Chapter 1

You Can’t Beat the Odds

Suppose, for the sake of argument, that you live in a large city such as Berlin or Hamburg. You are seated on a bus; a passenger departs, leaving behind an umbrella. You take the umbrella, with the idea that when you get home, you will pick up the phone and dial seven random numbers in the hope of reaching the owner of the umbrella.

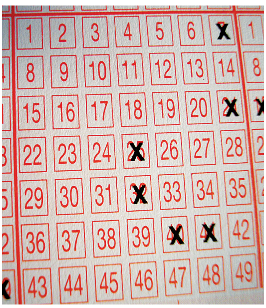
This is, of course, a made-up story, and such a plan in real life would be ridiculed as hopelessly naive. But don’t laugh too quickly, because many of your fellow citizens have the hope every Saturday evening of having chosen the correct lottery numbers, the probability of which is 1 in 13,983,816. Such odds are worse than those of locating the owner of the umbrella according to the plan described above, since there are “only” ten million random sequences of seven digits.

Many lottery players imagine that they can outwit chance by choosing numbers that have not appeared frequently in the past. Such a strategy is wholly without merit, for chance has no memory. Even if, say, the number 13 hasn't been drawn in a long time, in today's drawing it has exactly the same probability of being chosen as any of the other numbers. Other lottery players swear by their own cleverly devised systems for beating the odds, but all such attempts are nothing but wasted effort, for it has been many decades since it was proven that there is no system that can fool chance.

Let us close with a bit of advice: In fact, there is some positive action that a lottery player can take, and that is to choose a combination of numbers that is unlikely to be chosen by many other players. Then if, by some small chance, one wins, one is less likely to have to share the prize with a large number of other winners. That, however, is easier said than done. On one recent occasion, many lottery winners saw their dreams of millions greatly reduced when it turned out that the winning numbers, which formed a cross on the selection card, had been chosen by a surprisingly large number of people.

In the end, however, mathematics has nothing to say about the sweet feeling of expectation that inspires plans about all that one might do with one's fabulous winnings. I wish you good luck!

And Why Exactly 13,983,816?



How does a mathematician arrive at the precise number 13,983,816 of possible selections of lottery entries? Choose two numbers, let's call them n and k, and let us assume that n is larger than k. How many different k-element subsets are contained in a set of n objects?

While this may seem an abstract mathematical question, it concerns us directly in the question of the lottery. A lottery entry is, after all, nothing other than a selection of six numbers from among the numbers 1 through 49. So in this case we are dealing with the numbers $n = 49$ and $k = 6$.

We can easily find similar examples from our common experience:

- For $n = 52$ and $k = 5$ we are asking about the number of possible hands in poker.
- If at the close of a committee meeting each of the fourteen members shakes hands in parting with each of the others, how many handshakes are involved? Here we have the case $n = 14$, $k = 2$.

And now back to the general problem. The formula that we are seeking is a fraction with numerator $n \cdot (n-1) \cdots (n-k+1)$ and denominator $1 \cdot 2 \cdots k$. The numerator may look a bit frightening to the uninitiated, but it is simply the product of the k whole numbers counting down by 1 from n. (Those interested in learning about where this formula comes from will find an introduction in Chapter 29.)

Here are a few additional examples:

- For the lottery problem, we must divide $49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$ by $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$. That is where the number 13,983,816 comes from.
- For the poker problem, the quotient is $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ divided by $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, which leads to 2,598,960 different poker hands. Note that since only four of these hands are royal flushes, the probability of being dealt such a hand is $4/2{,}598{,}960 = 1/649{,}740$, or about three royal flushes out of every two million hands dealt.
- You can solve the handshake problem in your head: $14 \cdot 13$ divided by $1 \cdot 2$ is 91.

A Four-Mile-High Stack of Cards

The idea of randomly selecting the telephone number of an unknown person as an aid in coming to grips with the tiny probability of winning the lottery is not the only possible example. Here is another one.[1]

We begin with the observation that a deck of cards placed on the table is about an inch thick. It would take about 270,000 decks of

[1] Yet another way of picturing this small probability is offered in Chapter 83.

cards to assemble a stack of 13,983,816 cards. Since 270,000 inches is 22,500 feet, which is about 4.25 miles, a stack of 13,983,816 cards would be over four miles high. Suppose now that just one of those cards has a check mark on it. The odds of selecting that card at random on a single try from the four-mile-high stack of cards are about the same as the odds of choosing the correct six lottery numbers.

Chapter 2

Magical Mathematics: The Integers

I would like to introduce to you a little guessing game. Choose a three-digit number and write it twice in succession. For example, if you chose 761, then you should write down 761,761. The game begins by dividing your six-digit number by 7. The remainder, that is, whatever is left after the division, is your lucky number. This will be one of the numbers $0, 1, 2, 3, 4, 5, 6$, since these are the only possible remainders on division by 7. Now write your number and the remainder on a postcard and send it to the editor of this newspaper (*Die Welt*). By return post you will receive as many 100-euro notes as indicated by your lucky number.

$$\begin{array}{r}
108823 \\
7\overline{)761761} \\
\underline{7} \\
06 \\
\underline{0} \\
61 \\
\underline{56} \\
57 \\
\underline{56} \\
16 \\
\underline{14} \\
21 \\
\underline{21} \\
0
\end{array}$$

If you are unfortunate enough to have ended up with zero as your lucky number, you are in good company, since the same fate will have befallen all of your fellow readers. (If such were not the case, the publisher would never have agreed to print the newspaper article.)

The reason for this phenomenon rests in a well-hidden property of the set of whole numbers, or integers. Namely, placing a three-digit number next to itself is equivalent to multiplying it by 1,001, and since 1,001 is divisible by 7, the six-digit number will be divisible by 7 as well.

This idea can be packaged as a little magic trick for one's private use; one can replace the promise of 100-euro notes by predicting the remainder.

Indeed, it happens frequently that a mathematical fact somehow finds its way into a magician's hat. One simply has to find mathematical results that contradict everyday experience and that also have their basis hidden in the depths of some theory.

Here is a piece of advice: Magic is like perfume: the packaging is at least as important as the contents. No one should be suggesting that the chosen three-digit number is to be multiplied by 1,001; such a multiplication is equivalent to writing the number twice in succession, but then the whole trick would fall flat. Those looking for a variant from dividing by 7 can substitute 11 or 13, since 1,001 has these numbers as factors as well. It will just make the calculation of the remainder a bit more difficult.

Advanced Variants: 1,001, 100,001, ...

Is there a reason that we have to write down precisely a *three*-digit number? Could we achieve a similar result with two or four digits?

Let us consider a two-digit number n, written in the form xy. If we write the number twice in succession, then we obtain the four-digit number $xyxy$, which is equivalent to multiplying the original number by 101. But 101 is a prime number, and so the divisors of $xyxy$ are the divisors of xy together with 101. Since in performing this magic trick we know nothing about the number xy, we can say only that there will be zero remainder on division by 101. But asking for division by 101 gives the trick away, or at least strongly suggests what is at work, and furthermore, dividing by 101 may be too difficult for your friends and acquaintances. We conclude, then, that starting with a two-digit number is not such a good idea.

With four-digit numbers we are dealing with multiplication by 10,001. This number is not prime, since $10{,}001 = 73 \cdot 137$, with both factors being prime. Therefore, if you write down a four-digit number twice to form an eight-digit number, it is guaranteed that it is divisible by 73 and 137. But who is eager to divide by 73?

Since the number 100,001 has only the prime divisors 11 and 9,091, both inconvenient divisors, five-digit numbers are not optimal for our magic trick. And so it goes. We again find small divisors with 1,000,000,001 (it is divisible by 7). But do we really want to begin our magic act with, “choose a nine-digit number and write it down twice to form an eighteen-digit number”? My recommendation is that you stick with the original trick.

Here is a table of prime factors for the first several numbers of the form $10\ldots01$:

Number	**Prime Decomposition**
101	101
1,001	$7 \cdot 11 \cdot 13$
10,001	$73 \cdot 137$
100,001	$11 \cdot 9{,}091$
1,000,001	$101 \cdot 9{,}901$
10,000,001	$11 \cdot 909{,}091$
100,000,001	$17 \cdot 5{,}882{,}353$
1,000,000,001	$7 \cdot 11 \cdot 13 \cdot 19 \cdot 52{,}579$
10,000,000,001	$101 \cdot 3{,}541 \cdot 27{,}961$
100,000,000,001	$11 \cdot 11 \cdot 23 \cdot 4{,}093 \cdot 8{,}779$
1,000,000,000,001	$73 \cdot 137 \cdot 99{,}990{,}001$

Those wishing more information about the relations between mathematics and magic could do worse than to consult Martin Gardner’s book *Mathematics, Magic, and Mystery*. Additional magic tricks with a mathematical basis will be presented in Chapters 24 and 86.

Chapter 3

How Old Is the Captain?

Mathematics is considered—and rightly so—a particularly exact science. Its strictly logical construction has served as a model for many other fields in the natural sciences and humanities. A famous example of this is Isaac Newton's magnum opus, *Philosophiae Naturalis Principia Mathematica*. It begins with fundamental definitions and axioms about the world (what is force? what is mass? what are the fundamental laws of motion?), and out of this is derived—in a strictly deductive manner—a model of the world that revolutionized science.

After Newton, there arose a belief in the nature of progress that to us today seems rather naive: all phenomena should be reduced to the simplest possible mechanical model. Many of our fellow citizens still have the tendency to give particular credence to assertions that are couched in mathematical terminology, perhaps even embellished with a mathematical formula. But a good dose of skepticism is frequently in order, for usable results can be expected only when they are based on clear underlying concepts. Thus we will certainly all agree on a definition of "velocity," while "perceived temperature," on the other hand, is an altogether subjective matter. And therefore, the wind-chill formula, for example, is something that one may consider, depending on one's taste, as either amusing or annoying.

In this regard, one would do well to keep in mind the natural limits of mathematics. No matter how much intelligence is brought to bear on a topic, no valid result can be derived from insufficient information. Sometimes, the "result" is hidden so impossibly in the problem that the whole thing is to be taken as a joke: "A ship is 45 meters long. How old is the captain?"

In this form, it is clear to all that such questions are nonsensical. Nevertheless, one frequently encounters questions of the form, "What is the probability that Germany will become world champion?" And how is one to evaluate the odds of winning some soap company's sweepstakes when no one knows how many prizes are to be awarded and how many contestants there are?

Wind Chill and Related Matters

One formula that is used for perceived temperature due to wind chill is

$$T_{\text{wc}} = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16},$$

where T is the actual temperature in degrees Fahrenheit and v is the wind speed in miles per hour. For example, if $T = 32$ and $v = 12$, then T_{wc} is 22.8°F.

This wind-chill formula is a nice example of falsely conceived exactitude. Everyone agrees that it feels colder than it actually is when a strong wind is blowing. But it would be difficult to find two individuals for whom the "perceived temperature" at, say, 22 degrees Fahrenheit and a wind speed of 7 miles per hour was the same to four decimal places of accuracy. The temperature that is "felt" depends on the individual's constitution and clothing, as well as a host of other factors.

But the wind-chill calculators act as though they could calculate perceived temperature exactly. They even present a formula somehow cobbled together from the various parameters, and yet with an accuracy to four decimal places. To be sure, one expects monotonicity: when the wind is stronger, the temperature feels colder. But all in all, we would be better off with a very rough table of values, because the

$$h = Q(12 + 3s/8)$$

Figure 1. A mathematical attempt at humor. The formula supposedly gives the optimal height of a woman's high heels as a function of the number of drinks consumed.

formula leads to a wholly unjustified impression that we are dealing here with an exact science.

In the meanwhile, a host of imitators have appeared on the scene. For example, there have been reports of formulas for the height of stiletto heels (see Figure 1) and the degree of tension produced by a suspense novel. Such "scientific" attempts are frequently to be found in the "Features" sections of newspapers. And thus one can marvel, while eating one's breakfast toast and reading the morning paper, at how mathematics has been pressed into service as a source of humor.

Chapter 4

Vertiginously Large Prime Numbers

The simplest numbers are surely the so-called natural numbers, the ones we use to count: $1, 2, 3, \ldots$. Some of these numbers enjoy the special property that they cannot be written as a product of smaller numbers. Such is the case for 2, 3, and 5, but also for 101 and 1,234,271. Such numbers are said to be *prime*, and they have exercised particular fascination since the earliest beginnings of mathematics.

How big do the prime numbers get? Over two thousand years ago, Euclid gave a famous proof of the fact that there are infinitely many prime numbers, and therefore prime numbers of arbitrarily large size. The idea is the following: Euclid describes a sort of machine into which one deposits some prime numbers, and out comes a prime number that is different from all the prime numbers that were put in. It then follows that there cannot be a finite number of prime numbers.

The consequences of this fact are remarkable, some so remarkable that they can induce in certain individuals a feeling of vertigo. For example, Euclid's result guarantees that there exists a prime number so large that to print it would take more ink than has been produced in the history of the world; we shall, of course, never see such a monster in the flesh. The largest prime number that has been positively identified as such (in the year 2006) has almost ten million digits.

(To get an idea of the size of this number, if you wanted to publish a book in which this record-holding prime was to be printed, it would require over two thousand pages.) Large prime numbers are of use in cryptography, but for practical applications one can use "small" primes of only a few hundred digits.

One of the major tasks of the field of mathematics known as number theory is to discover new secrets about the primes. The great mathematician Carl Friedrich Gauss called number theory the "queen of mathematics."

The Prime Number Machine

Here we give a functional definition of Euclid's prime number machine. Suppose we are given n prime numbers, which we shall name $p_1, p_2, \ldots, p_n$. If such a description seems too abstract, then just keep in mind the four prime numbers $7, 11, 13, 29$, in which case $n = 4$ and $p_1 = 7$, $p_2 = 11$, $p_3 = 13$, and $p_4 = 29$.

We now form the product of these primes and add 1. We will call the result m. Thus

$$m = p_1 \cdot p_2 \cdots p_n.$$

In our special example, we have $m = 7 \cdot 11 \cdot 13 \cdot 29 + 1 = 29{,}030$.

Every number, and therefore m, has at least one prime divisor. Let us call such a divisor p. Observe that p must be different from all of p_1, p_2, $\ldots$, p_n, since if m is divided by one of these numbers, the remainder is 1. (In our example, we could choose $p = 5$, which is a prime divisor of 29,030; and indeed, the number 5 is not to be found among $7, 11, 13, 29$.)

Putting it all together, we see that for an arbitrary collection of prime numbers $p_1, p_2, \ldots, p_n$, a new prime number is produced that was not part of the input. Thus it cannot be the case that the collection of prime numbers is finite, because any finite set of primes fed into the machine produces yet another prime.

Figure 1 shows some additional examples in which now the output is *all* prime divisors of $p_1 \cdot p_2 \cdots p_n + 1$. Note particularly the second and third examples: the prime numbers that are input do not have to be distinct.

$$3,\ 7 \longrightarrow \boxed{3 \cdot 7 + 1 = 22} \longrightarrow 2,\ 11$$

$$2,\ 2,\ 5 \longrightarrow \boxed{2 \cdot 2 \cdot 5 + 1 = 21} \longrightarrow 3,\ 7$$

$$2,\ 2,\ 2,\ 2,\ 2 \longrightarrow \boxed{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 + 1 = 33} \longrightarrow 3,\ 11$$

$$7,\ 11,\ 13,\ 29 \longrightarrow \boxed{7 \cdot 11 \cdot 13 \cdot 29 + 1 = 29{,}030} \longrightarrow 2,\ 5,\ 2{,}903$$

Figure 1. Euclid's prime number machine in action.

Does Euclid's machine generate *all* prime numbers? What we mean by this is the following: We take it as known that 2 is a prime number. We feed it into the machine, and out comes the prime number 3 (the "product" of a set of a single number is the number itself, so the output is $2 + 1$). Now we can feed the machine with 2 and 3, which yields 7, and so we can now work with 2, 3, and 7. These three primes are possible inputs, and not all of them have to be input at once, and one or more of them may be used more than once. The question is, does every prime number appear as output from Euclid's machine?

The answer is yes, since for every prime number p, $p - 1$ is the product of certain (not necessarily distinct) primes $p_1, \dots, p_r$. Therefore, with input $p_1, \dots, p_r$, the prime p will be output, since $p_1 \cdot p_2 \cdots p_r + 1 = p$. This argument can be used to prove, using mathematical induction, the assertion that all prime numbers smaller than the number n are output by Euclid's machine, where n is an arbitrarily large number.

Chapter 5

Loss Plus Loss Equals Win

Mathematics, and probability theory in particular, is chock full of surprising phenomena. When a result stands in stark contrast to general expectation, it is called a *paradox*. Not long ago, the Spanish physicist Juan Parrondo enriched the menagerie of such paradoxes with a new example.

Consider two games of chance in which the player loses on average a small amount to the house. One has to pay an ante to enter the game, and then one wins or loses one euro in each round with probability $\frac{1}{2}$. For the second game, the odds of winning depend on the prior course of the game. There are more- and less-favorable rounds, but on average, the odds are 50–50.

And now for the surprise: If before each round one tosses a coin to determine whether to play game 1 or game 2, then the player has a winning strategy. If the house is willing to let the player continue for a sufficiently long time, one can become arbitrarily wealthy. After

Parrondo's discovery, one could read in various places that now there was a mathematical theory for all possible situations in which an apparent losing proposition ended in a win. Everyone has had such experiences. For example, in a game of chess you can sacrifice almost every piece and yet emerge the victor.

Of course, there is no such theory. However, it is of interest that mathematical results that find their way from the ivory tower of academia to the daily newspapers almost always lead to great expectations that can never be fulfilled. Such was the case, as many will recall, with fractals and chaos theory. Nevertheless, a number of interesting applications of Parrondo's paradox have been found. For example, it explains how a microorganism can alternate between two chemical reactions to swim against the current.

The Precise Rules for Game 2

The rules for the Parrondo's first game have already been described. Those for the second game are somewhat more complicated:

> If the total amount that the player has won thus far is divisible by 3, then the odds are unfavorable: the player loses one euro with probability 9/10 and wins a euro with probability 1/10.[1]
>
> The situation improves when the amount of winnings is not divisible by 3. Then the player wins with probability 3/4 and loses with probability 1/4.

Thus there are favorable and unfavorable situations for the player depending on the divisibility by 3 of the amount won. It can be shown that the game is perfectly fair. However, because of the ante, it is a losing game in the long run.

[1] For example, one card can be drawn at random from a pack of ten. On nine cards is written, "you have just lost one euro!" and on the tenth appears, "you have won one euro!"

A Paradox!

Paradoxes exist in many branches of mathematics. They are to be expected when there are phenomena that are inaccessible to our direct experience: very large or very small numbers, infinite sets, etc.[2]

It is a bit surprising that paradoxes appear so frequently in probability theory, since we have obtained in the course of human evolution a good feel for many aspects of chance. For example, we can gauge the mood of our interlocutor quite well based on facial expression alone, and we can estimate simple risks quite well.

A well-known paradox is one involving birthdays, which is described in Chapter 11. Another well-known example is the *permutation paradox*: A man writes ten letters and addresses ten accompanying envelopes. He puts the letters in the envelopes, but the envelopes are chosen at random. Will at least one letter end up in the correct envelope? A naive assessment would conclude that such a probability is extremely small. However, probability theory tells us that the likelihood is a full 63%. Try it! (In the guise of choosing partners in a game, this paradox appears in Chapter 29.)

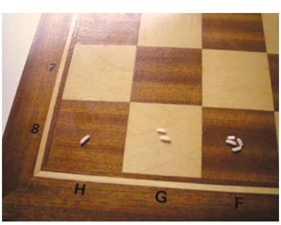
[2]Some paradoxes of the infinite can be found in Chapters 15 and 70.

Chapter 6

When It Comes to Large Numbers, Intuition Fails

The course of human evolution has ill prepared us for discoveries in physics and mathematics. Such things are of only marginal importance in reproduction and survival. Of interest are only such items as average velocity, lengths that are neither very large nor very small, and numbers of relatively small size. Thus just as it is difficult to understand the current view of the nature of the universe, where bizarre phenomena occur at very high velocities, so there is an almost inbuilt inability to grasp certain mathematical truths.

For example, let us talk about large numbers. In physics, there is at least the possibility of representing distances that go beyond our experience and intuition in terms of a suitable scale of measurement. Thus, for example, one can represent the solar system as a miniature model in which the sun is shrunk, say, to the size of an orange. With mathematics such opportunities of representation are fewer, and our ability to imagine what is going on is quickly left in the dust.



One thing that is particularly difficult to grasp is exponential growth. Many people have heard the parable of the grain of rice: If a grain of rice is placed on the first square of a chessboard, and twice that number on the second square, and twice *that* number on the third, and so on, then the numbers of grains of rice on the successive squares are $1, 2, 4, 8, \ldots, 2^{63}$. After these 64 steps, the number of grains is so large that it is far, far greater than the world's annual production of rice. To be sure, rice on a chessboard is not an everyday occurrence. Something analogous that is more in line with our experience is the phenomenon of chain letters. Say you receive a letter that already has been making the rounds, and you are supposed to send copies of the letter, with your name and address appended, to ten of your friends, who are to do the same in turn. Everyone whose name is more than five generations old on the letter is to be sent a picture postcard (or 100 euros, or whatever). This seems like a great idea, and from a naive point of view, there is profit (in postcards or euros) to be made. After all, one sends a single postcard to keep the system alive, and after a while, you receive a bushel basket of mail. (Surely a bushel basket would not suffice: if all the players do as they are supposed to, you can figure on over 100,000 cards.) However, such a game usually falls apart in the early stages, because too many people are sent too many letters by too many friends with a request to send ten letters.

Mathematicians have a particular respect for exponential growth. Problems whose degree of difficulty grows exponentially with the size of the input are considered especially difficult. Thus, for example, one attempts to show that the problem of breaking an encryption procedure is of an exponential nature.

Exponential Growth I: The Flood of Rice

How many grains of rice are actually involved in the rice parable? We need to sum $1 + 2 + 4 + \cdots + 2^{63}$. Such sums are easy to evaluate using the formula for a *geometric progression*:

$$1 + q + q^2 + \cdots + q^n = \frac{q^{n+1} - 1}{q - 1} \quad \text{for } q \neq 1 \text{ and } n = 1, 2, \ldots .$$

In our case, we obtain

$$\frac{2^{64}-1}{2-1} = 18{,}446{,}744{,}073{,}709{,}551{,}615 \approx 18 \cdot 10^{18}.$$

That is a lot of rice!

We have no intuition for such numbers. Indeed, even the fourteen million different combinations in the lottery leave our heads spinning. Let us at least try to get a handle on this amount of rice. A grain of rice is, after all, roughly a cylinder of diameter 1/20 of an inch and length about 1/3 inch. Thus about 1,200 grains of rice should fit in a cubic inch.[1]

Now we can do the math. If 1,200 grains fit in a cubic inch, then one would need $1{,}200 \cdot 12^3 \approx 2{,}000{,}000$ for a cubic foot, and $1{,}200 \cdot 12^3 \cdot 5{,}280^3 = 305{,}229{,}673{,}267{,}200{,}000 \approx 3 \cdot 10^{17}$ for a cubic mile. We should therefore divide the number of grains of rice by $3 \cdot 10^{17}$ to obtain the size of our rice pile in cubic miles: it turns out that one would need about 60 cubic miles.

To make such a number even more accessible, consider the following. Given that Germany has an area of 138,000 square miles, the amount of rice necessary to satisfy the requirements of the chessboard can be reformulated as follows: the rice would cause Germany to disappear under more than two feet of rice.

You don't believe it? I didn't believe it either, so I decided to try for myself. The result is pictured in Figures 1, 2, and 3.

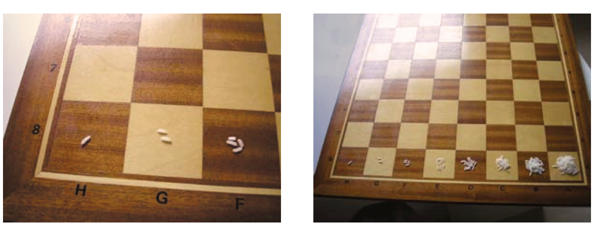


Figure 1. It began innocently enough...

[1]Even more if one could pack them optimally; here they lie every which way.

Figure 2. ... but then it went faster than I had expected...

Figure 3. ... and finally I had to give up.

Exponential Growth II: How Many Times Can You Fold a Sheet of Paper?

Before you read any further, answer the following question: How often do you think a piece of paper can be folded repeatedly in half? Most people guess wrong, giving much too high a number.

In folding, there are two aspects to consider. First, the thickness of the folded paper grows exponentially, doubling after each fold. After five folds, the folded paper is 32 pages thick, since $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$. That is about half an inch, and if you could manage another five folds, you would have about 16 inches.

But that is impossible: when several layers are arranged one above the other and have reached a thickness d, then the situation is different for the upper—that is, the inner after folding—layer than for the lower. Namely, the lower layer must be stretched, indeed enough to make a semicircle of radius r. The circumference of a circle is $2\pi r$, and so we are dealing here with the length πr. For example, if five folds represent half an inch thickness, then the lowest layer must compensate by being stretched by $\pi/2 \approx 1.6$ inches.

After a few more folds, the stretching has reached its limit. Experience shows that the limit is eight folds. (A Berlin radio station wanted to verify this, and on 12 September 2005 a public paper-folding was made with a sheet of paper of dimensions 33 by 49 feet. Even in this case, eight folds was the limit.)