because all of the different combinations of plus and minus signs occur once throughout the table, there are an equal number of each. The number of signs in the table is $1024 \times 10 = 10\,240$, and there are half as many plus signs, or 5120. George eats 5120 items over 1024 days.

We can also explain the fact that there are as many plus as minus signs in the table like this. Let's pair up one row of the table with another if each row is the result of changing every sign to the opposite. For example, $--+-+++---$ will be paired with $++-+----+++$. All of the rows will match up to form such pairs, and between each pair there will be 10 pluses and 10 minuses. Therefore pluses form half of the table.

We can also explain it like this. Half of the signs in the first column are pluses and half are minuses because George eats the first item on the menu half of the days and does not eat it the other half of the days. The same occurs in all of the columns of the table. Therefore, half of the table is composed of pluses and half of minuses.

Problem 9.8. The pigeonhole principle will be helpful here again. There are 40 pairs of numbers between 10 and 100 that sum to 100. These are $(10, 90)$, $(11, 89)$, $\ldots$, $(49, 51)$. That leaves out the 10 two-digit numbers $50, 91, 92, \ldots, 99$. Even if we take all of these 10, and one number from each pair, we still end up with only 50 numbers. If we take more than 50 numbers we will have to take both numbers from at least one pair, and they will sum to 100.

Problem 9.9. Our star consists of a regular pentagon, which is the central part of the star, and 5 adjacent triangles, which we will call its rays. Draw two diagonals in the central pentagon, as shown on the right. They divide the pentagon into three parts: two small congruent triangles, and one slightly bigger triangle. The latter one is congruent to a ray of the star since it forms, together with the adjacent ray, a parallelogram. You should prove this.



Now we see that the shaded part of the star consists of three triangles congruent to a ray and one small triangle. The unshaded part of the star also consists of three rays and one small triangle. Since the two smaller triangles are congruent, the area of shaded region is exactly half of the total area.

Problem 9.10. Our sum is equal to

$$
\begin{aligned}
&6\cdot(1+11+111+\cdots+\overbrace{11\ldots11}^{100})\\
&\quad=\tfrac{6}{9}\cdot(9+99+999+\cdots+\overbrace{99\ldots99}^{100})\\
&\quad=\tfrac{2}{3}\cdot(10-1+100-1+1000-1+\cdots+1\overbrace{00\ldots00}^{100}-1)\\
&\quad=\tfrac{2}{3}\cdot(\overbrace{11\ldots11}^{100}0-100)=\tfrac{2}{3}\cdot(\overbrace{11\ldots11}^{99}00-90)\\
&\quad=2\cdot(\overbrace{37037\ldots037}^{98}00-30)=\overbrace{740740\ldots740}^{99}0-60=\overbrace{740740\ldots740}^{96}7340.
\end{aligned}
$$

Problem 9.11. Let's examine the numbers $1, 11, 111, 1111, \ldots$. There are infinitely many such numbers while there are 2003 possible remainders on dividing numbers by 2003. Therefore, we can find two numbers consisting only of 1's which have identical remainders when divided by 2003. The difference of these two numbers will be of the form $11\ldots1100\ldots00$, and will be divisible by 2003.

We can get an additional result from our conclusion. Because 2003 is not divisible by 2 or 5, the 0's at the end of the resulting number can be thrown away. The remaining string of 1's must be divisible by 2003, so there is a number that consists of only 1's that is divisible by 2003.

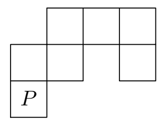
Problem 9.12. Suppose the conclusion is false, and we can't form a triangle from any three pieces. Arrange the pieces in ascending order of length. The two shortest ones are at least 17 cm long. Then the next one is at least 34 cm long because otherwise we could form a triangle from the three shortest pieces. Similarly, the next segment must be at least $17 + 34 = 51$ cm long, and the fifth must be at least $34 + 51 = 85$ cm long. Thus the five pieces together would form a bar at least $17+17+34+51+85 = 17(1+1+2+3+5) = 17\cdot 12 = 204$ cm long. This is a contradiction, since 204 centimeters is greater than 2 meters.

Note: The numbers 1, 1, 2, 3, 5 found in solving this problem are the beginning of an infinite sequence called the Fibonacci numbers. The members of this sequence are formed by adding the previous two members. The Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, appears often in mathematics.

Problem Set 28

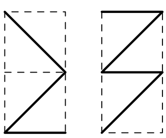
Problem 28.1. Given a water faucet and a cylindrical pot, how can you fill the pot exactly half-way with water?

Problem 28.2. Why are manhole covers circular rather than a square?

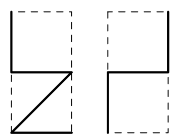


Problem 28.3. The letter P was painted in one of the squares of the assembly of seven congruent squares shown on the right. A cube with faces congruent to these squares was placed on the square with the P so that its face coincided with the square, and then the cube was rolled over its edges along the figure. In the process, the letter was imprinted on the face of the cube and also on all squares of the figure where this face landed. Which of the squares of the figure got an imprint of the letter, and what exactly do those imprints look like?

Problem 28.4. Is it possible to place six identical unsharpened pencils in a configuration where each pencil touches all the other ones?

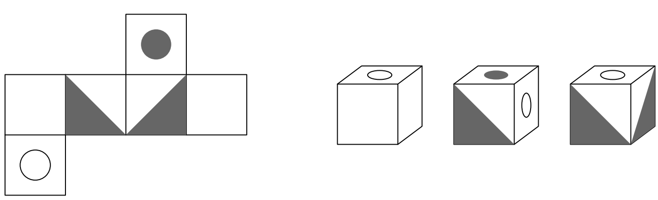


Problem 28.5. (a) The left half of the picture on the left shows the path of a fish as seen from the front of an aquarium. The picture next to it shows the same path as seen from the right side of the aquarium. Draw the path of the fish as seen from above. (b) Do the same problem for the paths shown on the right.



Problem 28.6. Is it true that 1-liter and 2-liter Coke bottles are proportional (one can be obtained from the other by scaling up all dimensions by the same factor)?

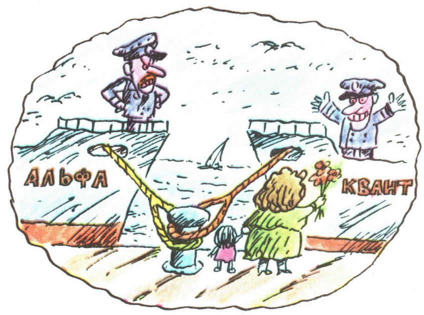
Problem 28.7. An unfolded cube is shown on the left half of the figure below. Which of the three cubes on the right can unfold in this way?



Problem 28.8. Can the shadow of an opaque cube be a regular hexagon?

Problem 28.9. Suppose we have three identical bricks. How can we find the length of the diagonal of a brick (i.e., the distance between two vertices not belonging to the same face) using only one measurement with a ruler?

Problem 28.10. (a) Is it possible to saw a figure out of a piece of wood so that every face of the figure is a square, but the figure is not a cube? (b) What if we ask, in addition, that the figure be convex?



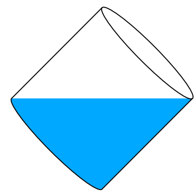
Problem 28.11. The picture shows that the yacht *Alpha* was moored at the dock before *Kvant*. Can *Alpha* sail first without having to take *Kvant*'s mooring cable off the bollard?

Additional Problems

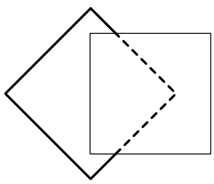
Problem 28.12. In three-dimensional space, we are given a point light source emitting light in all directions. Is it possible to choose (a) 100 opaque balls, (b) 4 opaque balls, and place them in space in such a way that they don't intersect each other, don't touch the light source, and completely block its light, i.e., every light ray emanating from the bulb meets one of these balls?

Problem 28.13. You have a compass, a ruler, a piece of paper, a pencil, and a ball. You can draw on the surface of the ball using the compass and the pencil, and you can draw on the paper using the compass, the pencil, and the ruler. Can you draw a line segment on the paper whose length is the radius of the ball?

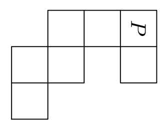
Solutions to Problem Set 28



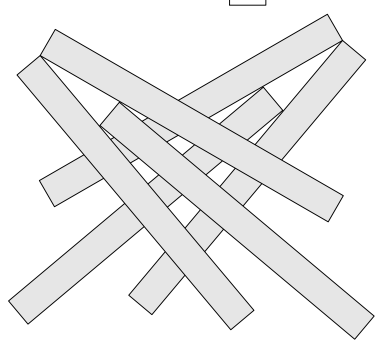
Problem 28.1. First fill the pot with water, then dump the extra water by tilting the pot until the bottom just starts to show, as in the diagram on the right.



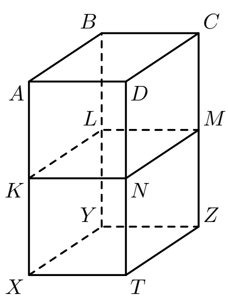
Problem 28.2. The diagonal of a square is longer than its side. Thus, if carelessly handled, a square cover can easily be dropped into a square manhole, as shown on the left (thick lines indicate the hole).



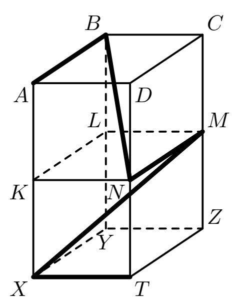
Problem 28.3. An imprint will only occur in the right top square of the figure, and the letter P will be turned 90 degrees clockwise, as shown on the right.



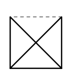
Problem 28.4. The answer is yes. Place three pencils on a table so that two of them touch at one end and a third is wedged in between. Then place three more on top of these, in the same configuration but rotated 90 degrees, as shown on the right.

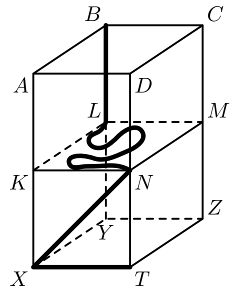


Problem 28.5. It should be clear from the way the problem is posed that the aquarium is a $1 \times 1 \times 2$ rectangular parallelepiped, and this is reflected in diagram on the left. Let's assume that the fish swims from the top down.

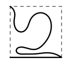


(a) From the leftmost diagram of the statement of the problem, we see that the fish started at a point on edge AB; the right rectangle then shows that it swam first from A to B along edge AB. Then we see that it swam directly along a straight line to point N, then along the line segment NM, then straight on to the point X, and finally along edge XT to the point T. The entire path of the fish is shown on the right.

The desired answer (the view from above) is this: 



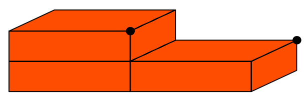
(b) It can be seen from the diagrams in the statement of the problem that the fish started at point B and swam along edge BL. Then it went to point N, but it did not necessarily swim along a straight line. It could swim along any curve as long as it remained in the plane $KLMN$. From point N, the fish swam straight to point X, and finally it swam along edge XT to point T. One possible path for the fish is shown on the right.

The view from above could be something like this: 

Problem 28.6. No, since the openings of the two bottles are the same size!

Problem 28.7. Only the leftmost cube. You should be able to prove it.

Problem 28.8. Yes it can. Let's place the cube so that one of its diagonals — a line connecting two most distant vertices of the cube — is perpendicular to the ground, and let light rays parallel to the diagonal shine on the cube from above.

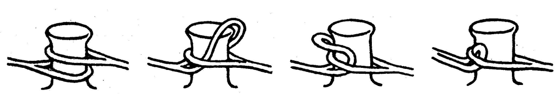


Problem 28.9. Let's place the bricks as shown in the diagram, with two bricks stacked up and the third one placed side by side with the bottom brick in the stack. In this configuration, there is an empty space above the third brick, with the same dimensions as each of the bricks. The two marked points on the diagram represent opposite vertices of this imaginary brick, and there are no obstacles to measuring the distance between these two points with a ruler. This distance is equal to the length of the diagonal.

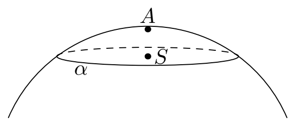
Problem 28.10. (a) Yes, it is possible. For example, we can take a cube and glue an identical cube to each of its faces.

(b) No, it is impossible. At every vertex of such a polyhedron, there are exactly three faces — in fact it's impossible to have fewer than three faces at a vertex of any polyhedron — while in the convex case it's impossible to have more than three square faces since otherwise the sum of the flat angles at the vertex would be at least $4 \times 90^\circ = 360^\circ$, which is impossible for a convex polyhedron.

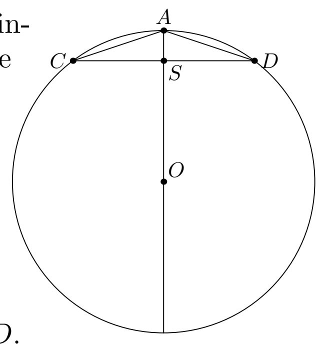
Problem 28.11. Yes, *Alpha* will be able to sail first without disturbing *Kvant*'s mooring cable. How to do this is shown below. The author of the problem, Nikolay Dolbilin, actually witnessed a similar situation at a port. At the last step, considerable force was required to jerk the cable out, so a dock worker used a winch.



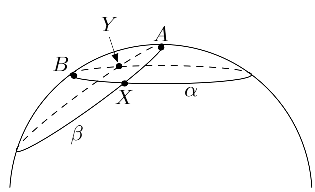
Problem 28.12. Let's solve part (b) right away. Consider a regular tetrahedron placed in space in such a way that our bulb is exactly at its center. Circumscribe a circle around one of the faces of the tetrahedron. Connect the bulb to every point of this circle with rays emanating from the bulb. We have constructed an infinite cone. We can inscribe a ball of any radius in this cone, and it will block all light rays from the bulb that are inside this cone. But what about the rays along the cone's surface? Are they blocked by the ball? To be on a safe side, let's slightly increase the size of the ball without changing the position of its center while making sure the new ball still does not touch the bulb. Now the rays along the cone's surface will also be blocked. In a similar manner, consider a second cone gotten by using a different face of the tetrahedron. Rays in this cone can be blocked by a ball of a suitable radius placed far enough from the bulb so that it does not intersect the first ball. Similarly, we can construct and block two more cones. Since every ray emanating from the bulb lies in one of the four cones, the four balls constructed in this way block all the light emitted by the bulb.



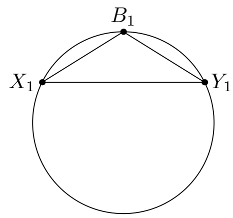
Problem 28.13. Let's draw a circle on the ball and call it α. To do so we use the compass (see the diagram on the right). Open the compass a distance a, and place the sharp point of the compass at a point A on the ball. What is the radius of α? At first glance it might appear that the radius of α is a, but this is not so. Where does the center of the circle α lie?



Let's mentally cut the ball with a plane containing α. This cut will separate off a "cap" of the ball, while the cross-section itself will be a disc bounded by α, whose center, S, will be the center of α. The straight line AS is perpendicular to the plane containing α, and it passes through the center, O, of the ball. If we mentally cut the ball into two halves with a plane containing the line AS, we get the picture on the right. In this picture the circle α appears as the line segment CD. The lengths of AC and AD are equal to a. If we knew the radius of α, i.e. the length of CS, we would be able to construct triangle CAD. But then we would also be able to find the center O of the circle circumscribed about triangle CAD — it is the intersection of the perpendicular bisectors of the sides of CAD — and thus we'd be able to find the radius of the ball which is equal to the length of OA.



But how can we find the radius of α? The idea is to construct on paper a triangle whose circumscribed circle is congruent to α. Draw a new circle, β, on the ball by placing the sharp point of the compass at some point, B, on the circle α and using the same compass opening, a (see drawing immediately above).



This new circle will pass through A and will intersect α at two points X and Y. Clearly, $BX = BY = a$. Place the compass's legs at points X and Y, and, without changing the opening of the compass, mark this distance on paper. This gives the picture in the last diagram, where the points X_1 and Y_1 are such that $X_1Y_1 = XY$. Now, with the opening of the compass back to a, construct on the paper a point B_1 such that $B_1X_1 = B_1Y_1 = a$. Triangle $X_1B_1Y_1$ is congruent to triangle XBY. Now we can construct the circle circumscribed about triangle $X_1B_1Y_1$ on the paper, and its radius is equal to the radius of α.

Note: For a plane, the following amusing fact is true. Let's draw a circle on a plane, mark on the circle a point, and start marking off points on the circle one by one without changing the compass's opening: first place the sharp point of the compass at the first marked point and mark the second one, then move the sharp point to the second mark and mark the third point, and so on. At the sixth step, we'll come back to the first point; moreover, the six marked points are the vertices of a regular hexagon. If we try doing the same thing on a sphere, we won't get a regular hexagon (think about the reason). So if you find yourself on an unknown surface, you can carry out this experiment — if you end up with something other than a regular hexagon, you can conclude that you are not on a plane!

Two and Two Is More Than Four: A Story

This text is a composite of several conversations among students, overheard by the author during actual circle sessions. It is meant to give both students and prospective circle leaders a feel for the atmosphere of lively discussion, collaboration, and teasing that characterizes many circles, and which can make teaching such a circle a worthwhile experience. It is immensely rewarding to watch students interact and build on each other's ideas.

Several of the problems were given in the text. Problem 1 of the story is 15.5 in the text, Problem 2 is 18.3, Problem 3 is 19.7, Problem 5 is 0.6, and Problem 6 is 16.2.

I. Sail Me Down the River[1]

Andy, Dan, Beth, and Ted came to their math circle after school. There was still time before the start, and the teacher wasn't there, but somebody had already written some problems on the blackboard. This was the first:

Problem 1. *Two boys simultaneously jumped off a raft floating on a river and started swimming in different directions: one downstream, the other upstream. In five minutes they turned around and soon were back on the raft. Assuming each kept a steady pace, which one got back first?*

"I wonder, is this problem for us?" asked Beth.

"Wouldn't it be so much more fun to go to the river than to do these problems?" replied Ted. "And this problem is wrong anyway."

"Why?" asked Beth.

"They also need to state that the boys' speeds were greater than the speed of the river's current," said Ted, "because otherwise the one who started upstream would not be able to swim away from the raft at all."

"Well, I think you're wrong!" Dan objected. "Even if his speed is the same as the river's current, he will swim away from the raft."

"Oh, come on!" protested Ted. "He will then flounder at the same spot! Once this happened to me. I kept swimming as fast as I could, and there

[1]A quote from the song "Down by the Sea", by the band Men at Work.

was a big pine tree opposite to me. I swam and swam, but still remained where the tree was. I hardly made it to the shore then. The river was too fast."

"You may have stayed at that spot, but the raft won't wait for you," said Dan, "it isn't on the shore like your pine tree, the river will pull it away."

"Now I'm totally confused," Ted said, "so, even if I'm slower than the river, I'll still be able to swim away from the raft, will I?"

"Sure! Because the raft isn't resisting the current, and you, though doing so weakly, are paddling against it."

"Well, then how do we solve the problem?"

Everyone was silent for a while with each trying to think it through, but no one came up with a solution.

"Oh, listen, I have an idea," Andy said suddenly. "Suppose it all happens on a lake. It is clear that both will return simultaneously. The raft stays still, everyone is swimming for five minutes away from it, and then returns for just as long."

"This isn't a lake though, it's a river," objected Dan. "It flows!"

"So what! The river does the same thing to everything in it; the river moves it downstream with the speed of the current. It is as if the current isn't there at all."

"This sounds right. Could it be so simple?"

"Of course. And in the fog, when you can't see the riverbank, you won't distinguish between a river and a lake at all."

"I've got it, I've got it!" Everyone looked at Ted, who, in his excitement, even started flapping his arms. "I can explain it in a simpler way. My father is a cameraman. He was filming a thriller, and in one scene he had to follow a boat in the river with a camera along the riverbank. So if he had always remained opposite to the raft while filming, what would we see on the screen?"

"Not much." Dan shrugged his shoulders. "In the middle of the screen there would be a raft, and the opposite shore with the trees would be moving, like in a train's window."

"At the moment, we don't care about the bank." Ted said sternly, "What's important is that there will be a still raft in still water. Two boys jumped from the raft in different directions. Each swam his distance for five minutes, and then he would swim back in the same five minutes."

"Yes, this is great," said Andy. "I think this is called changing the frame of reference to one that is based on the river. My brother studied this in his physics class."

"Don't be a smart aleck. But about the train, that was clever! Why didn't it occur to us?" Ted was glowing with pride. "The river moves everything forward like a train or an airplane. If we started running in opposite directions in the aisle of a flying airplane, and turned around in a couple of seconds, we would meet where we started."

"Yeah," agreed Dan. "And if you jump in an airplane, then you'll land in the same place, and, even though the airplane is moving at a huge speed, it won't fly out from under you."

"This is because," continued Andy, "the airplane's speed relative to the ground may be high, but you are moving at the same speed, so for you, the airplane doesn't move."

"Actually, when I was flying in a plane," said Ted, "during the takeoff I was being squished into the chair, and, if at that moment I was in the aisle and tried to jump, then I would have found myself in the tail of the plane."

"That is because, at that moment, the airplane was gaining speed and moving with acceleration," answered Dan. "That is a completely different thing. No wonder that walking is not allowed during takeoff and landing."

"Now I understand why cowboys in the movies aren't afraid to run across the tops of the trains and jump from car to car;" continued Ted, "it's the same as running on a stationary one."

"Well, not quite," said Andy, "in a train you have counter air pressing on you. When you jump, the air immediately slows you down — the air isn't moving with the train. If the train is moving very fast, the wind may blow you off the roof. In the time of cowboys the trains were very slow, even an average horse could outrun a train."

"While you were arguing about your cowboys, I solved the problem algebraically, with the help of equations," said Beth, waving a piece of paper. "Do you want me to show you?"

"Everything is clear anyway. Well, show us, so you haven't worked for nothing." The kids gathered around Beth's piece of paper.

While pointing to what she wrote, Beth continued. "It is short. In meters per minute, let v be the speed of the river, and x the speed of the first boy. He swims downstream for five minutes with the speed $x + v$ with respect to the bank. He will be $5(x+v)-5v = 5x$ meters from the raft. The next five minutes he will be moving at $x-v$ and will swim $5(x-v) = 5x-5v$ meters. So at this point he will meet the raft, which would have floated toward him $5v$ meters during these five minutes. The same can be checked for the second one."

"We've got the same thing, only without any x and v."

"Let's also solve the second problem."

Problem 2. *Two towns are located on a river, 10 miles apart along the water. Will it take a ship longer to go from one town to another and back, or to cover 20 miles on a lake?*

"Well, this is an even simpler problem," Ted declared at once. "When the boat sails downstream, the river helps the boat, and when it goes upstream, the river hinders it and eats up all the benefits gained by the help. Thus sailing downriver and back again upriver is the same as sailing on the lake."

"Something's wrong here," said Andy with a hint of doubt in his voice. "Imagine that the speed of the boat is equal to the speed of the river. Then the boat will never get back to the upriver town — it won't be able to fight the current. And on the lake, no problem! So it looks like it is always faster on the lake."

"But what's my error?"

"I think I've got it. You say that when the boat sails down the river, the current helps it, but what does this mean?"

"Well, isn't it clear?"

"'Helps' means that every second the current moves the boat some meters ahead; 'hinders' means that every second it moves the boat the same number of meters back compared to on the lake."

"And I said the same thing, only I said it shorter and clearer."

"Yes, but you forgot the main thing. The boat sails the same distance up and down the river, but it moves down faster than up, so the river helps the boat less than it hinders! Thus, there is less benefit than resistance."

"Oh, I am a fool! Yeah, great! So it will be faster on the lake."

"I've solved the problem algebraically," said Beth.

"Listen, you, 'guru-algebraist'," Ted needled her, "what is there to write if everything is already clear?"

"You got lost, so I wrote the equations. How could I know that you'd get it so quickly?"

"Okay, show us. Do we have to check it?"

Here was what Beth had written down:

v speed of the boat

u speed of the river (in km/h)

10 km downstream: $10/(v+u)$ hours

10 km upstream: $10/(v-u)$ hours

Total: $10/(v+u) + 10/(v-u) = 20v/(v^2-u^2)$ hours

20 km on the lake: $20/v$ hours

$$\frac{20v}{v^2-u^2} > \frac{20v}{v^2} = \frac{20}{v}$$

"I have trouble with algebraic computations," Ted said. "I'd rather do without them. Okay, do we have anything we have not solved yet?"

There was one more problem on the board.

Problem 3. *A boat goes downriver from town A to town B in three days, and goes upriver from B to A in five. How long will it take a raft to float from A to B?*

"Oh, now we're *really* going to need an equation." Ted became gloomy. "Here it says that the boat sails down the river for three days and up the river for five. And what does that mean?"

"Just that, relative to the ground, the ratio of their speeds is 5 to 3," said Andy.

"Whose speeds?" Ted sounded confused.

"Well, of the boat that sails downstream and one that sails upstream," Andy clarified. "One can even say that we sail two identical boats: one downstream, and another upstream. Then, if the first one sails 5 km on the river, the second will sail 3 km in the same time."

"Wow! I had a great idea!" Dan jumped in.

"What?"

"Send two boats simultaneously in different directions, but both starting from A. We also need to send the raft downstream."

"Why?" Ted was surprised.

"Because the raft will be midway between the boats all the time, if we measure distance along the river! Remember the problem with the boys jumping off the raft."

"Wow, cool! The boats are like those two boys! And since the boats are identical, they are sailing away from the raft with the same speed. But how does this help us?"

"Here's how. If the first boat sailed $5x$ km away from A, then the second one sailed $3x$ km. Do you agree? The raft is half way between them all the time. And where will this midpoint be?"

"The distance between the boats is $8x$ km; half of that is $4x$ km. So the raft will be x km downstream from A."

"And here we are. We want to know when the raft will reach B, so x should be taken equal to the distance between A and B. The first boat will sail 5 times the distance from A to B in this time. It takes 3 days to sail from A to B, and 5 times this time is 15 days. That means that 15 days is the answer."

"I wrote an equation; it happens to be easy, too." Beth said.

"You can't live without your equations," Ted said in frustration.

"Just look. It is only three lines. Let's say that the distance between A and B is 1."

"I didn't get it. One what?"

"Who cares! There is some distance between them, and we can say that's the unit of measurement. Remember that cartoon we once saw where the length of a snake was measured in parrots? We will measure lengths in distances between A and B. It is just as good as in kilometers. It is always done for convenience so as not to introduce an extra variable."

"Okay, what's next?"

v speed of boat
u speed of stream
$3(u+v) = 1$ and $5(v-u) = 1$
$u+v = 1/3$ and $u-v = 1/5$
subtract second equation from the first
$2u = 1/3 - 1/5 = 2/15$, so $u = 1/15$

"So the raft will float from A to B in 15 days."

"And our answer is the same."

"And I've invented a new problem myself! Hooray!" Andy ran to the board and grabbed a piece of chalk.

"Really?"

"Yes. I liked our first solution to the boat problem so much that I made up a problem of my own. Solve it!" And Andy wrote his problem on the board.

Problem 4. *From point A a raft and a boat sailed simultaneously downstream on a straight river, and at the same time another boat of the same type sailed from point C towards them. Prove that at the moment when the first boat reaches point C, the raft will be exactly midway between point A and the second boat.*

"It's almost the same problem," shouted Dan.

The children quickly solved Andy's problem, and were very fond of it.

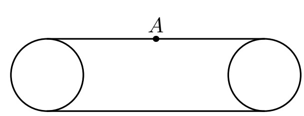
Can you solve it too?

II. Tanks and Escalators

At this moment Sergey, the math circle teacher, entered the room. The students started explaining their solutions to him, while vying with one another for his attention. Then they gave him the problem Andy posed, and carefully checked his solution.

"Well done," the teacher said. "It can be said that you managed to run a math circle without me, and you even came up with a great problem. But I also have something for you."

Problem 5. *The diagram shows the rolling track of a tank, seen from the side. The bottom is in contact with the ground. If the tank moves forward 10 cm, how many centimeters does the point marked A move?*



"It's a problem for first graders, isn't it?" wondered Ted. "If the tank traveled 10 centimeters, then point A also traveled 10 centimeters."