# Preface

*. . . no music is vulgar,*
*unless it is played in a way that makes it so.*
Herbert von Karajan
([**Mat82**, page 226])

This is a textbook for the mathematics curriculum of grades K–6, and is written for elementary school teachers. Specifically, it addresses the most substantial part of the curriculum, namely, numbers and operations.

How does this book differ from textbooks written for students in K–6? The most obvious difference is that, because adults have a longer attention span and a higher level of sophistication, the exposition of this book is more concise; it also offers coherent logical arguments instead of sound bites. Moreover, it does not hesitate to make use of symbolic notation to enhance the clarity of mathematical explanations whenever appropriate, although it must be said in the same breath that symbols are introduced very carefully and very gradually. Because the present consensus is that math teachers should know the mathematics beyond the level they are assigned to teach ([**NMP08a**, Recommendation 19, page xix]), this book also discusses topics that may be more appropriate for grades 7 and 8, such as rational numbers (positive and negative fractions), the Euclidean Algorithm, the *uniqueness* of the prime decomposition of a positive integer, and the conversion of a fraction to a (possibly infinite) decimal.[1]  Because teachers also have to

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[1]In the context of the current requirement that all elementary teachers be generalists, one may question whether it is realistic to expect all elementary teachers to possess this much mathematical knowledge. For this reason, the idea of having *mathematics teachers* in elementary school is being debated and examined (see, for example, [**NMP08a**, Recommendation 20, page xx], and [**Wu09b**]).

answer questions from students, some of which can be quite profound, their knowledge of what they teach must go beyond the minimal level. Ideally, they should know mathematics in the sense that mathematicians use the word "know": *knowing* a concept means knowing its precise definition, its intuitive content, why it is needed, and in what contexts it plays a role, and *knowing* a skill means knowing precisely what it does, when it is appropriate to apply it, how to prove that it is correct, the motivation for its creation, and, of course, the ability to use it correctly in diverse situations. For this reason, this book tries to provide such needed information so that teachers can carry out their duties in the classroom.

The most noticeable difference between this book and student texts is, however, its comprehensive and systematic *mathematical* development of the numbers that are the bread and butter of the K–12 curriculum: whole numbers, fractions, and rational numbers. Such a development acquires significance in light of the recent emphasis on *mathematical coherence* in educational discussions. Coherence in mathematics is not something ineffable like Mona Lisa's smile. It is a quality integral to mathematics with concrete manifestations affecting every facet of mathematics. If we want a coherent curriculum and a coherent progression of mathematics learning, we must have at least one default model of a logical, coherent presentation of school mathematics which respects students' learning trajectory. It is unfortunately the case that, for a long time, such a presentation has not been readily available. The mathematics community has been derelict in meeting this particular social obligation.

The result of this neglect is there for all to see: infelicities abound in mathematics textbooks and in the school mathematics curriculum. For example, it is common to see the use of fraction multiplication to "explain" why $\frac{m}{n} = \frac{cm}{cn}$ for any fraction $\frac{m}{n}$ and any nonzero whole number $c$ (see pages 206 and 270–271 for a discussion of this line of reasoning). Another example is the teaching of the arithmetic operations on fractions as if "fractions are a different kind of number" and as if these arithmetic operations have little or nothing to do with those on whole numbers. Yet another example is the teaching of decimals *parallel* to, and distinct from, the teaching of fractions in the upper elementary grades instead of correctly presenting the mathematics of decimals as part of the mathematics of fractions.

This book does not call attention to coherence per se, but tries instead to demonstrate coherence by example. Its systematic mathematical development makes it possible to point out the careful logical sequencing of the concepts and the multiple interconnections, large and small, among the concepts and skills.[2] Thus, it points out the fact that the usual algorithm for converting a fraction to a decimal by long division, if done correctly, is in

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[2]One should not infer from this statement that the systematic development presented in this book is the only one possible. This book follows the most common school model of going

fact a consequence of the product formula for fractions, $\frac{m}{n} \times \frac{k}{\ell} = \frac{mk}{n\ell}$. It also points out the overwhelming importance of the theorem on equivalent fractions (i.e., $\frac{m}{n} = \frac{cm}{cn}$) for the understanding of every aspect of fractions. On a larger scale, one sees in this systematic development the *continuity* in the evolution of the concepts of addition, subtraction, multiplication, and division from whole numbers to fractions, to rational numbers, and finally—in the context of school mathematics—to real numbers.[3] Although each arithmetic operation may look superficially different in different contexts, this book explains why it is fundamentally the same concept throughout. Thus with a systematic development in place, one can step back to take a global view of the entire subject of numbers and gain some perspective on how the various pieces fit together to form a whole fabric. In short, such a development is what gives substance to any discussion of coherence.

The universities and the education establishment[4] have been teaching teachers *mathematics-without-coherence* for quite some time now. In fact, the importance of content knowledge in the training of teachers has been (more or less) accepted as part of the education dogma only in the last few years. Thus far, we have not been serving our teachers very well in terms of providing them the minimal mathematical knowledge they need to carry out their teaching duties (see, for example, the discussion in [**Wu11**]). Although there seems to be an increasing awareness of this problem (to cite but one example, [**GW08**]), it is nevertheless worthwhile to note the danger of "teaching content" without also being alert to the multitude of flaws in school mathematics. It does not matter whether this is school mathematics handed down to us by tradition or recent reform, the flaws are there. The presence of these flaws is the inevitable consequence of the long separation between the mathematics and the education communities. For example, something as absurd as "$27 \div 6 = 4$ R 3" would have been caught decades ago by any competent mathematician had the two communities been in constant communication (see page 106). Since many such errors are mentioned in the text proper, there will be no need to overdo a good thing by repeating them here. How did this separation come about? I cannot speak for the education community. What I can say with some confidence is that mathematicians generally avoid getting involved in education for two reasons: they believe that education is a bottomless pit in which infinite hard work can lead nowhere, and that the mathematics is trivial. About the former I have nothing to say. The latter is wrong, however. School mathematics may be elementary, but trivial it is not, unless it is written

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from whole numbers to fractions and then to rational numbers, but it would be equally valid, for example, to go from whole numbers to integers and then to rational numbers.

[3]See Chapter 21 on the Fundamental Assumption of School Mathematics (FASM).

[4]They are responsible for teachers' pre-service and in-service professional development, respectively.

in a way that makes it so.[5] The fact that school mathematics has been trivialized in innumerable books and articles should actually be a rallying point for some truly competent mathematicians to step into the education arena and stop the bleeding. It is time for both communities to learn to minimize the damage that has resulted from this separation.

This book is one mathematician's attempt at a systematic presentation of the mathematics of K–6. Subsequent volumes written for middle school and high school teachers will round out the curriculum of the remaining grades. My fervent hope is that others will carry this effort further so that we can achieve an overhaul of the *mathematical education of teachers* as we know it today. Our teachers deserve better, and our children deserve no less.

I also hope that this book, together with its companion volumes for the higher grades, will serve two other purposes. One is to give mathematics educators a more solid starting point for their research. Doing education research on the basis of faulty mathematics is no different from trying to formulate a theory in physics on the basis of faulty experimental data. Regrettably, faulty mathematics is mostly what educators have to work with thus far,[6] and the time for change is now. A second purpose is to provide a resource for textbook publishers. The quality of our textbooks has to improve,[7] but publishers have a valid excuse that there is no literature to help them do better. Perhaps this and other volumes to follow will begin to give them the help they need.

This book is the product of over ten years of experimentation in my effort to teach mathematics to elementary and middle school teachers. The starting point was the workshop on fractions that I conducted in March of 1998 ([**Wu98**]). Part 2 of this book, and arguably its most important part, is nothing more than an expansion of [**Wu98**].[8] Over the years, I have taught from different parts of the book to in-service teachers, but in terms of a regular college course, there is too much material in the book for one semester. A suggested syllabus of such a course on numbers is the following:

Chapters 1–9, 12–22, 32.

A second course on numbers could be based on

Chapters 10–11, 25–42.

Elementary teachers also need to know some geometry; I plan to post a file on geometry on `www.ams.org/bookpages/mbk-79`, which may be downloaded for use as a supplementary textbook.

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[5]See von Karajan's remark at the beginning of the Preface.

[6]See, for example, the discussion on pages 33–38 of [**Wu08**].

[7]To get a very rough idea of the quality of school mathematics textbooks in the most scientifically advanced nation of the world, see Appendix B in [**NMP08b**].

[8]The reader may be startled by how little I have deviated from the original blueprint.

Solutions to the "Activities" scattered throughout this book as well as solutions to the exercises will be posted on the same website `www.ams.org/` `bookpages/mbk-79`. However, the latter will be accessible only to course instructors, and an instructor can receive a link to the solutions by sending a link to the department webpage that lists him or her as being on the teaching staff.

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