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SOURCES

Extension Theory

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Translator's Note

GRASSMANN's *Ausdehnungslehre* of 1862 is his most mature presentation of his system, and is unique in capturing the full sweep of his mathematical achievement. Stripped of the philosophical drapery of his earlier *Lineale Ausdehnungslehre* of 1844, the "Second Ausdehnungslehre" also contains an enormous amount of material not included (or only suggested) in the earlier book (which, as its title suggests, was deliberately restricted to the "lineale" aspects of the theory, by which he meant that part associated with *linear* constructions, or, in its two-dimensional realization, with constructions using the straightedge of elementary geometry). Among this new material is a detailed development of the inner product and its relation to the concept of angle, the "theory of functions" from the point of view of extension theory, and GRASSMANN's fundamental contributions to the problem of PFAFF. Curiously, one topic mentioned in the Foreword of the "First Ausdehnungslehre" that does *not* appear in the Second is the geometric exponential magnitude and the related geometric logarithm. In many ways, not least of which is its "rigorous Euclidean form", the Second Ausdehnungslehre is the version of GRASSMANN's system most accessible to contemporary readers.

This translation is based on the version of the *Ausdehnungslehre* ("A_2") appearing in *Hermann Grassmanns gesammelte mathematische und physikalische Werke*, published by B. G. TEUBNER under the general editorship of F. ENGEL (which volumes will be referred to here as the "Teubner Edition"). Since the editors of this volume (ENGEL and one of GRASSMANN's sons, Hermann Junior) hoped that it would become a working reference for active mathematicians rather than remain primarily of historical interest, they allowed themselves liberties with the A_2 that were not taken with the other publications in the collection (aside from the *Lehrbuch der Mathematik*, only excerpts of which were published in the Teubner Edition). Thus, in addition to the usual editorial work of correcting misprints, altering a word or construction here and there for clarity, and appending Editorial Comments as end notes, they *rearranged* some of GRASSMANN's theorems to smooth the flow of the argument (e.g. No. 132 was moved and renumbered 116b), *interpolated* new theorems of their own to flesh out and complete the discussion (e.g. No. 230a), *replaced* some of GRASSMANN's Proofs with versions of their own, where they regarded the originals as unsatisfactory (e.g. Part 1 of the Proof of No. 172), and in at least one instance *deleted* one of GRASSMANN's Remarks as being unclear (the Remark following No. 213). In fact, at the time of publication (1896) of this Volume their hope was not unreasonable. As Michael CROWE has documented in his magisterial *History of Vector Analysis* (Notre Dame, 1967; Dover reprint, 1985), there was at that time a "struggle for existence" between several vectorial systems, and such an "improved" A_2 might have been expected to promote the cause of the GRASSMANNian system. In the event, of course, the GRASSMANNian system was not generally adopted, and thus today interest in the A_2 is indeed more as a historical document (albeit an extraordinarily interesting one) than as a tool for mathematical research.

These circumstances left me with decisions to make about how to handle such editorial "improvements". Since my purpose has been to make GRASSMANN available to an English-speaking audience, I had no qualms about letting stand corrections to misprints and minor stylistic improvements; similarly, the rearrangements and even interpolations remained in place, although they are identified as such, either in Footnotes or by being enclosed {between braces}. Similarly, the Editorial Notes, excepting one or two lengthy digressions, have been included almost intact (more on the Editorial Notes below). On the other hand, the replacements and deletions could not pass muster as they stood. GRASSMANN's originals are restored to their proper places in this translation, but *they* are marked {with braces}, and the materials that replaced them in the Teubner Edition now appear in the Editorial Notes. Naturally there was no question of excluding the helpful figures that the editors of the Teubner Edition added to the Work.

GRASSMANN's nomenclature presents problems of an altogether different order. While a translator ought not inflict his technical problems on the patient reader, it is necessary to say at least a word or two here about my treatment of GRASSMANN's novel technical vocabulary. There is, unfortunately, no consistency among the experts about the proper English equivalents for GRASSMANN's technical terms. In fact GRASSMANN himself wavered somewhat in his native German; thus in the *Ausdehnungslehre* of 1844 (the "A_1") he used "Abschattung" for what, in the A_2, he called "Zurückleitung". Here and in my translation I have used "shadow" for this concept, which seems to have been the term preferred by WHITEHEAD in his *Universal Algebra*. Two other contentious terms are "Strecke" and "Verknüpfung", which I have translated as "displacement" and "conjunction", respectively. English-speaking GRASSMANN enthusiasts can and do argue about the best equivalents, and I am afraid my choices may not please them much; but if one were to wait until all such controversies were settled, at least another century would surely pass before an Englished GRASSMANN appeared. In general I have tried to be consistent with the choices used in my translation of the A_1. Finally, it is necessary to mention explicitly two expressions that might possibly lead to some confusion. First, the word "Hauptgebiet" is rendered throughout as "principal domain", and is so indicated in the Index of Technical Terms; it refers to the *underlying vector space* of the system, and must *not* be confused with the expression "principal ideal domain": Second, the word "Stufe", consistently translated as "order", is used in two distinct ways, the first as the "order of a (principal) domain", roughly the *dimension* of the underlying space, and second as the "order of an elementary magnitude", approximately its *rank* in tensor-like terms.

The Editorial Notes have their own interest, both as elaborations and elucidations of GRASSMANN's text, and as illustrating turn-of-the-century attitudes toward his work by competent and reasonably sympathetic mathematicians.

Following the Editorial Notes is a set of Supplementary Notes keyed to the Chapters in the two Parts of the book. These Notes are intended as a guide to help elucidate, in modern terms, what GRASSMANN actually did.

It is a pleasure to acknowledge the deep debt I owe to Prof. Alvin Swimmer and Prof. George B. Seligman, both of whom very kindly read through the entire manuscript and provided extensive lists of corrigenda together with invaluable

suggestions and comments. The addition of Supplementary Notes was inspired by Prof. Seligman. With their help this will be a much more useful work. Whatever defects remain are entirely my responsibility.

To conclude: This completes the second installment of my project of translating GRASSMANN's principal works into English. It is pleasant to contemplate collecting the ten pounds (plus accrued interest?) that R. W. GENESE offered to subscribe (*Nature* **48**, 517 (1893)) for the translation of GRASSMANN's *Ausdehnungslehre*; but that is, I suppose, rather too much to hope for. On the other hand, if these translations produce even a few new GRASSMANN enthusiasts in the English-speaking community, the effort will have been worth the trouble. This translation I dedicate to the memory of my paternal grandfather, my first German teacher.

L. Kannenberg
Weston, 1999

Foreword

The present work comprises the complete theory of extension, a mathematical science, the first volume of which I already published seventeen years ago under the title *Linear Extension Theory, a New Branch of Mathematics*; Otto Wigand Co., Leipzig 1844. In the Foreword to that work I had also indicated the principal topics which, according to my plan, were to be included in the contents of the second volume. Now, instead of publishing the second volume as a continuation of the first, and thus concluding the work in conformity with that plan, I have preferred to take up those topics previously treated in this new work as well, thus to provide a [single] coherent whole.

The principal reason that moved me to this decision is the difficulty which, according to all the mathematicians whose opinions I had the opportunity to hear, the study of that work caused the reader on account of what they believed to be its more philosophical than mathematical form. And this difficulty must in fact have been considerable, since the geometric articles that I have written on the interpretation of that work (*Crelles Journal* **31**, 111–137 (1846), **36**, 177–184 (1848), **42**, 187–212 (1851), **44**, 1–25 (1852), **49**, 1–65, 123–141 (1855), **52**, 254–275 (1856); *Geometrische Analyse*, Leipzig 1847)[1] are often cited and applied by other mathematicians, but the domain treated in that work itself is nowhere touched upon or applied, if I except an interesting short article by Kysaeus (*Bedeutung und Anwendung der Zahlen in Geometrie*, Siegen 1850). As a consequence of this, no review of the book, or statement of its contents, has ever appeared, indeed not a single mention of it outside of the publisher's list, except for one I wrote myself (*Grunerts Archiv* **6**, 237–250 (1845)).[2]

To remove that difficulty thus became an essential task for me if I wanted the book to be read and understood by others as well as myself. This difficulty could not however be removed without substantially changing the plan of the whole. For it is implicit not in some arbitrarily chosen form but in the plan I had envisaged: To formulate the whole from the ground up, independent of other branches of mathematics. The direct implementation of this plan, although it must be the most expeditious for science as such, must, unless it has also become subjective, offer considerable difficulty in that type of presentation, particularly in a science such as extension theory, which extends and intellectualizes the sensual intuitions of geometry into general, logical concepts, and, with regard to abstract generality, is not simply one among the other branches of mathematics, such as algebra, combination theory, and function theory, but rather far surpasses them, in that all fundamental elements are unified under this branch, which thus as it were forms the keystone of the entire structure of mathematics.

Thus I had to abandon this plan, and in the present work I have presupposed the other branches of mathematics, at least in their elementary development. In addition I have adopted exactly the opposite method in the form of presentation, as I have applied the most rigorous mathematical form we know, the Euclidean, to the present work, and have relegated to the Remarks everything that serves to illustrate or motivate the method chosen.

A necessary consequence of the plan as altered was that all the results of the first part, insofar as they did not include applications to physics, had to be taken up and derived anew according to the altered plan of the new edition (as is seen in Nos. 1–136, 216–329). Yet because of the differences in method of the two editions the same topic becomes so dissimilar between them that one scarcely finds any duplication, with the exception of the results obtained themselves, from which in the nature of the subject no deviation can appear. Thus the previous edition is by no means made superfluous by the new one. Indeed, even the new method is by no means to be preferred to the old; on the contrary, the methods of the first edition that derive from the original idea, and proceed thence completely independently, probe more deeply into the essence of the subject, and thus from a purely scientific viewpoint have a decided advantage relative to the latter. On the other hand this latter method is more acceptable to, and in any case more easily understood by mathematicians, who do not willingly see lying fallow the wealth of mathematical knowledge obtained otherwise by their studies. Thus the two presentations are mutually complementary and illustrative.

The presentation chosen here very closely follows arithmetic, in the sense that it assumes the numerical magnitude as a continuous quantity. Now just as arithmetic develops all other magnitudes from a single magnitude, arbitrarily chosen from the rest, which is defined as the unit and may be symbolized by e (see my *Lehrbuch der Arithmetik*; Enslin, Berlin 1861),[3] so, within the framework given here, extension theory first defines several unit magnitudes, e_1, e_2, ... , none of which is derivable from the others (thus for example e_2 cannot be developed from e_1 by multiplication of e_1 with some numerical magnitude), and then considers the magnitudes, which I have called extensive magnitudes, resulting from these units by multiplication with numerical magnitudes and addition of these products. In this way there easily follow the laws of addition, subtraction, multiples, and fractions, discussed in Chapter 1.

It may seem surprising that such a simple idea, which basically consists in no more than regarding multiple sums of different magnitudes (which is how extensive magnitudes appear) as autonomous magnitudes, can in fact be developed into a new science, and indeed in this connection it has been objected that all of extension theory is merely an abbreviated method of notation, and that it is erroneous to regard as magnitudes expressions that are not magnitudes at all. But this objection is based on a complete misunderstanding of the essence of mathematics and of magnitudes. According to this argument all of arithmetic, indeed, one may say all of pure mathematics, is merely an abbreviated method of notation; for number is just an abbreviated expression for a sum of units, the product for a sum of equal numbers, the power for a product of them, and so on; however, without this abbreviated method of notation, or better expressed, without this {formal} unification, no progress toward a unification of concepts is imaginable. Without this unification, for example, it would not be possible to attain the concepts of reductive types of calculation (subtraction, division, roots, logarithms) and the new numerical forms obtained by means of them: negative, fractional, irrational, and imaginary. Thus it is important above all that one actually unify that which in its essence forms a unity, and which therefore must also lead to new results to which one would not be led without that unification.

Now in fact extension theory leads to an inexhaustible wealth of such relations which, without the formation of that conceptual unity appearing in the extensive magnitude, would in no way be conceived or derived. Whether one regards the name "magnitude" as appropriate for this concept is in itself of very subordinate importance, since little depends on names here. The question is only whether this new concept unites with the general concept of magnitude in such a way that in their essence they amalgamate into a general concept, and that a dividing line drawn between the two domains would separate related matters arbitrarily, and divide the subject inconsistently. Thus in this latter case it would even be erroneous *not* to attribute the name "magnitude" to this new concept.

Now I believe that between that which I have called an extensive magnitude and the general numerical magnitude, and particularly the imaginary magnitude $(a + bi)$, there exists a relation so intimate that it would be absurd to treat one and not the other as a magnitude, since the imaginary magnitude is derivable from two units, 1 and $i = \sqrt{-1}$, by real numerical coefficients, just as are extensive magnitudes from two or more units (cf. No. 413, Remark). Thus it seems to me perfectly justified if I designate the extensive magnitude as a *magnitude*. But I go even further, in that I designate them not just as magnitudes generally but as *elementary* magnitudes. Thus they contrast with other magnitudes that likewise bear as decided a character of unified magnitudes as such, as do those elementary ones, and which only enter upon addition of higher structures (cf. Nos. 77, 364, and 377).

I now proceed to clarify the course of development of the present work.

To addition, subtraction, multiples and fractions is adjoined (Chapter 2) the general concept of the multiplication of extensive magnitudes, which is based on the relation of multiplication to addition (that is, that one may multiply the summands in place of the sum). The multiplication of these magnitudes thereby reduces to that of their units, $(e_1, e_2, \dots)$, and from consideration of the products of these units there then follow the different species of multiplication. Now it is possible to separate from these species two, to which all the others can be reduced.

The laws of one of these coincide precisely with ordinary multiplication in algebra, and therefore I will call it algebraic. However it is by far the more complicated with respect to the magnitudes generated and can only be brought to full clarity by consideration of functions, as a consequence of which I have relegated it to the second part of the work. The symbol for this algebraic multiplication must in the nature of the subject coincide with the usual symbol, since it would be absurd to symbolize differently conjunctions that are subject to the same laws in all relations.

The second of those multiplications, which is treated in Chapter Three, is shown to be characteristic of extension theory, and as essential in developing it, as it yields the different orders of elementary magnitudes that enter into extension theory. Thus it is characterized by the condition that two elementary factors of the product can only be interchanged if one simultaneously changes the algebraic sign $(+-)$ of the product. While the relation of this product to addition is indeed the same as with standard multiplication, its other laws deviate substantially from that of standard multiplication, whence it is necessary to distinguish it by its symbol. In this work I have therefore chosen as its symbol brackets enclosing the product, whence $[ab] = -[ba]$ if a and b are elementary factors of this product. This product leads to an extraordinary manifold of apparent forms {Erscheinungsformen}, and permits an enormous abundance of relations to appear which throw a new and unsuspected light on all branches of mathematics, so that this product forms the true core of the new science. Once the concept of the supplementary magnitude is introduced, that product appears in a completely new aspect as the inner product (Chapter 4), so that the range of the subject presented in the first edition (cf. the Foreword to that work) emerges fully in this form. The first part of this volume concludes with applications to geometry (Chapter 5).

Now in the second part there appear combined magnitudes that we can characterize collectively as functions of elementary magnitudes. The first chapter of this part treats functions in general linked by multiplication and division, the second differential calculus, the third the theory of series, and finally the fourth integral calculus, all of these only to the extent that they relate to extensive magnitudes, to be sure. Even so I believe that the corresponding branches of ordinary mathematics (relating to numerical magnitudes), and in particular integral calculus, are not only essentially simplified by this presentation, but are also variously supplemented and developed further.

Because the material is considerably increased from the first edition I have found it necessary to drop all applications to physics; but I hope, if my time and strength permit, to be able to pursue a mathematical treatment of the most important branches of physics in an independent work, in which I will make use of the science reported upon here.[4]

I have earnestly endeavored to avoid superfluous technical words and to limit myself to the smallest possible number of new technical expressions; but since one cannot speak without any words, and thus must either employ new word formations or word combinations for new concepts, or else bestow new meanings on old words, there unavoidably still remains a considerable collection of technical expressions. To aid in comprehensibility I have above all chosen the technical expressions so that, I hope, they immediately suggest the concepts they represent by their own forms; and at the end I have provided an alphabetical index, giving references to the places where they are defined.

It still remains for me to indicate related efforts of other mathematicians. Almost without exception these refer to those topics that I have designated as applications of extension theory to geometry (thus to §§ 24, 28–30, 37–40, 56, 74–79, 91, 92, 101, 102, 114–119, 144–148, 159–170 of the *Ausdehnungslehre* of 1844 and Nos. 216–347 of the present work). In the first edition (1844) the only investigation included there that was known to me was the celebrated work *Barycentrische Kalkül* by the founder of geometric analysis, Möbius, in which he treats the addition of points. Investigations on the geometric addition of displacements (of given length and direction) and of the significance of imaginary magnitudes were unknown to me.

This latter was first presented fully in an article by Gauss (*Göttingische gelehrte Anzeigen*, 1831), to which Gauss called my attention in a letter[5] because of a point handled in the same way in the Foreword to the *Ausdehnungslehre*. The concept of the geometric addition of displacements in the plane was already implicit in this presentation of imaginaries. The first to have dealt with the geometric addition of displacements in complete generality appears to have been Bellavitis, who had already formulated the relevant calculus in 1835 (*Annali delle scienze del regno Lombardo-Veneto* **5**, 244–259 (1835));[6] see No. 227, Remark.

Möbius independently developed the laws for the geometric addition of displacements and applied them to problems of celestial mechanics in his *Mechanik des Himmels* (1843). After the appearance of my *Ausdehnungslehre* (1844), investigations in the domain of geometric analysis proliferated. In particular, it was again Möbius and Bellavitis who materially advanced the science, and contributed significantly to its understanding and the wider circulation of the geometric methods of calculation on which I reported. There then appeared my own work on this subject, partly developed in my paper "*Geometrische Analyse*, geknüpft an die von Leibnitz erfundene geometrische Charakteristik, gekrönte Preisschrift, Leipzig 1847", which Möbius sought to make more accessible to mathematicians by a lucid presentation annexed thereto, and partly in *Crelle's Journal* (in the articles cited above). In addition, a year after the appearance of my *Lineale Ausdehnungslehre*, Saint-Venant published on the geometric addition of lines (*Comptes rendus* **21**, 620–625 (1845)),[7] which was identical to the outer multiplication of displacements presented by me in my book (§ 28–40). Obviously he was not familiar with this work, and I sent two copies of it to Cauchy with the request that one be delivered to Saint-Venant, whose address was unknown to me.

Later on, in several articles printed in the *Comptes rendus* of 1858*, Cauchy published a method for solving algebraic equations and related problems by means of certain symbolic magnitudes that he called *clefs algébriques*, a method that agreed precisely with that presented in my *Ausdehnungslehre* of 1844 (§ 45, 46 and 93). I was far from accusing the celebrated mathematician of plagiarism, but I did believe it was owed me and the subject that I raise a priority claim to the Paris Academy.[9] But the Commission to which this claim was referred in April 1854 for examination and report (*Comptes rendus* **38**, 743f) has never been heard from, and in addition Cauchy never afterward published anything on the subject.

These articles by Cauchy are the only ones that had any point of contact with my *Ausdehnungslehre* (of 1844) outside the domain of geometry. Since however these articles claim an independent origin, it appears as if the characteristic core of my book, excepting the geometric incidentals themselves, has stimulated no related efforts. And therefore with renewed courage I have undertaken this new book, which encompasses the old one and brings it to a conclusion.

For I have every confidence that the effort I have applied to the science reported upon here, which has occupied a considerable span of my lifetime and demanded the most intense exertions of my powers, is not to be lost. Indeed I well know that the form I have given the science is, and *must* be, imperfect. But I also know and must declare, even at the risk of sounding presumptuous, — I know, that even if this work as well should lie idle yet another seventeen years or more without influencing the living development of science, a time will come when it will be drawn forth from the dust of oblivion and the ideas laid down here will bear fruit. I know that if I do not succeed to a position, as yet yearned for in vain, where I can gather a circle of students about me upon whom I can impress these ideas and who I can stimulate to develop and enrich them further, yet some day these ideas, even if in an altered form, will reappear and with the passage of time will participate in a lively intellectual exchange. For truth is eternal, it is divine; and no phase in

****36,** 70–75, 129–135, 161–169.[8]

the development of truth, however small the domain it embraces, can pass away without a trace. It remains even if the garments in which feeble men clothe it fall into dust.

Stettin, 29 August 1861