Chapter 1

Cones

A wedge is any nonempty subset of a vector space that is closed under vector addition and scalar multiplication by nonnegative real numbers. A cone is a wedge that does not contain any negative multiple of its nonzero vectors. In every ordered vector space the set of vectors that dominate zero is a cone called the positive cone. Conversely, every cone induces a vector ordering on the vector space in which it is the positive cone. Therefore, cones and vector orderings are related in a one-to-one manner. In other words, an ordered vector space is a vector space equipped with cone.

This chapter studies the basic properties of cones. Like the rest of this book, the chapter talks about cones in the language of ordered vector spaces. The emphasis is to articulate features related to the order structure of an ordered vector space in terms of the geometry of the positive cone and to translate geometric properties of cones back into the properties of the vector ordering. The material in the chapter is presented without explicitly invoking any topological concepts.

In the first section, we introduce wedges and cones. In the second section, we study the special class of Archimedean cones. The important class of lattice cones (cones of vector lattices) is discussed in Section 1.3 and the basic lattice identities of the lattice cones are stated and proved here. Sections 1.4, 1.5, 1.6, and 1.7 introduce and investigate order bounded operators, positive linear functionals, cone bases, faces, and extremal rays.

The chapter concludes with a thorough discussion of the decomposition and interpolation properties of cones in vector spaces. These important properties are central to duality theory and are shared by lattice cones as well as certain function spaces. These properties provide a rich duality theory that can be described without invoking topological notions. The

decomposition and interpolation properties play an important role and will be featured frequently throughout the book.

1.1. Wedges and cones

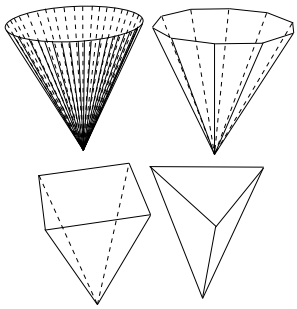
In this section we shall present some basic properties of wedges and cones in vector spaces. We start by recalling that a subset C of a vector space is called **convex** if $x, y \in C$ and $0 \leq \lambda \leq 1$ imply $\lambda x + (1-\lambda)y \in C$.

Definition 1.1. *A nonempty subset $\mathcal{W}$ of a vector space is said to be a* ***wedge*** *if it satisfies the following two properties:*

(1) $\mathcal{W} + \mathcal{W} \subseteq \mathcal{W}$,[1]

(2) $\alpha\mathcal{W} \subseteq \mathcal{W}$ *for all* $\alpha \geq 0$.

Clearly, every wedge is a convex set. Moreover, if $\mathcal{W}$ is a wedge, then it is not difficult to see that the set $\mathcal{W} \cap (-\mathcal{W})$ is a vector subspace. When this vector subspace is trivial, the wedge is called a cone. Precisely, we have the following definition.



Definition 1.2. *A nonempty subset $\mathcal{K}$ of a vector space is said to be a* ***cone*** *if it satisfies the following three properties:*

(1) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$,

(2) $\alpha\mathcal{K} \subseteq \mathcal{K}$ *for all* $\alpha \geq 0$,

(3) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

A ***cone with vertex*** *x is any set of the form $x + \mathcal{K}$, where $\mathcal{K}$ is a cone.*[2]

Obviously, every cone is a wedge but a wedge need not be a cone. For instance, every vector subspace is a wedge but only the trivial vector subspace is a cone. Clearly, every cone (being a wedge) is automatically a convex set.

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[1] Throughout this monograph we shall use the following standard notation. If A and B are subsets of a vector space X and $\alpha, \beta \in \mathbb{R}$, then we let

$$\alpha A + \beta B = \{x \in X : \exists\ a \in A \text{ and } b \in B \text{ such that } x = \alpha a + \beta b\}.$$

[2] Many authors define a cone to be simply a nonempty subset $\mathcal{K}$ of a vector space such that $\alpha x \in \mathcal{K}$ holds for all $x \in \mathcal{K}$ and all $\alpha \geq 0$. In the literature a wedge is quite often referred to as a **convex cone**. A cone as defined in our Definition 1.2 is also known as a **pointed convex cone** or as a **pointed convex cone with vertex at zero**or even as a **cone with vertex at zero**.

Recall that a binary relation $\geq$ on a set X is said to be an **order relation** on the set X (and X equipped with $\geq$ is called a **partially ordered set**) if it satisfies the following three properties:

(i) **reflexivity**: $x \geq x$ for all $x \in X$,

(ii) **antisymmetry**: $x \geq y$ and $y \geq x$ imply $x = y$,

(iii) **transitivity**: $x \geq y$ and $y \geq z$ imply $x \geq z$.

If a binary relation $\geq$ on a set X satisfies properties (i) and (iii), i.e., if $\geq$ is reflexive and transitive but not necessarily antisymmetric, then $\geq$ is called a **pre-order relation** on the set X (and X equipped with $\geq$ is called a **pre-partially ordered set**).

Definition 1.3. *In a pre-ordered set, the notation $x > y$ means $x \geq y$ and $x \neq y$. We shall also express $x \geq y$ by saying that x **dominates** y, and when $x > y$ is true, we shall say that x **strictly dominates** y.*

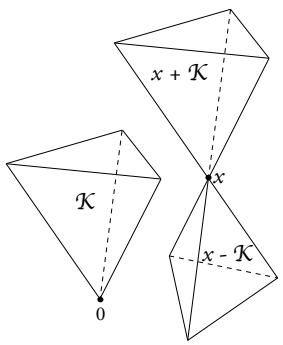
Alternatively, the symbol $x \geq y$ will be denoted by $y \leq x$, and $x > y$ will be denoted by $y < x$. A subset A of a pre-ordered set is said to **majorize** another set B if for each $b \in B$ there exists some $a \in A$ such that $a \geq b$.

Wedges are associated with vector pre-orderings and cones with vector orderings. An order relation (resp. a pre-order relation) $\geq$ on a vector space L is said to be a **vector ordering** (resp. a **vector pre-ordering**) if, in addition to being reflexive, antisymmetric and transitive (resp. reflexive and transitive), $\geq$ is also compatible with the algebraic structure of L in the sense that if $x \geq y$, then

(a) $x + z \geq y + z$ for each $z \in L$ and

(b) $\alpha x \geq \alpha y$ for all $\alpha \geq 0$.

A vector space L equipped with a vector ordering $\geq$ is called an **ordered vector space** or a **partially ordered vector space**, denoted $(L, \geq)$ or simply L if $\geq$ is well understood. In an ordered vector space $(L, \geq)$ any vector satisfying $x \geq 0$ is known as a **positive vector** and the collection of all positive vectors $L_+ = \{x \in L\colon\ x \geq 0\}$ is referred to as the **positive cone** or simply as the **cone** of L. Notice that L_+ is indeed a cone in L. Similar definitions can be given for a vector pre-ordering. The cone L_+ also will be denoted by L^+.



An arbitrary cone $\mathcal{K}$ of a vector space X defines a vector ordering $\geq$ on X by letting $x \geq y$ whenever $x - y \in \mathcal{K}$. Equivalently, $x \geq y$ whenever $x \in y + \mathcal{K}$ or $y \in x - \mathcal{K}$; see the companion figure. It is easy to check that $\geq$ is indeed a vector ordering whose positive cone coincides with $\mathcal{K}$, that is, $X_+ = \mathcal{K}$. Consequently,

the vector orderings and cones are in one-to-one correspondence. That is, every cone $\mathcal{K}$ of a vector space X induces a vector ordering $\geq$ on X whose positive vectors are precisely the vectors in $\mathcal{K}$. If we wish to emphasize that the vector order on X is generated by the cone $\mathcal{K}$, then we shall denote it by $\geq_{\mathcal{K}}$.

Similarly, if $\geq$ is a vector pre-ordering on a vector space X, then the set of positive vectors $X_+ = \{x \in X\colon\ x \geq 0\}$ is a wedge. Conversely, every wedge $\mathcal{W}$ of X gives rise to a vector pre-ordering $\geq$ on X (by letting $x \geq y$ if $x - y \in \mathcal{W}$) that satisfies $X_+ = \mathcal{W}$. Again, if there is a need to emphasize that the vector pre-ordering on X is generated by the wedge $\mathcal{W}$, then we shall denote it by $\geq_{\mathcal{W}}$. In summary, cones give rise to vector orderings and wedges to vector pre-orderings.

It turns out that wedges are simply sums of vector subspaces with cones. The proof of the next result is straightforward and is omitted.

Lemma 1.4. *A nonempty convex subset $\mathcal{W}$ of a vector space is a wedge if and only if it is the sum of a vector subspace and a cone, that is, if and only if there exists a vector subspace V and a cone $\mathcal{K}$ such that $\mathcal{W} = V + \mathcal{K}$.*

Moreover, whenever $\mathcal{W}$ is a wedge, we have the following.

(1) *If we let $V = \mathcal{W} \cap (-\mathcal{W})$ and $\mathcal{K} = \{0\} \cup (\mathcal{W} \setminus V)$, then V is a vector subspace and $\mathcal{K}$ is a cone satisfying $\mathcal{W} = V + \mathcal{K}$.*

(2) *If $\mathcal{W} = V + \mathcal{K}$, where V is a vector subspace and $\mathcal{K}$ is a cone such that $V \cap \mathcal{K} = \{0\}$, then $V = \mathcal{W} \cap (-\mathcal{W})$.*

(3) *The wedge $\mathcal{W} = V + \mathcal{K}$ is a cone, i.e., $V = \{0\}$, if and only if $\mathcal{W}$ contains no straight line passing through the origin.*

Now let $\mathcal{W}$ be a wedge of a vector space X, and let x be any vector in the vector subspace generated by $\mathcal{W}$. Choose vectors $x_1, \ldots, x_n \in \mathcal{W}$ and real numbers $\alpha_1, \ldots, \alpha_n$ such that $x = \sum_{i=1}^n \alpha_i x_i$. If we let $I_1 = \{i\colon\ \alpha_i \geq 0\}$ and $I_2 = \{i\colon\ \alpha_i < 0\}$, then we have

$$x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i \in I_1} \alpha_i x_i - \sum_{i \in I_2} (-\alpha_i) x_i \in \mathcal{W} - \mathcal{W}\,.$$

From this, it easily follows that the vector subspace generated by $\mathcal{W}$ is precisely $\mathcal{W} - \mathcal{W}$.

Definition 1.5. *A wedge $\mathcal{W}$ of a vector space X is said to be **generating** if $X = \mathcal{W} - \mathcal{W}$, i.e., if the vector subspace generated by $\mathcal{W}$ coincides with X.*

It should be clear that a wedge is generating if and only if it majorizes the space. That is, we have the following simple result.

Lemma 1.6. *A wedge $\mathcal{W}$ of a vector space X is generating if and only if for each $x \in X$ there exists some $y \in \mathcal{W}$ satisfying $y \geq_{\mathcal{W}} x$.*

Let $\mathcal{W}$ be a wedge of a vector space X. We shall say that a vector $e \in \mathcal{W}$ is an **order unit** (or more precisely a $\mathcal{W}$-**order unit** in case the wedge $\mathcal{W}$ must be indicated) if for each $x \in X$ there exists some $\lambda > 0$ such that $x \leq \lambda e$. Clearly, if $\mathcal{W}$ has an order unit, then the wedge $\mathcal{W}$ is automatically generating. Also, if $e \in \mathcal{W}$ is an order unit, then so are αe for all $\alpha > 0$ and $e + x$ for each $x \in \mathcal{W}$.

The order units of a wedge coincide with its internal points.[3]

Lemma 1.7. *Let $\mathcal{W}$ be a wedge of some vector space X. A vector $e \in \mathcal{W}$ is an order unit if and only if it is an internal point of $\mathcal{W}$.*

Proof. Assume first that e is an internal point of $\mathcal{W}$ and let $x \in X$. Then there exists some $\alpha > 0$ such that $e + \alpha(-x) \geq 0$. This implies $x \leq \frac{1}{\alpha} e$, so that e is an order unit.

For the converse, suppose that e is an order unit and let $x \in X$. Fix some $\lambda_0 > 0$ such that $\lambda_0(-x) \leq e$. Now notice that for each $0 \leq \lambda \leq \lambda_0$ we have

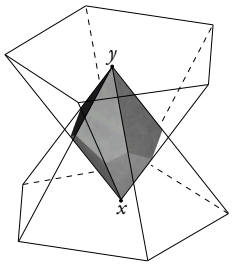
$$\lambda(-x) = \left(\tfrac{\lambda}{\lambda_0}\right)\lambda_0(-x) \leq \tfrac{\lambda}{\lambda_0} e \leq e\,.$$

Consequently, $e + \lambda x \geq 0$ for all $0 \leq \lambda \leq \lambda_0$, which shows that e is an internal point of $\mathcal{W}$. ■

There are some useful sets associated with an ordered vector space that will play an important role in our study in this book. They are introduced next. If x and y are two vectors in an ordered vector space L satisfying $x \leq y$, then the **order interval** $[x, y]$ is the set defined by $[x, y] = \{z \in L\colon\ x \leq z \leq y\}$. If $x \leq y$ is not true, then we let $[x, y] = \emptyset$, so that the order interval is defined for all vectors x and y.[4] Clearly, for all x and y we have

$$[x, y] = (x + L^+) \cap (y - L^+) = x + [0, y - x]\,.$$

In particular, each order interval is a convex set.



Definition 1.8. *A subset A of an ordered vector space L is said to be* ***full*** *(or* ***order-convex****) if for every $x, y \in A$ we have $[x, y] \subseteq A$.*

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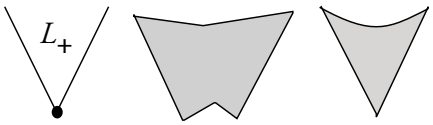
[3] Recall that a point $a \in A$, where A is a subset of a vector space X, is called an **internal point** of A if for each $x \in X$ there exists some $\lambda_0 > 0$ such that $a + \lambda x \in A$ for all $0 \leq \lambda \leq \lambda_0$. Notice that a point $a \in A$ is an internal point if and only if for each $x \in X$ there exists some $0 < \alpha_0 < 1$ such that $\alpha a + (1 - \alpha)x \in A$ for all $\alpha_0 < \alpha < 1$.

If A is a convex set, then a vector $a \in A$ is an internal point of A if and only if for each $x \in X$ there exists some $\alpha > 0$ such that $a + \alpha x \in A$.

[4] This "order interval" definition is applicable to any pre-ordered set as well.



Full sets that are not convex

Clearly, the intersection of a family of full sets is a full set. From this, we easily see that every subset B of L is included in a smallest (with respect to inclusion) full subset—called the **full hull** of B and denoted $[B]$. A moment's thought reveals that

$$[B] = (B + L^+) \cap (B - L^+) = \bigcup_{x,y \in B} [x, y] \,.$$

From the first formula we conclude that

(1) the full hull of a convex set is convex, and

(2) the full hull of a circled set is circled.[5]

A subset A of an ordered vector space L is said to be **bounded above** if there exists some vector x (called an **upper bound** of A) such that $a \leq x$ holds true for all $a \in A$. Similarly, A is **bounded below** if there exists some vector x (called a **lower bound** of A) satisfying $x \leq a$ for all $a \in A$. A subset A of L is **order bounded** if A is bounded from above and below or, equivalently, if it is included in an order interval.

A vector u in an ordered vector space L is called the **least upper bound** (or the **supremum**) of a nonempty subset A of L, written $u = \sup A$, if

(a) u is an upper bound of A, i.e., $a \leq u$ holds for all $a \in A$, and

(b) u is the least upper bound of A in the sense that for any upper bound v of A we have $u \leq v$.

It should be clear from the above definition that a nonempty subset of an ordered vector space can have at most one supremum. The definition of the **greatest lower bound** (or the **infimum**) of a nonempty set A, denoted $\inf A$, is introduced analogously.

The classical lattice notations for the supremum and infimum of a finite set $\{x_1, \ldots, x_n\}$ are

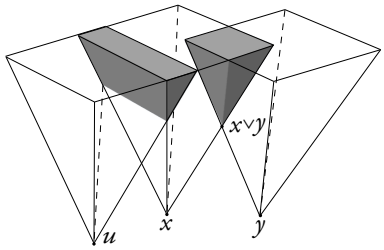
$$\sup\{x_1, \ldots, x_n\} = \bigvee_{i=1}^{n} x_i \quad \text{and} \quad \inf\{x_1, \ldots, x_n\} = \bigwedge_{i=1}^{n} x_i \,.$$

We also write $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$.

[5] Recall that a subset S of a vector space is called **circled** (or **balanced**) whenever $\lambda x \in S$ holds for all $x \in S$ and all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

Of course, in any ordered vector space $x \wedge x = x \vee x = x$. In very special ordered vector spaces any finite set of points has a supremum and an infimum. Other ordered vector spaces that are equally rare have the property that some pairs of vectors do not have a supremum or an infimum while other pairs do have least upper and greatest lower bounds. In most ordered vector spaces for each pair of points x, y that are not comparable (i.e., $x \not\geq y \not\geq x$) the supremum $x \vee y$ and infimum $x \wedge y$ do not exist.

There is a useful geometric interpretation of the supremum and the infimum of a set of two points. Suppose that $\mathcal{K}$ is a cone of a vector space L. Then for any two vectors $x, y \in L$ there exists some $z \in L$ satisfying $(x+\mathcal{K}) \cap (y+\mathcal{K}) = z+\mathcal{K}$ if and only if $z = x \vee y$. Similarly, there exists some $w \in L$ such that $(y-\mathcal{K}) \cap (x-\mathcal{K}) = w-\mathcal{K}$ if and only if $w = x \wedge y$. Notice that in the companion figure $x \vee y$ exists but $x \vee u$ does not exist. We shall return to the lattice properties in Section 1.3.



We continue by recalling the notions of monotone nets. A net $\{x_\alpha\}$ in a partially ordered set is said to be **increasing**, written $x_\alpha \uparrow$ (resp. **decreasing**, written $x_\alpha \downarrow$) if $\alpha \geq \beta$ implies $x_\alpha \geq x_\beta$ (resp. $x_\alpha \leq x_\beta$). The symbol $x_\alpha \uparrow x$ means that $x_\alpha \uparrow$ and $x = \sup\{x_\alpha\}$. Likewise the notation $x_\alpha \uparrow\leq x$ (resp. $x \leq x_\alpha \downarrow$) means that the net $\{x_\alpha\}$ is increasing (resp. decreasing) and $x_\alpha \leq x$ (resp. $x \leq x_\alpha$) holds for each index α. The meaning $x_\alpha \downarrow x$ is analogous. The decreasing and increasing nets are referred to as **monotone nets**.

A nonempty subset D of a partially ordered set is said to be **directed upward**, in symbols $D\uparrow$, if for every pair $x, y \in D$ there exists some $z \in D$ satisfying $x \leq z$ and $y \leq z$. The **directed downward** sets are defined similarly.

Next, we shall indicate how one can obtain new ordered vector spaces from old ones. First, notice that every subspace of an ordered vector space can be viewed as an ordered vector space in its own right. Indeed, if Y is a vector subspace of an ordered vector space X, then it should be clear that Y equipped with the induced order from X is automatically an ordered vector space. Its positive cone is simply the cone $Y_+ = Y \cap X_+$. Unless otherwise stated, the vector subspaces of ordered vector spaces will be considered as ordered vector spaces with the induced vector ordering.

Another way of constructing ordered vector spaces is by means of Cartesian products. If $\{(L_i, L_i^+)\}_{i\in I}$ is a family of ordered vector spaces, then it should be easy to see that the product $L^+ = \prod_{i\in I} L_+^i$ is a cone of the

Cartesian product $L = \prod_{i\in I} L_i$, called the **product cone** of the family of cones $\{L_i^+\}_{i\in I}$, and so (L, L^+) is an ordered vector space. Notice that $(x_i)_{i\in I} \geq (y_i)_{i\in I}$ holds in L if and only if $x_i \geq y_i$ holds in L_i for each i. An important particular case of a Cartesian product ordered vector space is the ordered vector space $\mathbb{R}^\Omega$, where Ω is a nonempty set—and, as above, $f \geq g$ holds in $\mathbb{R}^\Omega$ if and only if $f(\omega) \geq g(\omega)$ for each $\omega \in \Omega$. More generally, if L is an ordered vector space and I is any nonempty index set, then the Cartesian product L^I is likewise an ordered vector space, where $(x_i)_{i\in I} \geq (y_i)_{i\in I}$ in L^I is equivalent to $x_i \geq y_i$ in L for each $i \in I$.

Finally, we shall close the section with the notion of an order isomorphism. Recall that a function $T\colon X \to Y$ is called an **operator** (or a **linear transformation**) if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ holds for all $x, y \in X$ and all scalars $\alpha, \beta \in \mathbb{R}$.

Definition 1.9. *Let L and M be two ordered vector spaces. An operator $T\colon L \to M$ is called an **order isomorphism** (or an **order-embedding**) if*

(a) *T is one-to-one and*

(b) *$x \geq 0$ holds in L if and only if $Tx \geq 0$ holds in M.*

*If there exists an onto (i.e., surjective) order isomorphism from L to M, then L and M are called **order isomorphic** ordered vector spaces.*[6]

That is, two ordered vector spaces are order isomorphic if and only if there exists a one-to-one correspondence between the elements of the ordered vector spaces that preserves both the algebraic and the order structures. From the point of view of the theory of ordered vector spaces two order isomorphic ordered vector spaces are viewed as identical objects.

We shall also say that an ordered vector space L is **order-embeddable** into another ordered vector space M if there exists an order isomorphism from L to M. In this case, the subspace $T(L)$ ordered with the induced vector ordering from M can be identified with L and for this reason $T(L)$ is referred to as a **copy** of L in M.

Exercises

1. Show that a nonempty convex subset of a vector space is a wedge if and only if it is closed under nonnegative scalar multiplication.

2. If $\mathcal{W}$ is a wedge, then prove that $\mathcal{W} \cap (-\mathcal{W})$ is a vector subspace.

3. If two vectors x and y of a cone satisfy $x+y = 0$, then show that $x = y = 0$.

[6] Since the structures under consideration here are the vector orderings, according to the general notion of an isomorphism (as explained following the Preface of the book) the "order isomorphism" can be very well referred to simply as an "isomorphism."

4. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are two wedges of a vector space, then show that $\lambda\mathcal{W}_1$ for $\lambda \in \mathbb{R}$, $\mathcal{W}_1 \cap \mathcal{W}_2$, and $\mathcal{W}_1 + \mathcal{W}_2$ are all wedges. Is the sum of two cones a cone?

5. If two cones $\mathcal{K}_1$ and $\mathcal{K}_2$ of a vector space satisfy $\mathcal{K}_1 \cap (-\mathcal{K}_2) = \{0\}$, then show that $\mathcal{K}_1 + \mathcal{K}_2$ is a cone.

6. Prove that if $\mathcal{K}$ is a cone and $\mathcal{W}$ is a wedge of a vector space, then $\mathcal{K} \cap \mathcal{W}$ is a cone—and so, in particular, the intersection of two cones is a cone.

7. Consider the wedge $\mathcal{W}$ of $\mathbb{R}^2$ defined by $\mathcal{W} = \{(x, y) \in \mathbb{R}^2 \colon\ x + y \leq 0\}$. According to Lemma 1.4 we know that $\mathcal{W}$ has a decomposition of the form $\mathcal{W} = V + \mathcal{K}$, where V is a vector subspace and $\mathcal{K}$ is a cone. Find three different such decompositions of $\mathcal{W}$.

8. Establish that if e is an order unit of an ordered vector space L, then so are $e + x$ and λe for all $x \in L_+$ and $\lambda > 0$.

9. Following the terminology in this section, we shall say that a vector subspace X of an ordered vector space L is **majorizing** if for each $u \in L$ there exists some $x \in X$ such that $u \leq x$. (A majorizing vector subspace is also called a **cofinal** vector subspace.)

For an ordered vector space L establish the following.

(a) A vector subspace X of L majorizes L if and only if $L = X - L_+$.
(b) The cone L_+ is generating if and only if the vector space $L_+ - L_+$ majorizes L.
(c) If a vector subspace contains an order unit, then it majorizes L.
(d) A vector subspace that majorizes L need not have an order unit.

10. For an operator $T \colon X \to Y$ between vector spaces show the following.

(a) If $\mathcal{W}$ is a wedge of X, then $T(\mathcal{W})$ is a wedge of Y.
(b) If $\mathcal{W}$ is a wedge of Y, then $T^{-1}(\mathcal{W})$ is a wedge of X.
(c) If T is one-to-one and $\mathcal{W}$ is a cone of X, then $T(\mathcal{W})$ is a cone of Y.
(d) If T is one-to-one and $\mathcal{W}$ is a cone of Y, then $T^{-1}(\mathcal{W})$ is a cone of X.

11. Prove Lemma 1.4.

12. Can a cone contain a straight line?

13. Establish that a positive vector e of an ordered vector space is an order unit if and only if the vector zero is an internal point of the convex set $e - L_+ = \{x \in L \colon\ x \leq e\}$. [*Hint*: Notice that e is an internal point of L_+ if and only if 0 is an internal point of $e - L_+$ and then apply Lemma 1.7.]

14. If $[a, b]$ and $[c, d]$ are order intervals of an ordered vector space, then establish that $[a, b] + [c, d] = [a + c, b + d]$, $\lambda[a, b] = [\lambda a, \lambda b]$ for all $\lambda \geq 0$, and $\lambda[a, b] = [\lambda b, \lambda a]$ for all $\lambda < 0$.

15. Prove that translations, arbitrary intersections, and scalar multiples of full sets are full sets.

16. Show that if B is a nonempty subset of an ordered vector space, then its full hull $[B]$ is given by

$$[B] = (B + L^+) \cap (B - L^+) = \bigcup_{x,y \in B} [x, y]\,.$$

Use the formula $[B] = (B + L^+) \cap (B - L^+)$ to conclude that
(i) the full hull of a nonempty convex set is convex and
(ii) the full hull of a circled set is circled.

17. Let B be a nonempty subset of an ordered vector space consisting of "incomparable" vectors, i.e, if $x, y \in B$ and $x \neq y$, then neither $x \geq y$ nor $y \geq x$ is true. Find the full hull of B. [*Hint*: Note that $[x, x] = \{x\}$.]

18. If a nonempty subset A of an ordered vector space L has a supremum (resp. an infimum), then show that for each $x \in L$ the set $x + A$ has a supremum (resp. an infimum) and the following lattice identities hold:

$$x + \sup A = \sup(x + A) \quad \text{and} \quad x + \inf A = \inf(x + A)\,.$$

Similarly, with an appropriate interpretation establish the following useful lattice identities:

$$x - \sup A = \inf(x - A) \quad \text{and} \quad x - \inf A = \sup(x - A).$$

$$\lambda \sup A = \sup(\lambda A) \quad \text{and} \quad \lambda \inf A = \inf(\lambda A) \ \text{ for all } \ \lambda \geq 0\,.$$

19. Assume that two non-empty subsets A and B of an ordered vector space have suprema. Then show that the set $A + B$ also has a supremum and that $\sup(A + B) = \sup A + \sup B$ holds true.

20. For each $n \in \mathbb{N}$ let $f_n \colon [0, 1] \to \mathbb{R}$ be the continuous function whose graph consists of the two line segments joining the points $(0, 0)$ with $(\frac{1}{n}, 1)$ and $(\frac{1}{n}, 1)$ with $(1, 1)$. Show that $f_n \uparrow \mathbf{1}$ holds in $C[0, 1]$, where $\mathbf{1}$ denotes the constant function one on $[0, 1]$.

Likewise, if $\{g_n\} \subseteq C[0, 1]$ is defined by $g_n(t) = \sqrt[n]{t}$, then show that $g_n \uparrow \mathbf{1}$ in $C[0, 1]$.

21. For each $n \in \mathbb{N}$ let $h_n \colon [0, 1] \to \mathbb{R}$ be the continuous function whose graph consists of the polygonal line joining the points $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2} + \frac{1}{2n}, 1)$, and $(1, 1)$. Show that $h_n \uparrow$ holds true in $C[0, 1]$ and that the sequence $\{h_n\}$ does not have a supremum in $C[0, 1]$.

22. Show that every bounded from above nonempty subset A of $\mathbb{R}^\Omega$, where Ω is a fixed nonempty set, has a supremum in the ordered vector space $\mathbb{R}^\Omega$ and that $[\sup A](\omega) = \sup_{f \in A} f(\omega)$ holds for all $\omega \in \Omega$.[7]

Use this conclusion to find the suprema of the sequences appearing in Exercises 20 and 21 above in $\mathbb{R}^{[0,1]}$.

23. For two nets $\{x_\alpha\}$ and $\{y_\delta\}$ of an ordered vector space show the following.
(a) If $x_\alpha \uparrow x$ and $y_\delta \uparrow y$, then $x_\alpha + y_\delta \uparrow x + y$, $\lambda x_\alpha \uparrow \lambda x$ for all $\lambda \geq 0$, and $\lambda x_\alpha \downarrow \lambda x$ for all $\lambda < 0$.
(b) If $x_\alpha \uparrow x$, $y_\delta \uparrow$, and $x_\alpha + y_\delta \uparrow z$, then $y_\delta \uparrow z - x$.

24. Show that for two vectors x and y in an ordered vector space their infimum $x \wedge y$ exists if and only if their supremum $x \vee y$ exists—in which case we have the identity $x \wedge y + x \vee y = x + y$. [*Hint*: Use Exercise 18 above.]

[7] An ordered vector space for which every increasing bounded from above net has a supremum is called **Dedekind** (or **order**) **complete**. It is not difficult to see that, in this terminology, our conclusion can be stated as follows: *The ordered vector space* $\mathbb{R}^\Omega$ *is Dedekind complete.*

1.2. Archimedean cones

The objective here is to introduce and study the special class of Archimedean cones. We start with their definition.

Definition 1.10. *An ordered vector space L is said to be* ***Archimedean*** (*or we say that L has the* ***Archimedean property***) *whenever it follows from $y \in L$, $x \in L_+$, and $ny \leq x$ for all $n = 1, 2, \ldots$ that $y \leq 0$.*

A cone $\mathcal{K}$ of a vector space L is said to be ***Archimedean*** *if the order induced by $\mathcal{K}$ on L makes L an Archimedean ordered vector space.*

The vector $x \in L_+$ in the preceding definition can be assumed to lie in L.

Lemma 1.11. *An ordered vector space L is Archimedean if and only if it follows from $x, y \in L$ and $ny \leq x$ for all $n = 1, 2, \ldots$ that $y \leq 0$.*

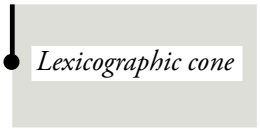
Proof. If the condition holds true, then L is clearly Archimedean. For the converse, assume that L is Archimedean and that for some $x, y \in L$ we have $ny \leq x$ for all $n \in \mathbb{N}$. This implies $(n+1)y \leq x$ or $ny \leq x - y$ for all $n \in \mathbb{N}$. Since $x - y \geq 0$, it easily follows that $y \leq 0$. ■

A classical example of a non-Archimedean ordered vector space is provided by the lexicographic plane. The **lexicographic plane** is the vector space $L = \mathbb{R}^2$ equipped with the following vector ordering (known as the **lexicographic ordering**):

$$(x_1, x_2) \geq (y_1, y_2) \text{ if either } x_1 > y_1 \text{ or else } x_1 = y_1 \text{ and } x_2 \geq y_2 .$$

The lexicographic ordering is induced by the cone

$$L_+ = \big\{(x, y) \in \mathbb{R}^2 : \text{ either } x > 0 \text{ or else } x = 0 \text{ and } y \geq 0\big\},$$



which is referred to as the **lexicographic cone**. To see that L equipped with the cone L_+ is a non-Archimedean ordered vector space, note, for instance, that $n(0,1) \leq (1,0)$ holds for all n while $(0,1) \notin -L_+$. Observe also that the lexicographic ordering is **total**. That is, any two arbitrary vectors $u, v \in L$ are comparable in the sense that either $u \geq v$ or $v \geq u$ is true.

The next two results contain some basic "order continuity" properties of Archimedean ordered vector spaces.

Theorem 1.12. *Let $x_1, \ldots, x_k$ be positive vectors in an Archimedean ordered vector space. Assume that the real sequences $\{\alpha_n^i\}$ $(i = 1, \ldots, k)$ satisfy $\alpha_n^i \xrightarrow[n\to\infty]{} \alpha_i$ in $\mathbb{R}$ for each $i = 1, \ldots, k$ and $\alpha_n^1 x_1 + \cdots + \alpha_n^k x_k \leq 0$ for each n. Then we have $\alpha_1 x_1 + \cdots + \alpha_k x_k \leq 0$.*

Proof. Fix any $m \in \mathbb{N}$ and then pick some n such that $\alpha_i - \alpha_n^i < \frac{1}{m}$ holds for all $i = 1, \dots, k$. Now from

$$\begin{aligned} \alpha_1 x_1 + \cdots + \alpha_k x_k &= (\alpha_1 - \alpha_n^1)x_1 + \cdots + (\alpha_k - \alpha_n^k)x_k + \alpha_n^1 x_1 + \cdots + \alpha_n^k x_k \\ &\leq (\alpha_1 - \alpha_n^1)x_1 + \cdots + (\alpha_k - \alpha_n^k)x_k \\ &\leq \tfrac{1}{m}x_1 + \cdots + \tfrac{1}{m}x_k \,, \end{aligned}$$

we see that $m(\alpha_1 x_1 + \cdots + \alpha_k x_k) \leq x_1 + \cdots + x_k$ holds for all $m \in \mathbb{N}$. Now invoke the Archimedean property to see that $\alpha_1 x_1 + \cdots + \alpha_k x_k \leq 0$. ■

An easy consequence of the preceding result is the following.

Corollary 1.13. *Let $x_1, \dots, x_k$ and $y_1, \dots, y_m$ be vectors in an Archimedean ordered vector space L. Assume that the sequences of real numbers $\{\alpha_n^i\}$ $(i = 1, \dots, k)$ and $\{\beta_n^j\}$ $(j = 1, \dots, m)$ satisfy*

(a) *$\alpha_n^i \xrightarrow[n\to\infty]{} \alpha_i$ for each $i = 1, \dots, k$,*

(b) *$\beta_n^j \xrightarrow[n\to\infty]{} \beta_j$ for each $j = 1, \dots, m$, and*

(c) *$\alpha_n^1 x_1 + \cdots + \alpha_n^k x_k \leq \beta_n^1 y_1 + \cdots + \beta_n^m y_m$ for each n.*

If the x_i and the y_j belong to $L_+ - L_+$, then

$$\alpha_1 x_1 + \cdots + \alpha_k x_k \leq \beta_1 y_1 + \cdots + \beta_m y_m \,.$$

Exercises

1. Show that an ordered vector space L is Archimedean if and only if for each $x \in L_+$ we have $\frac{1}{n}x \downarrow 0$.

2. Show that a totally ordered Archimedean ordered vector space is either trivial or order isomorphic to $\mathbb{R}$. [*Hint*: Assume L is nontrivial. Fix some $0 < u \in L$ and for each $x \in L$ let $T(x) = \sup\{\alpha \in \mathbb{R}\colon\ \alpha u \leq x\}$. Establish that $T(x)$ is well defined such that $T(x)u = x$, and infer from this that $T\colon L \to \mathbb{R}$ is an order isomorphism.]

3. Show that the Cartesian product of a family of Archimedean ordered vector spaces is an Archimedean ordered vector space.

4. Let $\mathcal{K}$ be a cone of a vector space X and let $Y = \mathcal{K} - \mathcal{K}$ be the vector subspace generated by the cone $\mathcal{K}$ in X. Show that the cone $\mathcal{K}$ is an Archimedean cone of X if and only if it is an Archimedean cone of Y.

5. Is a subcone of an Archimedean cone necessarily Archimedean?

6. A cone $\mathcal{K}$ of a vector space L is said to be **almost Archimedean** if it follows from $-\frac{1}{n}x \leq_{\mathcal{K}} y \leq_{\mathcal{K}} \frac{1}{n}x$ for all $n \in \mathbb{N}$ and some $x, y \in L$ that $y = 0$. An ordered vector space L is called **almost Archimedean** if its cone L_+ is almost Archimedean.

(a) Prove that every subcone of an arbitrary Archimedean cone is almost Archimedean—and conclude from this that every Archimedean cone is almost Archimedean.

(b) Give an example of an almost Archimedean ordered vector space that is not Archimedean.

1.3. Lattice cones

In this section we shall discuss briefly the class of Riesz spaces (or vector lattices). Since there is a vast literature dealing with Riesz spaces (see for instance [**14, 15, 66, 68, 86, 88, 93**]), we shall mainly exhibit some of their basic properties. However, we shall introduce more properties of vector lattices in subsequent sections as they are needed. We start with the definition of a Riesz space.

Definition 1.14. *An ordered vector space L is said to be a* ***Riesz space*** *(or a* ***vector lattice****) if every nonempty finite subset of L has a supremum (least upper bound) and an infimum (greatest lower bound).*

The cone of a Riesz space is called a ***lattice cone****. We shall also say that a cone $\mathcal{K}$ of a vector space X is a* ***lattice cone*** *if X partially ordered by the cone $\mathcal{K}$ is a Riesz space.*

In other words, the cone L_+ of an ordered vector space L is a lattice cone if and only if L is a Riesz space. Using an easy inductive argument, it is easy to see that

- *an ordered vector space L is a Riesz space if and only if every two vectors of L have a supremum and an infimum in L.*

Also, we shall say that a vector ordering $\geq$ on an arbitrary vector space X is a **lattice ordering** if its positive cone $X^+ = \{x \in X\colon\ x \geq 0\}$ is a lattice cone.

We also have the following useful characterization of Riesz spaces.

Lemma 1.15. *An ordered vector space L is a Riesz space if and only if for every pair of vectors $x, y \in L$ their infimum $x \wedge y$ exists in L. Moreover, if x and y are elements in a Riesz space, then*

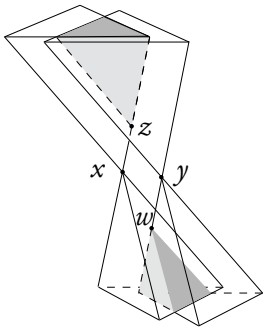
$$x \vee y = -\big[(-x) \wedge (-y)\big] \quad \text{and} \quad x \wedge y = -\big[(-x) \vee (-y)\big]\,.$$

Likewise, an ordered vector space L is a Riesz space if and only if for every pair of vectors $x, y \in L$ their supremum $x \vee y$ exists in L.

Proof. Assume first that every pair in an ordered vector space L has an infimum, and let $x, y \in L$. Put $z = (-x) \wedge (-y)$. We shall prove that $x \vee y = -z$. To see this, note first that we have $z \leq -x$ and $z \leq -y$, or $x \leq -z$ and $y \leq -z$. That is, $-z$ is an upper bound of the set $\{x, y\}$. To see that $-z$ is the least upper bound of the set $\{x, y\}$, suppose that a vector t satisfies $x \leq t$ and $y \leq t$. This implies $-t \leq -x$ and $-t \leq -y$, and therefore

$-t \leq (-x) \wedge (-y) = z$, i.e., $t \geq -z$. Hence $x \vee y = -z$ holds. The converse is straightforward. ■

There is another important geometrical characterization of cones that are lattice cones. This characterization will be used extensively in this work.



Theorem 1.16. *A cone $\mathcal{K}$ in a vector space X is a* ***lattice cone*** *if and only if for any $x, y \in X$ there exists some $z \in X$ satisfying*

$$(x + \mathcal{K}) \cap (y + \mathcal{K}) = z + \mathcal{K}\,,$$

in which case $z = x \vee y$ (or, equivalently, for any $x, y \in X$ there exists some $w \in X$ satisfying

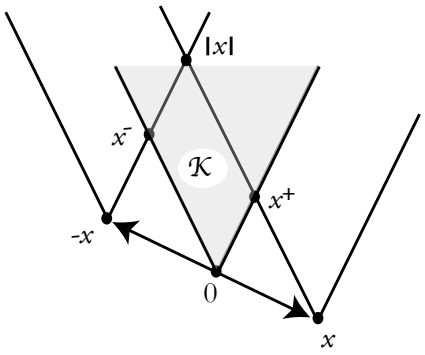
$$(x - \mathcal{K}) \cap (y - \mathcal{K}) = w - \mathcal{K}\,,$$

in which case we have $w = x \wedge y$).

In every Riesz space there are three important vectors associated with each vector x. Its **positive part** x^+, its **negative part** x^-, and its **absolute value** $|x|$. These vectors are defined via the formulas

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0 \quad \text{and} \quad |x| = x \vee (-x)\,.$$

Their geometrical meaning is shown in the figure below.



The following theorem contains a list of several important lattice identities that hold true for elements in Riesz spaces. These identities will be used freely throughout this monograph.

Theorem 1.17 (Lattice identities)**.** *For any elements x, y, and z in a Riesz space we have the following:*

(1) $x + y \vee z = (x + y) \vee (x + z)$ *and* $x + y \wedge z = (x + y) \wedge (x + z)$,

(2) $x - y \wedge z = (x - y) \vee (x - z)$ *and* $x - y \vee z = (x - y) \wedge (x - z)$,

(3) $x \vee y = (x-y)^+ + y = (y-x)^+ + x$,

(4) $\lambda(x \vee y) = (\lambda x) \vee (\lambda y)$ *and* $\lambda(x \wedge y) = (\lambda x) \wedge (\lambda y)$ *for* $\lambda \geq 0$,

(5) $|\lambda x| = |\lambda|\,|x|$ *for all* $\lambda \in \mathbb{R}$,

(6) $x \vee y = \frac{1}{2}\big(x+y+|x-y|\big)$ *and* $x \wedge y = \frac{1}{2}\big(x+y-|x-y|\big)$,

(7) $x + y = x \vee y + x \wedge y$,

(8) $x = x^+ - x^-$ *and* $x^+ \wedge x^- = 0$,

(9) $|x| = x^+ + x^-$ (*and so* $|x| = 0$ *if and only if* $x = 0$),

(10) $|x-y| = x \vee y - x \wedge y$,

(11) $|x+y| \vee |x-y| = |x| + |y|$,

(12) $|x| \vee |y| = \frac{1}{2}\big(|x+y|+|x-y|\big)$ *and* $|x| \wedge |y| = \frac{1}{2}\big||x+y|-|x-y|\big|$.

Proof. (1) The first identity should be interpreted as follows:

- *if two elements* y *and* z *in an ordered vector space have a supremum* $y \vee z$, *then for every element* x *the pair of vectors* $x+y$ *and* $x+z$ *also has a supremum and* $(x+y) \vee (x+z) = x + (y \vee z)$.

To see this, let $t = y \vee z$ and fix x. Clearly, $x+y \leq x+t$ and $x+z \leq x+t$. Now assume that $x+y \leq s$ and $x+z \leq s$. It follows that $y \leq s-x$ and $z \leq s-x$, and so $t = y \vee z \leq s-x$ or $x+t \leq s$. This shows that $x+t$ is the supremum of the set $\{x+y, x+z\}$, and this proves the validity of the first identity. The validity of the second identity can be proven in a similar manner.

(2) It follows from part (1) and Lemma 1.15.

(3) To establish the first identity in this part note that

$$(x-y)^+ + y = (x-y) \vee 0 + y = \big[(x-y)+y\big] \vee (0+y) = x \vee y\,.$$

The second identity can be proven in a similar manner.

(4) Assume that $\lambda > 0$. From $x \leq x \vee y$ and $y \leq x \vee y$, we get $\lambda x \leq \lambda(x \vee y)$ and $\lambda y \leq \lambda(x \vee y)$. Now assume that $\lambda x \leq z$ and $\lambda y \leq z$. It follows that $x \leq \frac{1}{\lambda} z$ and $y \leq \frac{1}{\lambda} z$, and so $x \vee y \leq \frac{1}{\lambda} z$. Consequently $\lambda(x \vee y) \leq z$, and from this we see that $(\lambda x) \vee (\lambda y) = \lambda(x \vee y)$. The other identity can be established with a similar argument.

(5) If $\lambda \geq 0$, then $|\lambda x| = (\lambda x) \vee (-\lambda x) = \lambda\big[x \vee (-x)\big] = |\lambda||x|$ and if $\lambda < 0$, then $|\lambda x| = (\lambda x) \vee (-\lambda x) = \big[(-\lambda)(-x)\big] \vee (-\lambda x) = (-\lambda)\big[(-x) \vee x\big] = |\lambda||x|$.

(6) For the first identity in this part note that

$$\begin{aligned} \tfrac{1}{2}\big(x+y+|x-y|\big) &= \tfrac{1}{2}\big[x+y+(x-y) \vee (y-x)\big] \\ &= \tfrac{1}{2}\big[(2x) \vee (2y)\big] = x \vee y\,. \end{aligned}$$

(7) Adding the identities in (6) yields the desired identity.

(8) Letting $y = 0$ in (7) yields

$$x = x \vee 0 + x \wedge 0 = x \vee 0 - (-x) \vee 0 = x^+ - x^- .$$

For the second identity in this part, note that

$$\begin{aligned} x^+ \wedge x^- &= (x^+ - x^-) \wedge 0 + x^- = x \wedge 0 + x^- \\ &= -\big[(-x) \vee 0\big] + x^- = -x^- + x^- = 0 . \end{aligned}$$

(9) Observe that

$$|x| = x \vee (-x) = (2x) \vee 0 - x = 2x^+ - (x^+ - x^-) = x^+ + x^- .$$

(10) Using the definition of the absolute value, (1) and (7), we get

$$\begin{aligned} |x - y| &= (x - y) \vee (y - x) = (2x) \vee (2y) - (x + y) \\ &= 2(x \vee y) - (x \vee y + x \wedge y) \\ &= x \vee y - x \wedge y . \end{aligned}$$

(11) Notice that

$$\begin{aligned} |x + y| \vee |x - y| &= \big[(x + y) \vee (-x - y)\big] \vee \big[(x - y) \vee (y - x)\big] \\ &= \big[(x + y) \vee (x - y)\big] \vee \big[(-x - y) \vee (y - x)\big] \\ &= \big[x + y \vee (-y)\big] \vee \big[-x + (-y) \vee y\big] \\ &= \big[x + |y|\big] \vee \big[-x + |y|\big] \\ &= \big[x \vee (-x)\big] + |y| = |x| + |y| . \end{aligned}$$

(12) For the first identity in this part note that

$$\begin{aligned} \tfrac{1}{2}\big[|x + y| + |x - y|\big] &= \tfrac{1}{2}\big[(x + y) \vee (-x - y) + |x - y|\big] \\ &= \tfrac{1}{2}\big[\big(x + y + |x - y|\big) \vee \big(-x - y + |x - y|\big)\big] \\ &= \tfrac{1}{2}\Big(\big[2(x \vee y)\big] \vee \big[2\big\{(-x) \vee (-y)\big\}\big]\Big) \\ &= x \vee y \vee (-x) \vee (-y) \\ &= \big[x \vee (-x)\big] \vee \big[y \vee (-y)\big] \\ &= |x| \vee |y| . \end{aligned}$$

Using (10), (7), and (11) and the preceding identity, we also get

$$\begin{aligned} &\big||x + y| - |x - y|\big| \\ &\quad = |x + y| \vee |x - y| - |x + y| \wedge |x - y| \\ &\quad = |x + y| \vee |x - y| - \big(|x + y| + |x - y| - |x + y| \vee |x - y|\big) \\ &\quad = 2\big(|x + y| \vee |x - y|\big) - \big(|x + y| + |x - y|\big) \\ &\quad = 2\big(|x| + |y|\big) - 2\big(|x| \vee |y|\big) = 2\big(|x| \wedge |y|\big) , \end{aligned}$$

and the proof is finished. ■

The interpretation given in the proof of property (1) of Theorem 1.17 also yields the following simple (but very useful) result.

Corollary 1.18. *A cone $\mathcal{K}$ of a vector space X is a lattice cone for the vector subspace $\mathcal{K}-\mathcal{K}$ generated by $\mathcal{K}$ in X if and only if for any two vectors $x, y \in \mathcal{K}$ their supremum $x \vee y$ exists in the partially ordered set $\mathcal{K}$.*[8]

Moreover, if $x, y, z, u \in \mathcal{K}$, then

$$(x-y) \vee (z-u) = (x+u) \vee (z+y) - (y+u)$$

holds in $\mathcal{K}-\mathcal{K}$.

We can generalize Theorem 1.16 by presenting another useful geometrical characterization of the Archimedean lattice cones.

Theorem 1.19. *In an Archimedean ordered vector space L, if $L_+ \cap (x+L_+)$ is a cone with vertex y, then $y = x^+$ [and so $L_+ \cap (x+L_+) = x^+ + L_+$].*

In particular, an Archimedean ordered vector space L is a Riesz space if and only if for each $x \in L_+$ the set $L_+ \cap (x+L_+)$ is a translation of a cone.

Proof. Assume that $L_+ \cap (x+L_+) = y + \mathcal{K}$, where $\mathcal{K}$ is a cone of L. Since $y \in y+\mathcal{K}$, we see that $y \in L_+$ (i.e., $y \geq 0$) and $y \in x+L_+$ (i.e., $y \geq x$). That is, y is an upper bound of the set $\{0, x\}$. To see that y is the least upper bound of the set $\{0, x\}$, suppose that another vector $z \in L$ satisfies $z \geq x$ and $z \geq 0$.

From $z = x + (z-x) \in x + L_+$ and $z \in L_+$, we get $z \in y + \mathcal{K}$. This implies $z - y \in \mathcal{K}$ and so $n(z-y) \in \mathcal{K}$ for each $n \in \mathbb{N}$. But then from $y + n(z-y) \in y + \mathcal{K}$ we get $y + n(z-y) - x \geq 0$ or $\frac{1}{n}(y-x) + z - y \geq 0$ for all n. Since $y - x \in L_+$ and $z - y \in L_+ - L_+$, it follows from the Archimedean property of L_+ (see Corollary 1.13) that $z - y \geq 0$. That is, $z \geq y$ and consequently $y = x^+$ holds true in L.

If this property is true for each vector x, then L must be a Riesz space; see Exercise 2 at the end of the section. ■

Recall that an ordered vector space for which every increasing bounded from above net has a supremum is called **Dedekind** (or **order**) **complete**. Likewise, an ordered vector space for which every increasing bounded from above sequence has a supremum is called **Dedekind σ-complete** or simply **σ-order complete**.

The Dedekind complete and Dedekind σ-complete Riesz spaces enjoy the following important properties.

[8] It is not difficult to see that the supremum of x and y exists in $\mathcal{K}$ if and only if it exists in X, in which case the suprema coincide.

Theorem 1.20. *In a Dedekind complete Riesz space*

(a) *every nonempty bounded from above set has a supremum and*

(b) *every nonempty bounded from below set has an infimum.*

Similarly, in a Dedekind σ-complete Riesz space every nonempty countable bounded from above (resp. from below) set has a supremum (resp. an infimum).

Proof. We consider the Dedekind complete case only. Let D be a nonempty bounded from above subset of a Dedekind complete Riesz space. Then the set of all finite suprema of elements from D can be viewed as an increasing net that is bounded from above. It is not difficult to see that the supremum of this net is also the supremum of D. ■

A vector subspace M of a vector lattice L is said to be

(1) a **Riesz subspace** or a **vector sublattice** if for each $x, y \in M$ we have $x \vee y$ and $x \wedge y$ in M (where the supremum and infimum are taken in L) and

(2) an **ideal** if $|y| \leq |x|$ and $x \in M$ implies $y \in M$.

An ideal is always a vector sublattice but a vector sublattice need not be an ideal.

For every nonempty subset A of a vector lattice, the symbol $A^{\wedge}$ will denote the collection of all vectors that can be written as infima of finite subsets of A. That is, a vector a belongs to $A^{\wedge}$ if there exist vectors $a_1, a_2, \ldots, a_k \in A$ such that $a = \bigwedge_{i=1}^{k} a_i$. Similarly, $A^{\vee}$ is the set consisting of all suprema of the finite subsets of A. We write $A^{\vee\wedge}$ for $(A^{\vee})^{\wedge}$ and $A^{\wedge\vee}$ for $(A^{\wedge})^{\vee}$. It always turns out that $A^{\vee\wedge} = A^{\wedge\vee}$. This is a consequence of the distributive law (see Exercise 11 at the end of the section) of the lattice operations in vector lattices and is left as an exercise.

If A is a nonempty subset of a Riesz space E, then the collection of all Riesz subspaces of E that include A is nonempty. The intersection of this collection of Riesz subspaces is the smallest Riesz subspace that includes A. This Riesz subspace is called the **Riesz subspace** (or the **vector sublattice**) **generated by** A and will be denoted $\mathcal{R}(A)$. The next result presents a description of $\mathcal{R}(A)$. Its proof is straightforward and is left as an exercise.

Lemma 1.21. *The Riesz subspace $\mathcal{R}(Y)$ generated by a vector subspace Y of a Riesz space coincides with $Y^{\wedge\vee}$ and also with $Y^{\vee\wedge}$. That is,*

$$\mathcal{R}(Y) = Y^{\wedge\vee} = Y^{\vee\wedge}.$$

Consequently, the Riesz subspace generated by a nonempty subset A of a Riesz space is $\mathcal{R}(A) = [A]^{\wedge\vee}$, where $[A]$ denotes the linear span of A.

Another interesting class of vector subspaces of an ordered vector space is that of lattice-subspaces.

Definition 1.22. *A vector subspace M of an ordered vector space L is said to be a* ***lattice-subspace*** *if M under the induced ordering from L is a vector lattice in its own right.*

That is, M is a lattice-subspace if for every $x, y \in M$ the least upper bound and greatest lower bound of the set $\{x, y\}$ exist in M when ordered by the cone $M \cap L^+$.

Every vector sublattice of a vector lattice is automatically a lattice-subspace, but a lattice-subspace need not be a vector sublattice. For details about lattice-subspaces see [**6, 69, 78, 79**].

An operator $T\colon L \to M$ between two vector lattices is said to be a **lattice homomorphism** (or a **Riesz homomorphism**) whenever we have $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in L$. A one-to-one lattice homomorphism $T\colon L \to M$ between two vector lattices is called a **lattice isomorphism** (or a **Riesz isomorphism**) and the Riesz subspace $T(L)$ of M is called a **lattice copy** of L in M. We also say that a Riesz space L is **lattice embeddable** into another Riesz space M if there exists a lattice isomorphism $T\colon L \to M$ (in which case $T(L)$ is a lattice copy of L in M). Two Riesz spaces are said to be **lattice isomorphic** (or **Riesz isomorphic**) if there exists a surjective lattice isomorphism from one to another.

The next result is simple but very useful.

Lemma 1.23. *Let $T\colon L \to M$ be a surjective one-to-one linear operator. If M is a vector lattice and we define the vector ordering $\geq$ on L by letting $u \geq v$ whenever $Tu \geq Tv$, then L is a vector lattice and $T\colon L \to M$ is a lattice isomorphism.*

Corollary 1.24. *Every copy of a vector lattice L in a partially ordered vector space M is a lattice-subspace of M.*

Exercises

1. Show that the lexicographic cone is a lattice cone.

2. Show that an ordered vector space L is a Riesz space if and only if for each $x \in L$ the supremum $x^+ = x \vee 0$ exists in L. [*Hint*: Use property (1) of Theorem 1.17 to establish that $x \vee y = (x - y)^+ + y$.]

3. Prove Corollary 1.18.

4. Let $L = C^1[0, 1]$ denote the vector space of all continuously differentiable functions on $[0, 1]$ and endow L with the pointwise vector ordering.

(a) Show that L is an Archimedean ordered vector space with order units—and so the cone L_+ of L is generating.

(b) Prove that L is not a Riesz space.
(c) Give an example of two noncomparable functions f and g of L such that their supremum $f \vee g$ exists in L.

[*Hint*: For (b) show that the functions $\phi, \psi \in C^1[0,1]$, defined by $\phi(x) = x$ and $\psi(x) = 1 - x$, do not have a supremum in $C^1[0,1]$. For part (c) consider the functions $f(t) = (t - \frac{1}{2})^3$ and $g(t) = -(t - \frac{1}{2})^3$.]

5. For a Riesz space L show that the following statements are equivalent.
(a) The Riesz space L is Archimedean.
(b) The Riesz space L is almost Archimedean.
(c) For each $x \in L_+$ we have $\frac{1}{n}x \downarrow 0$.
(d) If $x, y \in L_+$ satisfy $ny \leq x$ for all n, then $y = 0$.

(For the definition of an almost Archimedean ordered vector space see Exercise 6 of Section 1.2.)

6. Show that every Dedekind σ-complete (and hence every Dedekind complete) Riesz space is Archimedean. [*Hint*: Use the preceding exercise.]

7. If $x + y = u + v$ holds true in a Riesz space, then show that

$$x \vee u + y \vee v = x \wedge u + y \wedge v = x + y\,.$$

8. Show that a vector subspace X of a Riesz space is a Riesz subspace if and only if for each $x, y \in X$ we have $x \wedge y \in X$ (and also if and only if $x, y \in X$ implies $x \vee y \in X$). [*Hint*: Use the lattice identities $x \vee y = -[(-x) \wedge (-y)]$ and $x \wedge y = -[(-x) \vee (-y)]$.]

9. Let $\mathcal{W}$ be a wedge of a Riesz space. Show that the vector subspace generated by $\mathcal{W}$ (i.e., $\mathcal{W} - \mathcal{W}$) is a Riesz subspace if and only if for each $w_1, w_2 \in \mathcal{W}$ there exists some $w_3 \in \mathcal{W}$ such that $w_3 + w_1 \wedge w_2 \in \mathcal{W}$.

In particular, infer from this that if $w_1, w_2 \in \mathcal{W}$ implies $w_1 \wedge w_2 \in \mathcal{W}$, then $\mathcal{W} - \mathcal{W}$ is a Riesz subspace of L.

[*Hint*: Note that $(x_1 - x_2) \wedge (y_1 - y_2) = (x_1 + y_2) \wedge (y_1 + x_2) - (x_2 + y_2)$.]

10. Let X be a vector subspace of a Riesz space L and let $X_+ = X \cap L_+$. Establish that X is a Riesz subspace of L if and only if the cone X_+
(a) generates X, i.e., $X = X_+ - X_+$ and
(b) is closed under finite infima, i.e, $x, y \in X_+$ implies $x \wedge y \in X_+$.

11. Show that a Riesz space (as a lattice) satisfies the "**infinite distributive laws**." That is, show that if $\sup A$ exists for a nonempty subset A of a Riesz space L, then $\sup\{x \wedge a\colon\ a \in A\}$ exists for each $x \in L$ and

$$x \wedge \sup A = \sup\big\{x \wedge a\colon\ a \in A\big\}\,.$$

Likewise, if $\inf A$ exists for a nonempty subset A of some Riesz space L, then show that $\inf\{x \vee a\colon\ a \in A\}$ exists for each $x \in L$ and

$$x \vee \inf A = \inf\big\{x \vee a\colon\ a \in A\big\}\,.$$

In particular, if $x, x_1, x_2, \ldots, x_n$ are vectors in a Riesz space, then show that

$$x \wedge \Big[\bigvee_{i=1}^{n} x_i\Big] = \bigvee_{i=1}^{n}(x \wedge x_i) \quad \text{and} \quad x \vee \Big[\bigwedge_{i=1}^{n} x_i\Big] = \bigwedge_{i=1}^{n}(x \vee x_i)\,.$$

12. If x, y, and z are arbitrary vectors in a Riesz space, then show that
(a) (**Birkhoff's identity**)

$$|x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x - y|\,,$$

(b) (**Birkhoff's inequalities** [**30**, p. 296])

$$|x \vee z - y \vee z| \leq |x - y| \quad \text{and} \quad |x \wedge z - y \wedge z| \leq |x - y|\,.$$

13. Establish the validity of the following **lattice inequalities** in Riesz spaces.
(a) If two vectors x and y in a Riesz space satisfy $x \leq y$, then

$$x^+ \leq y^+ \quad \text{and} \quad y^- \leq x^-\,.$$

(b) (**The triangle inequality**) If x and y are any two vectors in a Riesz space, then

$$\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|\,.$$

(c) If $x, x_1, x_2, \ldots, x_n$ are positive vectors in a Riesz space, then

$$x \wedge (x_1 + x_2 + \cdots + x_n) \leq x \wedge x_1 + x \wedge x_2 + \cdots + x \wedge x_n\,.$$

If $x \wedge x_i \wedge x_j = 0$ holds for all $i \neq j$, then equality holds.
(d) If $x_1, x_2, \ldots, x_n$ are vectors in a Riesz space, then

$$n\big(x_1^+ \wedge \cdots \wedge x_n^+\big) = n(x_1 \wedge \cdots \wedge x_n)^+ \leq (x_1 + \cdots + x_n)^+\,.$$

14. Two vectors x and y in a Riesz space are called **disjoint** or **orthogonal**, in symbols $x \perp y$, if $|x| \wedge |y| = 0$. Establish the following disjointness properties in Riesz spaces.
(a) If $x \perp y$ and $x \perp z$ hold in a Riesz space, then $x \perp (\lambda y + \mu z)$ for all $\lambda, \mu \in \mathbb{R}$.
(b) Two vectors x and y in an arbitrary Riesz space are disjoint if and only if $|x+y| = |x-y|$. [*Hint*: Use property (12) of Theorem 1.17 .]
(c) If $x \perp y$ holds in a Riesz space, then

$$|x + y| = |x - y| = |x| + |y| = \big||x| - |y|\big| = |x| \vee |y|\,.$$

[*Hint*: Use property (11) of Theorem 1.17 .]
(d) Every subset of a Riesz space consisting of pairwise disjoint nonzero vectors is linearly independent.

15. Prove Lemma 1.23 and Corollary 1.24.

16. Show that two vector lattices are lattice isomorphic if and only if they are order isomorphic.

17. Let Σ be a σ-algebra of subsets of a nonempty set Ω and let $\mathrm{ca}(\Sigma)$ be the vector space of all countably additive signed measures on Σ having finite total variation. Recall that a set function $\mu\colon \Sigma \to [-\infty, \infty]$ that takes only at most one of the values $-\infty$ and ∞ is called a **signed measure** if $\mu(\emptyset) = 0$ and is σ-**additive** (or **countably additive**) in the sense that

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$$

holds for each pairwise disjoint sequence $\{A_n\}$ of Σ. A signed measure that takes only nonnegative extended values is called a **measure**. The **total variation** of a signed measure μ is the nonnegative extended real number $V(\mu)$ defined by

$$V(\mu)=\sup\Big\{\sum_{i=1}^{n}|\mu(A_n)|\colon\ A_i\in\Sigma \text{ for all } i \text{ and } A_i\cap A_j=\emptyset \text{ for } i\neq j\Big\}.$$

The signed measure μ is said to have **finite total variation** if $V(\mu)<\infty$.

On $\mathrm{ca}(\Sigma)$ define the binary relation $\geq$ by letting $\mu\geq\nu$ if $\mu(A)\geq\nu(A)$ for all $A\in\Sigma$. Show that $\geq$ is a vector ordering on $\mathrm{ca}(\Sigma)$ and that $\mathrm{ca}(\Sigma)$ under $\geq$ is a vector lattice whose lattice operations for each $\mu,\nu\in\mathrm{ca}(\Sigma)$ and each $A\in\Sigma$ are given by

$$[\mu\vee\nu](A)=\sup\{\mu(B)+\nu(A\setminus B)\colon\ B\in\Sigma \text{ and } B\subseteq A\}$$

and

$$[\mu\wedge\nu](A)=\inf\{\mu(B)+\nu(A\setminus B)\colon\ B\in\Sigma \text{ and } B\subseteq A\}.$$

18. Consider the vectors subspace V of $C[0,1]$ consisting of all linear functions, i.e., a function $x\in C[0,1]$ belongs to V if and only if there exist real numbers b and m such that for each $t\in[0,1]$ we have $x(t)=mt+b$. Show that V is a lattice-subspace of $C[0,1]$ but not a Riesz space.

19. Prove Lemma 1.21.

20. Let L denote the vector subspace of $\mathbb{R}^{[0,1]}$ consisting of all continuous functions of bounded total variation and let

$$\mathcal{K}=\{f\in L\colon\ f(x)\geq 0 \text{ for all } x\in[0,1]\} \text{ and}$$

and

$$L_+=\{f\in\mathcal{K}\colon\ f \text{ is nondecreasing}\}.$$

Establish the following.

(a) $\mathcal{K}$ is a lattice cone. Describe also the lattice operations of the Riesz space $(L,\mathcal{K})$.
(b) The Riesz space $(L,\mathcal{K})$ is not Dedekind σ-complete.
(c) L_+ is a lattice subcone of $\mathcal{C}$.
(d) The Riesz space (L,L_+) is Dedekind complete. Also describe the lattice operations of this Riesz space.

[*Hint*: See for details and more [**16**, Section 39, p. 366] and [**12**, Section 9.8, p. 365].]

1.4. Positive and order bounded operators

Recall that a function $T\colon X \to Y$ between two vector spaces is said to be

(1) **additive** if for all $x, y \in X$ we have $T(x+y) = T(x) + T(y)$,

(2) **homogeneous** whenever for all $x \in X$ and all $\lambda \in \mathbb{R}$ we have $T(\lambda x) = \lambda T(x)$,

(3) **linear** if it is both additive and homogeneous.

A linear function $T\colon X \to Y$ between two vector spaces is referred to as an **operator**.[9] It is very common in operator theory to denote the vector $T(x)$ by Tx and occasionally we shall be doing so. The vector space of all (linear) operators from a vector space X to a vector space Y will be denoted $\mathcal{L}(X,Y)$.

We now introduce some classes of operators between ordered spaces that are related to the order structures of the spaces.

An operator $T\colon L \to M$ between two ordered vector spaces is called

(a) **positive**, in symbols $T \geq 0$, if $T(x) \geq 0$ holds for all $x \in L^+$,

(b) **strictly positive**, in symbols $T \gg 0$, if $T(x) > 0$ for all $x > 0$,

(c) **regular** if T can be written as a difference of two positive operators,

(d) **order bounded** if T carries order bounded subsets of L to order bounded subsets of M.

It should be clear that every strictly positive operator is a positive operator, every positive operator is regular and that every regular operator is order bounded. So, if for any two ordered vector spaces L and M we denote the vector space of all order bounded operators from L to M by $\mathcal{L}_{\mathrm{b}}(L,M)$ and the vector space of all regular operators from L to M by $\mathcal{L}_{\mathrm{r}}(L,M)$, then we have the vector subspace inclusions

$$\mathcal{L}_{\mathrm{r}}(L,M) \subseteq \mathcal{L}_{\mathrm{b}}(L,M) \subseteq \mathcal{L}(L,M)\,.$$

The inclusion $\mathcal{L}_{\mathrm{r}}(L,M) \subseteq \mathcal{L}_{\mathrm{b}}(L,M)$ can be proper; see Exercise 3 at the end of this section. Also notice that the collection of all positive operators from L to M is a wedge of $\mathcal{L}(L,M)$. In case L_+ is a generating cone, this wedge is of course a cone.

As usual, the general additive functions are defined as follows.

Definition 1.25. *A mapping $T\colon \mathcal{W}_1 \to \mathcal{W}_2$ between two wedges is said to be **additive** whenever $T(x+y) = T(x) + T(y)$ holds for all $x, y \in \mathcal{W}_1$.*

[9] More precisely, it should be called a *linear operator*. In this monograph the term "operator" is synonymous with "linear operator."

The following is a basic extension result dealing with additive functions between cones. The result informs us that quite often the additive mappings extend to positive operators.

Lemma 1.26. *Suppose that L and M are two ordered vector spaces with M Archimedean. For an additive mapping $T\colon L^+ \to M^+$ we have the following.*

(1) *The mapping T extends to a positive operator from L to M.*

(2) *Any extension of T to all of L is uniquely determined on $L_+ - L_+$, the vector subspace generated by L_+.*

(3) *If L_+ is generating, then T extends uniquely to a positive operator from L to M.*

Proof. Assume that $T\colon L^+ \to M^+$ is an additive mapping. The extension of T to $X = L_+ - L_+$ is achieved as follows. For each $x \in X$ pick $x_1, x_2 \in L_+$ with $x = x_1 - x_2$ and put

$$S(x) = T(x_1) - T(x_2)\,. \tag{$\star$}$$

Notice that if $x = y_1 - y_2$ with $y_1, y_2 \in L_+$, then $y_1 + x_2 = x_1 + y_2$, and so by the additivity of $T\colon L_+ \to M_+$ we get

$$T(y_1) + T(x_2) = T(y_1 + x_2) = T(x_1 + y_2) = T(x_1) + T(y_2)\,,$$

and so $T(y_1) - T(y_2) = T(x_1) - T(x_2)$. This shows that the mapping $S\colon X \to M$ as defined by $(\star)$ is well defined—and also that it is the only possible extension of $T\colon L_+ \to M_+$ to a positive operator on all of X. We shall show that S is linear. That is, we shall prove that S is additive and homogeneous on X.

For the additivity of S note that if $x = x_1 - x_2$ and $y = y_1 - y_2$ holds with the $x_1, x_2, y_1, y_2 \in L_+$, then we have

$$\begin{aligned} S(x+y) &= S\big((x_1 + y_1) - (x_2 + y_2)\big) \\ &= T(x_1 + y_1) - T(x_2 + y_2) \\ &= T(x_1) + T(y_1) - \big[T(x_2) + T(y_2)\big] \\ &= \big[T(x_1) - T(x_2)\big] + \big[T(y_1) - T(y_2)\big] \\ &= S(x) + S(y)\,. \end{aligned}$$

In particular, the additivity of S implies that S is $\mathbb{Q}$-homogeneous, i.e., $S(rx) = rS(x)$ holds for all $x \in X$ and all rational numbers r.

Next, we shall show that S is homogeneous. For this, we need to prove first that S is monotone. That is, $x \geq y$ in X implies $S(x) \geq S(y)$ in M. Indeed, if $x \geq y$ holds in X, then $x - y \in L^+$ and the additivity of S yields

$$S(x) = S\big((x-y)+y\big) = S(x-y) + S(y) = T(x-y) + S(y) \geq S(y)\,.$$

Now fix $x \in L^+$ and let $\lambda \in \mathbb{R}$. Pick two sequences of rational numbers $\{r_n\}$ and $\{t_n\}$ such that $r_n \uparrow \lambda$ and $t_n \downarrow \lambda$. The inequalities $r_n x \leq \lambda x \leq t_n x$ and the monotonicity of S coupled with the $\mathbb{Q}$-homogeneity of S imply

$$r_n S(x) = S(r_n x) \leq S(\lambda x) \leq S(t_n x) = t_n S(x)$$

for each n. Since $S(\lambda x) \in M_+ - M_+$, $S(x) = T(x) \geq 0$ and M is Archimedean, it follows from Corollary 1.13 that $\lambda S(x) = S(\lambda x)$. Finally, if $\lambda \in \mathbb{R}$ and $x \in X$, then write $x = x_1 - x_2$ with $x_1, x_2 \in L_+$ and note that

$$\begin{aligned} S(\lambda x) &= S(\lambda x_1 + (-\lambda)x_2) = S\big(\lambda x_1\big) + S\big((-\lambda)x_2\big) \\ &= \lambda S(x_1) - \lambda S(x_2) = \lambda\big[T(x_1) - T(x_2)\big] = \lambda S(x)\,. \end{aligned}$$

Consequently, S is also homogeneous, and so $S\colon X \to M$ is the unique extension of the additive mapping $T\colon L_+ \to M_+$ to a positive operator.

Next, let Y be any algebraic complement of X, i.e., let Y be any vector subspace of L such that $L = X \oplus Y$, and let $S_1\colon Y \to M$ be any operator. For instance, we can let $S_1(y) = 0$ for all $y \in Y$. Then, the operator $S_2\colon L \to M$ defined by $S_2(x \oplus y) = S(x) + S_1(y)$ is a positive linear extension of $T\colon L_+ \to M_+$ to all of L. Now the desired conclusions follow immediately from the above discussion. ■

The rest of the section is devoted to presenting several important results concerning extensions of positive operators. Our first extension result is a generalization of the classical Hahn–Banach theorem (due to H. Hahn [**50**] and S. Banach [**27**]) to the setting of ordered vector spaces. This theorem plays a fundamental role in modern analysis and without any doubt its generalization will be of great importance to us here.

In order to state and prove this result, we need to recall a few definitions. First, recall that a function $p\colon X \to M$, where X is a (real) vector space and M is an ordered vector space, is called **sublinear** whenever it is *subadditive* and *positively homogeneous*, i.e., if

(a) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and

(b) $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and all $\lambda \geq 0$.

Also, a function $f\colon C \to M$, where C is a convex subset of a vector space and M is an ordered vector space, is said to be **convex** if for all $x, y \in C$ and all $0 \leq \alpha \leq 1$ we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y)\,.$$

Clearly, every sublinear function is a convex function.

We are now ready to state and prove the most general version of the classical Hahn–Banach extension theorem.

Theorem 1.27 (Hahn–Banach)**.** *Let X be a (real) vector space, M a Dedekind complete Riesz space, and let $p\colon X \to M$ be a convex function—in particular, let p be sublinear. If Y is a vector subspace of X and $S\colon Y \to M$ is an operator satisfying $S(y) \leq p(y)$ for all $y \in Y$, then there exists some operator $T\colon X \to M$ such that*

(1) *$T = S$ on Y, i.e., T is a linear extension of S to all of X and*

(2) *$T(x) \leq p(x)$ holds for all $x \in X$.*

Proof. The critical step is to show that S has a linear extension satisfying (2) on an arbitrary vector subspace generated by Y and one extra vector. If this is done, then an application of Zorn's lemma guarantees the existence of an extension of S to all of X with the desired properties.

To see this, let $u \notin Y$, and let $V = \{y + \lambda u\colon \ y \in Y \text{ and } \lambda \in \mathbb{R}\}$. If $T\colon V \to M$ is a linear extension of S, then

$$T(y + \lambda u) = S(y) + \lambda T(u)$$

must hold true for all $y \in Y$ and all $\lambda \in \mathbb{R}$. Put $w = T(u)$. To complete the proof, we must establish the existence of some $w \in M$ such that

$$S(y) + \lambda w \leq p(y + \lambda u) \tag{1}$$

holds for all $y \in Y$ and $\lambda \in \mathbb{R}$. It is a routine matter to verify that (1) is true if and only if there exists some vector $w \in M$ satisfying

$$\tfrac{1}{\lambda}\big[S(x) - p(x - \lambda u)\big] \leq w \leq \tfrac{1}{\mu}\big[p(y + \mu u) - S(y)\big] \tag{2}$$

for all $x, y \in Y$ and all $\lambda, \mu > 0$. Now notice that (2) is true for some $w \in M$ if and only if

$$\sup_{x \in Y,\, \lambda > 0} \tfrac{1}{\lambda}\big[S(x) - p(x - \lambda u)\big] \leq \sup_{y \in Y,\, \mu > 0} \tfrac{1}{\mu}\big[p(y + \mu u) - S(y)\big],$$

which in turn is equivalent to

$$\tfrac{1}{\lambda}\big[S(x) - p(x - \lambda u)\big] \leq \tfrac{1}{\mu}\big[p(y + \mu u) - S(y)\big] \tag{3}$$

for all $x, y \in Y$ and all $\lambda, \mu > 0$. Rearranging terms, we see that (3) is equivalent to

$$S(\mu x + \lambda y) \leq \mu p(x - \lambda u) + \lambda p(y + \mu u) \tag{4}$$

for all $x, y \in Y$ and all $\lambda, \mu > 0$. Thus, a linear extension of S to all of X exists if and only if (4) is true. For the validity of (4) note that if $x, y \in Y$

and $\lambda, \mu > 0$, then using the convexity of p, we obtain

$$\begin{aligned} S(\mu x + \lambda y) &= (\lambda+\mu) S\big(\tfrac{\mu}{\lambda+\mu} x + \tfrac{\lambda}{\lambda+\mu} y\big) \\ &\leq (\lambda+\mu) p\big(\tfrac{\mu}{\lambda+\mu} x + \tfrac{\lambda}{\lambda+\mu} y\big) \\ &= (\lambda+\mu) p\big(\tfrac{\mu}{\lambda+\mu}[x-\lambda u] + \tfrac{\lambda}{\lambda+\mu}[y+\mu u]\big) \\ &\leq (\lambda+\mu)\big[\tfrac{\mu}{\lambda+\mu} p(x-\lambda u) + \tfrac{\lambda}{\lambda+\mu} p(y+\mu u)\big] \\ &= \mu p(x-\lambda u) + \lambda p(y+\mu u)\,, \end{aligned}$$

as desired. This complete the proof of the theorem. ■

Recall that a vector subspace G of a Riesz space E is said to be a **Riesz subspace** (or a **vector sublattice**) whenever G is closed under the lattice operations of E, i.e., whenever for each pair $x, y \in G$ the vector $x \vee y$ (taken in E) belongs to G.

As a first application of the Hahn–Banach extension theorem we present the following useful extension property of positive operators.

Theorem 1.28. *Let $T\colon L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. Assume also that G is a Riesz subspace of L and that $S\colon G \to M$ is an operator satisfying $0 \leq S(x) \leq T(x)$ for all $x \in G^+$. Then S can be extended to a positive operator from L to M such that $0 \leq S(x) \leq T(x)$ holds for all $x \in L^+$.*

Proof. Define $p\colon L \to M$ by $p(x) = T(x^+)$, and note that p is sublinear and satisfies $S(x) \leq p(x)$ for all $x \in G$. By Theorem 1.27 there exists a linear extension of S to all of L (which we denote by S again) satisfying $S(x) \leq p(x)$ for all $x \in L$. Now if $x \in L^+$, then

$$-S(x) = S(-x) \leq p(-x) = T\big((-x)^+\big) = T(0) = 0\,,$$

and so $0 \leq S(x) \leq p(x) = T(x)$ holds, as desired. ■

A subset A of a Riesz space is called **solid** whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. As defined earlier, a solid vector subspace of a Riesz space is referred to as an **ideal**. From the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, it follows immediately that every ideal is a Riesz subspace.

The next result deals with restrictions of positive operators to ideals.

Theorem 1.29. *If $T\colon L \to M$ is a positive operator between two Riesz spaces with M Dedekind complete, then for every ideal A of L the formula*

$$T_A(x) = \sup\big\{T(y)\colon\ y \in A \ \text{ and } \ 0 \leq y \leq x\big\}\,,\ \ x \in L^+\,,$$

defines a positive operator from L to M. Moreover, we have

(a) $0 \leq T_A \leq T$,

(b) $T_A = T$ *on* A,

(c) *if* B *is another ideal with* $A \subseteq B$, *then* $T_A \leq T_B$ *holds.*

Proof. Note first that

$$T_A(x) = \sup\{T(x \wedge y) \colon\ y \in A^+\}$$

holds for all $x \in L^+$. According to Lemma 1.26 it suffices to show that T_A is additive on L^+.

To this end, let $x, y \in L^+$. If $z \in A^+$, then the inequality

$$(x + y) \wedge z \leq x \wedge z + y \wedge z$$

implies that $T\big((x + y) \wedge z\big) \leq T(x \wedge z) + T(y \wedge z) \leq T_A(x) + T_A(y)$, and hence

$$T_A(x + y) \leq T_A(x) + T_A(y)\,.$$

On the other hand, the inequality $x \wedge u + y \wedge v \leq (x + y) \wedge (u + v)$ implies

$$T_A(x) + T_A(y) \leq T_A(x + y)\,.$$

Therefore, $T_A(x + y) = T_A(x) + T_A(y)$ holds, so that T_A is additive on L^+.

Properties (a), (b) and (c) are now easy consequences of the formula defining the operator T_A. ■

Recall that a vector subspace G of an ordered vector space L **majorizes** L whenever for each $x \in L$ there exists some $y \in G$ with $x \leq y$ (or, equivalently, if for each $x \in L$ there exists some $y \in G$ with $y \leq x$).

It is important to know that every positive operator whose domain is a majorizing vector subspace and whose values are in a Dedekind complete Riesz space always has a positive extension. This is a classical result due to L. V. Kantorovich [**57**].

Theorem 1.30 (Kantorovich)**.** *Let* L *and* M *be two ordered vector spaces with* M *a Dedekind complete Riesz space. If* G *is a majorizing vector subspace of* L *and* $T \colon G \to M$ *is a positive operator, then* T *has a positive linear extension to all of* L.

Proof. Fix $x \in L$ and let $y \in G$ satisfy $x \leq y$. Since G is majorizing, there exists a vector $u \in G$ with $u \leq x$. Hence, $u \leq y$ and the positivity of T implies $T(u) \leq T(y)$ for all $y \in G$ with $x \leq y$. In particular, it follows that $\inf\{T(y) \colon\ y \in G \text{ and } x \leq y\}$ exists in M for each $x \in L$. Thus, a mapping $p \colon L \to M$ can be defined via the formula

$$p(x) = \inf\{T(y) \colon\ y \in G \text{ and } x \leq y\}\,.$$

Clearly, $T(x) = p(x)$ holds for each $x \in G$ and an easy argument shows that p is also sublinear.

Now, by the Hahn–Banach Extension Theorem 1.27, the operator T has a linear extension S to all of L satisfying $S(z) \leq p(z)$ for each $z \in L$. If $z \in L^+$, then $-z \leq 0$, and so from

$$-S(z) = S(-z) \leq p(-z) \leq T(0) = 0\,,$$

we see that $S(z) \geq 0$. This shows that S is a positive linear extension of T to all of L. ■

We close the section with an interesting local approximation property of positive operators—which also shows that there is an abundance of positive operators between Riesz spaces.

Theorem 1.31. *Let $T\colon L \to M$ be a positive operator between two Riesz spaces with M Dedekind σ-complete. Then for each $x \in L^+$ there exists a positive operator $S\colon L \to M$ such that*

(1) $0 \leq S \leq T$,
(2) $S(x) = T(x)$,
(3) $S(y) = 0$ *for all* $y \in L$ *with* $|y| \wedge x = 0$.

Proof. Let $x \in L^+$ be fixed and define $S\colon L^+ \to M^+$ by

$$S(y) = \sup\bigl\{T(y \wedge nx)\colon\ n = 1, 2, \dots\bigr\}\,.$$

(The supremum exists since M is Dedekind σ-complete and the sequence $\{T(y \wedge nx)\}$ is bounded above in M by Ty.) We claim that S is additive.

To see this, let $y, z \in L^+$. From $(y+z) \wedge nx \leq y \wedge nx + z \wedge nx$ we get

$$T\bigl((y+z) \wedge nx\bigr) \leq T(y \wedge nx) + T(z \wedge nx) \leq S(y) + S(z)\,,$$

and so $S(y+z) \leq S(y) + S(z)$. On the other hand, for each m and n we have $y \wedge nx + z \wedge mx \leq (y+z) \wedge (n+m)x$, and thus

$$T(y \wedge nx) + T(z \wedge mx) \leq T\bigl((y+z) \wedge (n+m)x\bigr) \leq S(y+z)$$

holds for all n and m. This implies $S(y) + S(z) \leq S(y+z)$, and hence $S(y+z) = S(y) + S(z)$, so that S is additive.

By Lemma 1.26 the mapping S extends uniquely to all of L as a positive operator. Now it is a routine matter to verify that the operator S satisfies the desired properties. ■

Exercises

1. Let $T\colon L \to M$ be a surjective (onto) positive operator between two ordered vector spaces. Show that if e is an order unit of L, then $T(e)$ is an order unit of M.

2. Show that Lemma 1.26 is not true if M is not Archimedean. [*Hint*: Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be an additive function that is not linear, i.e., not of the form $\phi(x) = cx$, and let M be the lexicographic plane. Consider the mapping $T\colon \mathbb{R}^+ \to M^+$ defined by $T(x) = \big(x, \phi(x)\big)$ for each x in $\mathbb{R}^+$. Note that T is additive and that if T could be extended to an operator from $\mathbb{R}$ to M, then ϕ should be linear.]

3. This exercise presents an example (due to H. P. Lotz) of an order bounded operator that is not a regular operator—and consequently the inclusion $\mathcal{L}_{\mathrm{r}}(L,M) \subseteq \mathcal{L}_{\mathrm{b}}(L,M)$ can be proper.

Let $L = M = C[-1,1]$ and define the operator $T\colon L \to M$ for all $f \in L$ and all $-1 \le t \le 1$ by

$$[Tf](t) = \begin{cases} f(\sin\frac{1}{t}) - f(\sin(t+\frac{1}{t})) & \text{if} \quad 0 < |t| < 1, \\ 0 & \text{if} \quad t = 0\,. \end{cases}$$

Show that T defines an order bounded operator that fails to be regular. (For details about this problem see also [**15**, Example 1.16, p. 13].)

4. Let $L = L_p[0,1]$ with $0 < p < 1$ and let $G = L_1[0,1]$. Clearly, $G \subseteq L$ and G is an ideal of L. (Here $f \ge g$ means that $f(x) \ge g(x)$ holds for almost all x with respect to the Lebesgue measure.) Now consider the operator $T\colon G \to \mathbb{R}$ defined by

$$T(f) = \int_0^1 f(x)\,dx\,.$$

Show that T does not have a positive linear extension to all of L. [*Hint*: To see this, assume by way of contradiction that T is extendable to a positive operator from L to $\mathbb{R}$. In particular, this implies that if $f \in L$ is defined by $f(x) = \frac{1}{x}$, then the set of real numbers

$$D = \big\{T(g)\colon\ g \in G \ \text{ and } \ 0 \le g \le f\big\}$$

is bounded. On the other hand, if $g_n = f\chi_{(\frac{1}{n},1)}$, then $T(g_n) = \ln n \in D$ holds for each n. Therefore, D must be unbounded, a contradiction.]

5. A positive operator $T\colon G \to M$, where G is a vector subspace of an ordered vector space L and M is an ordered vector space, is said to have a **smallest extension** whenever there exists a positive extension S of T to all of L satisfying $S \le R$ for all other positive extensions R of T to all of L, in which case S is called the **smallest extension** of T.

Now let L and M be two Riesz spaces with M Dedekind complete, let A be an ideal of L, and let $T\colon A \to M$ be a positive operator. Show that if T has a positive extension to all of L, then T has a smallest extension. Moreover, if in this case S is the smallest extension, then

$$S(x) = \sup\big\{Ty\colon\ y \in A \ \text{ and } \ 0 \le y \le x\big\}$$

holds for all $x \in L^+$.

6. Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. For each $T \in \mathcal{L}_{\mathrm{b}}(L,M)$ let $\Pi(T)$ denote the restriction

of T to A. Show that the positive operator $\Pi\colon \mathcal{L}_{\mathrm{b}}(L,M) \to \mathcal{L}_{\mathrm{b}}(A,M)$ satisfies

$$\Pi(S \vee T) = \Pi(S) \vee \Pi(T) \quad \text{and} \quad \Pi(S \wedge T) = \Pi(S) \wedge \Pi(T)$$

for all $S, T \in \mathcal{L}_{\mathrm{b}}(L,M)$.[10] (In other words, Π is a lattice homomorphism.)

1.5. Positive linear functionals

The **algebraic dual** of a vector space X (i.e., the vector space of all linear functionals on X) will be denoted by X^*. That is, a function $f\colon X \to \mathbb{R}$ belongs to X^* if and only if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ holds for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$. A linear functional $f \in L^*$, where L is now an ordered vector space, is said to be

(1) **positive**, denoted $f \geq 0$, if $f(x) \geq 0$ holds for all $x \in L^+$,[11]

(2) **strictly positive**, denoted $f \gg 0$, if $f(x) > 0$ holds for all $x > 0$,[12]

(3) **regular** if f can be written as a difference of two positive linear functionals,

(4) **order bounded** if f carries order intervals of L to bounded subsets of $\mathbb{R}$.

It should be clear that

(a) every strictly positive linear functional is positive,

(b) every positive linear functional is regular,

(c) every positive functional (and hence every regular linear functional) is order bounded.

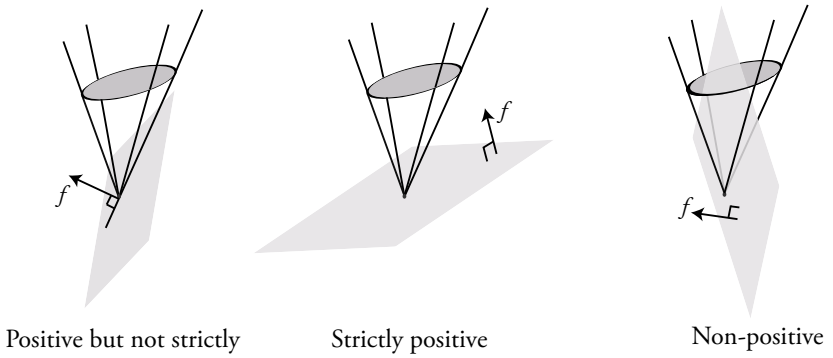
The geometrical meanings of positive, strictly positive and nonpositive linear functional are as follows. For any linear functional f define the half spaces $H = \{x\colon f(x) \geq 0\}$ and $G = \{x\colon f(x) > 0\}$. Clearly, f is positive if and only if $L^+ \subseteq H$. It is strictly positive if and only if $L^+ \subseteq G$. Notice that for any vector z we have $H_z = z + H = \{x\colon f(x) \geq f(z)\}$ and, of course, $G_z = z + G = \{x\colon f(x) > f(z)\}$. Therefore, f is positive if and only if there

[10] An operator between spaces of operators is referred to as a **transformer**. So, the operator Π is an example of a transformer.

[11] Recalling that a set A of a vector space is **supported** at a point $a \in A$ by a linear functional f (called a **supporting functional** of A at a) if f attains its minimum value over A at a, i.e., $f(x) \geq f(a)$ holds for all $x \in A$, we see that the positive linear functionals are precisely the linear functionals that support the cone at zero.

[12] Recalling that an extreme point a of a convex subset A of a vector space is called an **exposed point** of A if there exists a linear functional f (called a linear functional that **exposes** A at a) if $f(x) > f(a)$ holds for all $x \in A \setminus \{a\}$, we see that the strictly positive linear functionals are precisely the linear functionals that expose the cone at zero.

exists some z such that $z + L^+ \subseteq H_z$ and it is strictly positive if and only if there exists some z such that $z + L^+ \subseteq G_z$.



The reader should also notice immediately that we can use the same positivity notions for wedges as for cones. If $\mathcal{K}$ is a cone (or a wedge) in a vector space X, then a linear functional $f \in X^*$ is $\mathcal{K}$-**positive** whenever $x \in \mathcal{K}$ implies $f(x) \geq 0$. The notions of $\mathcal{K}$-strictly positive, $\mathcal{K}$-regular, and $\mathcal{K}$-order bounded linear functionals are defined analogously. The geometrical meanings of positive, strictly positive and nonpositive linear functional are shown in the figure above.

Our first result in this section is a characterization of the positive linear functionals. But first, we need a definition. A nonempty subset A of an ordered vector space L is said to be **comprehensive from above** (or simply **comprehensive**) if $x \leq y$ with $x \in A$ implies $y \in A$. Equivalently, a nonempty subset A is comprehensive if $A + L_+ = A$.

Lemma 1.32. *A linear functional on an ordered vector space is positive if and only if it is bounded below on a comprehensive set.*

In particular, every linear functional that supports a comprehensive subset of an ordered vector space is positive.

Proof. Let $f\colon L \to \mathbb{R}$ be a linear functional on an ordered vector space. If f is positive, then f is bounded below from 0 on the comprehensive set L_+. For the converse, assume that there exists a constant $c \in \mathbb{R}$ such that $f(a) \geq c$ holds true for all a in a comprehensive set A. Fix some $a \in A$ and let $x \in L_+$. Then $a + \lambda x \in A$ for all $\lambda > 0$ and so from our condition we get $f(x) \geq \frac{c - f(a)}{\lambda}$ for all $\lambda > 0$. Letting $\lambda \to \infty$ yields $f(x) \geq 0$ for all $x \in L_+$, i.e., f is a positive linear functional. ■

We now introduce the order and regular duals of an ordered vector space.

Definition 1.33. *For a given ordered vector space L, the vector spaces*

$$L^\sim := \mathcal{L}_{\mathrm{b}}(L, \mathbb{R}) \quad \text{and} \quad L^r := \mathcal{L}_{\mathrm{r}}(L, \mathbb{R})$$

are called the ***order dual*** *and the* ***regular dual****, respectively, of* L*.*

In other words, the order dual $L^{\sim}$ is the vector space consisting of all order bounded linear functionals on L and L^r is the vector space consisting of all regular linear functionals on L. Clearly, we have the following vector subspace inclusions: $L^r \subseteq L^{\sim} \subseteq L^*$.

The proof of the next result is straightforward and is omitted.

Lemma 1.34. *If* L *is an ordered vector space, then the collection of all positive linear functionals on* L *is a wedge in* L^* *(and also in* L^r *and* $L^{\sim}$*).*

When the cone L_+ *is generating, the wedge of all positive linear functionals is a cone.*

We shall close the section with two results regarding the extension of positive linear functionals. The first one (due to H. Bauer [**28**] and I. Namioka [**71**]) presents a necessary and sufficient condition for a positive linear functional defined on a vector subspace to have a positive linear extension to the whole space.

Theorem 1.35 (Bauer–Namioka)**.** *Let* X *be a vector subspace of an ordered vector space* L *and let* $f\colon X \to \mathbb{R}$ *be a positive linear functional, that is,* $0 \leq x \in X$ *implies* $f(x) \geq 0$*. Then* f *has a positive linear extension to all of* L *if and only if there is a nonempty convex and absorbing*[13] *subset* A *of* L *such that*

$$x \in X \cap (A - L_+) \implies f(x) \leq 1\,.$$

Proof. If f has a positive linear extension, say g, to all of L, then the set $A = \{x \in L\colon\ g(x) \leq 1\}$ is nonempty convex and absorbing and satisfies (as can be easily checked) the desired property.

For the converse, assume that there exists a nonempty convex and absorbing subset A of L such that if $x \in X \cap (A - L_+)$, then $f(x) \leq 1$. From $A \subseteq A - L_+$, it follows that the nonempty convex set $A - L_+$ is also absorbing. So, if we let $p\colon L \to \mathbb{R}$ be the Minkowski functional of $A - L_+$, i.e., for each $x \in L$ we have

$$p(x) = \inf\{\lambda > 0\colon\ x \in \lambda(A - L_+)\}\,,$$

then p is a sublinear function. Now notice that if $x \in X$ and some $\lambda > 0$ satisfies $x \in \lambda(A - L_+)$, then $\frac{x}{\lambda} \in X \cap (A - L_+)$ and so $\frac{f(x)}{\lambda} \leq 1$ or $f(x) \leq \lambda$. This implies that $f(x) \leq p(x)$ for all $x \in X$. By the Hahn–Banach extension theorem there exists a linear extension $h\colon L \to \mathbb{R}$ of f satisfying $h(x) \leq p(x)$ for all $x \in L$.

[13] Recall that a subset S of a vector space V is said to be **absorbing** if for every $u \in V$, there exists some $\lambda_0 > 0$ such that $\lambda u \in S$ holds true for all λ with $|\lambda| \leq \lambda_0$.

To complete the proof, we shall show that h is positive. To see this, fix $x \in L_+$. Then for each $\alpha > 0$ we have $-\alpha x \in -L_+ \subseteq A - L_+$ and so $\alpha p(-x) = p(-\alpha x) \leq 1$. It follows that $-\alpha h(x) = h(-\alpha x) \leq p(-\alpha x) \leq 1$ and thus $-h(x) \leq \frac{1}{\alpha}$ for all $\alpha > 0$. Letting $\alpha \to \infty$, we get $-h(x) \leq 0$ or $h(x) \geq 0$. This shows that h is a positive linear functional. ■

Our second extension theorem for positive linear functionals is the special case of Theorem 1.30 (when $M = \mathbb{R}$). We state it as a theorem simply because of its wide range of applications.

Theorem 1.36 (Kantorovich)**.** *If X is a majorizing vector subspace of an ordered vector space L and $f\colon X \to \mathbb{R}$ is a positive linear functional, then f has a positive linear extension to all of L.*

Exercises

1. Show that an arbitrary intersection of a family of comprehensive from above (resp. from below) subsets of an ordered vector space is a comprehensive from above (resp. from below) set.

2. If L is an ordered vector space with an order unit u, then prove that the order bounded linear functionals on L are precisely the linear functionals on L that carry the order interval $[0, u]$ to bounded subsets of $\mathbb{R}$, i.e.,

$$L^{\sim} = \left\{ f \in L^* \colon \ f([0,u]) \text{ is a bounded subset of } \mathbb{R} \right\}.$$

3. Assume that a Riesz space L admits a strictly positive linear functional. Show that if a nonempty subset A of L has a supremum, then there exists an at most countable subset B of A such that $\sup A = \sup B$.[14] [*Hint*: Fix a strictly positive linear functional f on L and assume that $x_\alpha \downarrow 0$ holds in L. Let $f(x_\alpha) \downarrow s$ in $\mathbb{R}$ and then pick an increasing sequence $\{\alpha_n\}$ of indices such that $f(x_{\alpha_n}) \downarrow s$. Now verify that if $x \leq x_{\alpha_n}$ holds for each n, then $x \leq x_\alpha$ for each α and deduce that $x \leq 0$. This shows that $x_{\alpha_n} \downarrow 0$. The general case can be reduced to this by replacing A by the collection of all finite suprema of A.]

4. Let L be an ordered vector space. Show that the collections $L^{\sim}$ (of all order bounded linear functionals on L) and L^r (of all regular linear functionals on L) are both vector subspaces of L and that $L^r \subseteq L^{\sim}$.

5. Prove Lemma 1.34.

6. If L is an ordered vector space and $f \in L^{\sim}$, then for each $x \in L_+$ show that

$$\sup f([0,x]) + \inf f([0,x]) = f(x).$$

[14] An ordered vector space where for every nonempty subset A that has supremum there exists an at most countable subset B of A with $\sup B = \sup A$ is known as an ordered vector space with the **countable sup property**. So, with this terminology, the result in this exercise states that: *every Riesz space that admits a strictly positive linear functional has the countable sup property.*

7. Let $\mathcal{K}$ be a cone of a vector space X and let $Y = \mathcal{K} - \mathcal{K}$, the linear span of $\mathcal{K}$. Show that a linear functional $f \in X^*$ is $\mathcal{K}$-order bounded if and only if its restriction to Y is $\mathcal{K}$-order bounded.

8. Let $L = B[0,1]$, the ordered vector space of all (uniformly) bounded real-valued functions on $[0,1]$, and let $X = C[0,1]$, the ordered vector space of all real-valued continuous functions on $[0,1]$. Establish the following.
   (a) X majorizes L.
   (b) If $f\colon X \to \mathbb{R}$ is the positive linear functional defined via the formula $f(x) = \int_0^1 x(t)\,dt$ (the Riemann integral), then show that f has a positive linear extension to all of L.

9. Let X denote the vector subspace of $L = \ell_\infty$ consisting of all convergent (real) sequences and consider the positive linear functional $f\colon X \to \mathbb{R}$ defined by $f(x_1, x_2, \ldots) = \lim_{n\to\infty} x_n$. Show that f has a positive linear extension to all of L.

10. Let L be the collection of all (uniformly) bounded real functions on $\mathbb{R}$ that vanish outside closed intervals. That is, a function $x\colon \mathbb{R} \to \mathbb{R}$ belongs to L if and only if there exist some $M > 0$ and an interval $[a,b]$ (both depending on x) such that $|x(t)| \leq M$ for all $t \in \mathbb{R}$ and $x(t) = 0$ for each $t \notin [a,b]$. Establish the following.
   (a) The collection L under the usual pointwise algebraic operations and the pointwise ordering is a Riesz space—and so its cone is generating.
   (b) The Riesz space L has no order units.
   (c) The vector subspace X of L consisting of all continuous real functions on $\mathbb{R}$ with compact support (i.e., vanishing off closed intervals) majorizes L.
   (d) The function $f\colon X \to \mathbb{R}$, defined by $f(x) = \int_{-\infty}^{\infty} x(t)\,dt$, is a strictly positive linear functional on X. (See also Exercise 4 of Section 2.5.)
   (e) The strictly positive linear functional f has a positive linear extension to all of L.

11. Show that every ordered vector space with an order unit admits a nonzero positive linear functional. [*Hint*: Let L be an ordered vector space having an order unit e, let $X = \{\lambda e\colon\ \lambda \in \mathbb{R}\}$ and consider the nonzero positive linear functional $f\colon X \to \mathbb{R}$ defined by $f(\lambda e) = \lambda$. Now note that X majorizes L. Alternatively, notice first that e is an internal point of the convex set $L_+ \setminus \{0\}$ and then use the Separation Theorem 8.15 to separate by a nonzero linear functional the disjoint convex sets $\{0\}$ and $L_+ \setminus \{0\}$.]

12. Assume that X is a majorizing vector subspace of an ordered vector space L and let $f\colon X \to \mathbb{R}$ be a positive linear functional. Give an alternative proof of Kantorovich's Extension Theorem 1.36 by using Theorem 1.35. [*Hint*: Let $f\colon X \to \mathbb{R}$ be a positive linear functional. Consider the nonempty convex set $B = \{x \in X\colon\ f(x) \leq 1\}$ and let $A = B - L_+$. Clearly, A is convex and we claim that A is also absorbing. To see this, let $x \in L$. Since X majorizes L, there exists some $y \in X$ such that $y \geq x$. So, if we let $u = y - x \in L_+$, then $x = y - u$. Next, choose some $\alpha_0 > 0$ such that $\alpha f(y) = f(\alpha y) \leq 1$ (or $\alpha y \in B$) holds for all $0 \leq \alpha \leq \alpha_0$ and notice that for any such α we have $\alpha x = \alpha y - \alpha u \in B - L_+$ proving that A

is also absorbing. Now notice that $A - L_+ = B - L_+$ and that for each $x \in X \cap (A - L_+)$ we have $f(x) \leq 1$, and then invoke Theorem 1.35.]

1.6. Faces and extremal vectors of cones

We start our discussion with the notion of a face of a convex set.

Definition 1.37. *A nonempty convex subset F of a convex set C is called a* ***face*** *if $\alpha x + (1-\alpha)y \in F$ with $x, y \in C$ and $0 < \alpha < 1$ imply $x, y \in F$.*

The reader should verify easily that a convex subset F of a convex set C is a face of C if and only if it follows from $\frac{1}{2}x + \frac{1}{2}y \in F$ with $x, y \in C$ that $x, y \in F$.

Some of the basic properties of faces are incorporated in the next result. Its straightforward proof is left for the reader.

Lemma 1.38. *For a nonempty convex subset C of a vector space X we have the following.*

(1) *A face of a face of C is itself a face of C.*

(2) *A nonempty intersection of a family of faces of C is a face of C.*

(3) *If $f\colon X \to \mathbb{R}$ is a nonzero linear functional that attains a minimum (or a maximum) value over the convex set C, say m, then the set $F = \{x \in C\colon\ f(x) = m\}$ is a face of C.*

Recall that a point e of a convex set C is called an **extreme point** of C if $e = \alpha x + (1-\alpha)y$ with $x, y \in C$ and $0 < \alpha < 1$ imply $x = y = e$.[15] Clearly, a point e of a convex set C is an extreme point of C if and only if the singleton $\{e\}$ is a face of C.

The following simple (but very useful result) should be obvious.

Lemma 1.39. *Every extreme point of any face of a convex set C is an extreme point of C.*

Every cone has only one extreme point—namely the zero vector.

Lemma 1.40. *If $\mathcal{K}$ is a cone of a vector space, then zero is the only extreme point of $\mathcal{K}$.*

Proof. If x is a nonzero vector of $\mathcal{K}$, then the identity $x = \frac{1}{2}0 + \frac{1}{2}(2x)$ shows that x cannot be an extreme point of $\mathcal{K}$. On the other hand, if $0 = \lambda x + (1-\lambda)y$ with $0 < \lambda < 1$ and $x, y \in \mathcal{K}$, then it is easy to see that $x = y = 0$. That is, 0 is the one and only extreme point of $\mathcal{K}$. ■

[15] It is not difficult to see that a point e of a convex set C is an extreme point of C if and only if the set $C \setminus \{e\}$ is likewise a convex set.

The faces of a cone are very special sets.

Lemma 1.41. *Every face of a cone $\mathcal{K}$ is a subcone*[16] *of $\mathcal{K}$.*

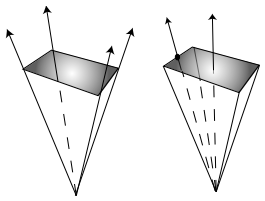
Proof. Let F be a face of a cone $\mathcal{K}$ and let $x \in F$. If $\lambda \geq 1$, then from the convex combination $x = \frac{1}{\lambda}(\lambda x) + (1 - \frac{1}{\lambda})0$ and $\lambda x, 0 \in \mathcal{K}$, it follows that $0 \in F$ and $\lambda x \in F$ for all $\lambda \geq 1$. On the other hand, for $0 < \lambda < 1$ the convexity of F implies $\lambda x = \lambda x + (1 - \lambda)0 \in F$. Thus, $\lambda x \in F$ holds for all $x \in F$ and all $\lambda \geq 0$.

The preceding conclusion in conjunction with the convexity of F shows that F is a wedge. Now use the fact that F is a subset of the cone $\mathcal{K}$ to conclude that F is indeed a subcone of $\mathcal{K}$. ■

Among the special faces of a cone are its extremal rays and they will be discussed next. Recall that if x is a vector in a linear space, then the set $\mathcal{A}_x = \{\alpha x\colon\ \alpha \geq 0\}$ is called the **half-ray** generated by x.

Definition 1.42. *Let $\mathcal{K}$ be a cone of a vector space X. A nonzero vector $e \in \mathcal{K}$ is said to be an* ***extremal vector*** *(or a* ***discrete vector****) of the cone $\mathcal{K}$ if $0 \leq x \leq e$ implies that $x = \lambda e$ for some $\lambda \geq 0$. In this case, the half-ray $\mathcal{A}_e$ is called an* ***extremal ray*** *of the cone $\mathcal{K}$.*

The geometrical meaning of the extremal rays can be seen in the companion figure, where four extreme rays are shown for the cone in the left figure and then two rays that are not extreme in the right figure. It should be obvious that if e is an extremal vector of a cone, then so are the vectors λe for $\lambda > 0$. It turns out that the extremal vectors of a cone are precisely the ones whose half-rays are faces of the cone.



This and another important "algebraic" characterization of the extremal vectors are included in the next result.

Lemma 1.43. *For a nonzero vector e of a cone $\mathcal{K}$ the following statements are equivalent.*

(1) *The vector e is an extremal vector of the cone $\mathcal{K}$.*

(2) *If two vectors $x_1, x_2 \in \mathcal{K}$ satisfy $e = x_1 + x_2$, then the vectors x_1 and x_2 are linearly dependent.*

(3) *The half-ray $\mathcal{A}_e$ is a face of the cone $\mathcal{K}$.*

Proof. (1) $\Longrightarrow$ (2) Assume that two nonzero vectors $x_1, x_2 \in \mathcal{K}$ satisfy $e = x_1 + x_2$. From $0 < x_1 < e$ and $0 < x_2 < e$ (where $\leq$ denotes the vector

[16] As usual, a cone $\mathcal{K}_1$ is called a **subcone** of another cone $\mathcal{K}$ if $\mathcal{K}_1 \subseteq \mathcal{K}$.

ordering induced by the cone $\mathcal{K}$), it follows that there exist $\lambda > 0$ and $\mu > 0$ such that $x_1 = \lambda e$ and $x_2 = \mu e$. This implies $\mu x_1 - \lambda x_2 = 0$ so that x_1 and x_2 are linearly dependent.

(2) $\Longrightarrow$ (3) Assume that two vectors $x_1, x_2 \in \mathcal{K}$ and some $0 < \alpha < 1$ satisfy $\alpha x_1 + (1-\alpha)x_2 \in \mathcal{A}_e$, that is, there exists some $\lambda \geq 0$ such that $\alpha x_1 + (1-\alpha)x_2 = \lambda e$. If $\lambda = 0$, then $x_1 = x_2 = 0 \in \mathcal{A}_e$. Thus, we can suppose that $\lambda > 0$, and so $e = \frac{\alpha}{\lambda} x_1 + \frac{1-\alpha}{\lambda} x_2$.

From our hypothesis the vectors x_1 and x_2 are linearly dependent. We can assume that $x_2 \neq 0$ and that $x_1 = \mu x_2$ for some $\mu \geq 0$. Therefore, $x_2 = \frac{\lambda}{1-\alpha+\alpha\mu} e$ and $x_1 = \frac{\lambda\mu}{1-\alpha+\alpha\mu} e$ so that $\mathcal{A}_e$ is a face of $\mathcal{K}$.

(3) $\Longrightarrow$ (1) Assume that $0 < x < e$, i.e., $x > 0$ and $e - x > 0$. From $\frac{1}{2}x + \frac{1}{2}(e-x) = \frac{1}{2}e \in \mathcal{A}_e$ and our hypothesis, it follows that $x \in \mathcal{A}_e$. Consequently, for some $\lambda \geq 0$ we have $x = \lambda e$. ■

For the characterization of the extremal vectors of a polyhedral cone see Theorem 3.34. We shall close the section with a few more properties of extremal vectors.

Lemma 1.44. *Any three nonzero extremal vectors of a cone in a vector space that generate three distinct extremal rays are linearly independent.*

Proof. Let x_1, x_2, and x_3 be three nonzero extremal vectors of a cone $\mathcal{K}$ that generate three distinct extremal rays. Assume by way of contradiction that the vectors x_1, x_2, x_3 are linearly dependent.

It is easy to see (by relabeling the vectors if necessary) that there exist positive scalars λ and μ such that $x_1 = \lambda x_2 + \mu x_3$. But then a glance at Lemma 1.43 reveals that x_2 and x_3 must be linearly dependent from which it follows that $\mathcal{A}_{x_2} = \mathcal{A}_{x_3}$, a contradiction. Consequently, the vectors x_1, x_2, x_3 are linearly independent. ■

The preceding conclusion is not true for four or more extremal vectors. In other words, we have the following.

- *If $\mathcal{A}_{x_1}, \mathcal{A}_{x_2}, \ldots, \mathcal{A}_{x_k}$ are distinct extremal rays of a cone in a finite dimensional vector space X with $\dim X \geq k \geq 4$, then the vectors $x_1, x_2, \ldots, x_k$ need not be linearly independent.*

For an example, see Exercise 8 at the end of the section.

However, in spite of the above negative conclusion, any set of extremal vectors of a lattice cone that generates distinct extremal rays is linearly independent.

Theorem 1.45. *If $\mathcal{A}_{x_1}, \ldots, \mathcal{A}_{x_k}$ are nonzero distinct extremal rays of a lattice cone, then the vectors $x_1 \ldots, x_k$ are linearly independent.*

Proof. To see this, note that for $i \neq j$ we have $x_i \wedge x_j = 0$. Indeed, from $0 \leq x_i \wedge x_j \leq x_i$ and $0 \leq x_i \wedge x_j \leq x_j$ we get $x_i \wedge x_j = \lambda x_i$ and $x_i \wedge x_j = \mu x_j$, and so if $x_i \wedge x_j > 0$, then $\mathcal{A}_{x_i} = \mathcal{A}_{x_j}$, a contradiction. To complete the proof, note that in a vector lattice any collection of pairwise disjoint vectors is linearly independent; see part (d) of Exercise 14 of Section 1.3. ■

For a complete version of this result see Theorem 3.21.

Exercises

1. Let S be a nonempty subset of a vector space and let $\operatorname{co} S$ be the convex hull of S. Show that every extreme point of $\operatorname{co} S$ belongs to S.

2. Prove Lemmas 1.38 and 1.39.

3. Let $\mathcal{K}$ be a cone of a vector space X. Assume that there exists a positive linear functional $f \in X^*$ such that $\mathcal{K} \cap \operatorname{Ker} f = \mathcal{A}_e$, where $e > 0$. Show that e is an extremal vector of the cone $\mathcal{K}$.

4. Show that a nonzero vector e of a cone $\mathcal{K}$ is an extremal vector of the cone $\mathcal{K}$ if and only if $\mathcal{K} \setminus \mathcal{A}_e$ is a convex set.

5. Show that the standard cone of $C[0,1]$ does not have any extremal rays.

6. Find the extremal rays of the standard cone of $\mathbb{R}^\Omega$.

7. Find all extremal vectors of the $L_p(\mu)$-spaces (ordered by $f \geq g$ whenever $f(\omega) \geq g(\omega)$ holds for μ-almost all ω) for any $0 \leq p \leq \infty$. [*Hint*: The extremal vectors are the positive multiples of the functions χ_A, where A is an atom. (Recall that an **atom** is any measurable set A of positive measure such that whenever B is a measurable set satisfying $B \subseteq A$, then either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$.)]

8. Give an example of four distinct extremal vectors of a cone in $\mathbb{R}^4$ that are not linearly independent. [*Hint*: Consider the cone $\mathcal{K}$ of $\mathbb{R}^4$ given by $\mathcal{K} = \{(\lambda_1 + \lambda_4, \lambda_2 - \lambda_4, \lambda_3 + \lambda_4, 0)\colon\ \lambda_i \geq 0 \text{ for each } i\}$ and note that the four linearly dependent vectors $x_1 = (1,0,0,0)$, $x_2 = (0,1,0,0)$, $x_3 = (0,0,1,0)$, and $x_4 = (1,-1,1,0)$ are (up to positive scalar multiples) the only extremal vectors of $\mathcal{K}$.]

9. Does the infimum of two extremal vectors exist? [*Hint*: If two essentially distinct extremal vectors u and v have an infimum, then $u \wedge v = 0$.]

1.7. Cone bases

In this section, we shall discuss some special convex subsets of a cone known as *bases*. We start with their definition.

Definition 1.46. *Let $\mathcal{K}$ be a cone in a vector space. A nonempty convex subset $\mathcal{B}$ of $\mathcal{K} \setminus \{0\}$ is said to be a* ***base*** *for the cone $\mathcal{K}$ if for each $x \in \mathcal{K} \setminus \{0\}$ there exist $\lambda > 0$ and $b \in \mathcal{B}$ both uniquely determined such that $x = \lambda b$.*

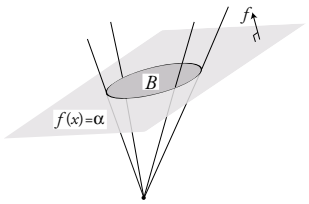
The existence of a base is associated with the existence of strictly positive linear functionals.

Theorem 1.47. *A cone $\mathcal{K}$ of a vector space X has a base if and only if X admits a $\mathcal{K}$-strictly positive linear functional. Moreover, if $f\colon X \to \mathbb{R}$ is any $\mathcal{K}$-strictly positive linear functional, then for each fixed $\alpha > 0$ the convex set*

$$\mathcal{B} = \big\{x \in \mathcal{K}\colon\ f(x) = \alpha\big\}$$

is a base for the cone $\mathcal{K}$.

The intuition for this result is shown in the figure below.



Proof. Let $\mathcal{B}$ be a base for a cone $\mathcal{K}$. Define the function $f\colon \mathcal{K} \to [0,\infty)$ by $f(0) = 0$ and for $x > 0$ let $f(x)$ be the unique positive real number for which there exists the unique vector $b \in \mathcal{B}$ with $x = f(x)b$. Clearly, f is strictly positive on $\mathcal{K}$.

To see that f is additive, let $x, y \in \mathcal{K} \setminus \{0\}$ and let $b, b_1 \in \mathcal{B}$ satisfy $x = f(x)b$ and $y = f(y)b_1$. Then the vector $b_2 = \frac{f(x)}{f(x)+f(y)}b + \frac{f(y)}{f(x)+f(y)}b_1 \in \mathcal{B}$ satisfies $x + y = [f(x) + f(y)]b_2$, and so by the definition of the base we get $f(x + y) = f(x) + f(y)$.

Therefore, f is also additive. By Lemma 1.26, the linear functional f extends to a $\mathcal{K}$-strictly positive linear functional on all of X.

For the converse, assume that $f\colon X \to \mathbb{R}$ is a $\mathcal{K}$-strictly positive linear functional. Fix $\alpha > 0$ and put $\mathcal{B} = \big\{x \in \mathcal{K}\colon\ f(x) = \alpha\big\}$. We claim that $\mathcal{B}$ is a base for $\mathcal{K}$. To see this, note first that if $x >_{\mathcal{K}} 0$, then $b = \frac{\alpha}{f(x)}x \in \mathcal{B}$ and $x = \frac{f(x)}{\alpha}b$ holds true. Now assume that $x = \lambda_1 b_1 = \lambda_2 b_2$, with $\lambda_i > 0$ and $b_i \in \mathcal{B}$ for each i. Then we have

$$\lambda_1 = \tfrac{\lambda_1}{\alpha} f(b_1) = \tfrac{1}{\alpha} f(\lambda_1 b_1) = \tfrac{1}{\alpha} f(\lambda_2 b_2) = \tfrac{\lambda_2}{\alpha} f(b_2) = \lambda_2\,,$$

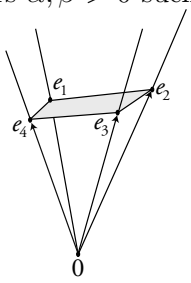
and from this we get $b_1 = b_2$. This shows that $\mathcal{B}$ is a base of $\mathcal{K}$. ∎

We now turn our attention to extremal rays of cones. Recall that if x is a vector in a linear space, then the set $\mathcal{A}_x = \big\{\alpha x\colon\ \alpha \geq 0\big\}$ is called the **half-ray** generated by x. A half-ray $\mathcal{A}_x$ is an **extremal ray** of a cone $\mathcal{K}$ if the vector x is an extremal vector of the cone $\mathcal{K}$.

If a cone has a base, the extremal vectors of the cone are in essence the extreme points of the base.

Theorem 1.48. *Let $\mathcal{B}$ be a base of a cone $\mathcal{K}$. Then a vector $b \in \mathcal{B}$ is an extremal vector if and only if b is an extreme point of the convex set $\mathcal{B}$.*

Proof. Assume first that $b \in \mathcal{B}$ is an extremal vector of the cone $\mathcal{K}$ and let $b = \lambda b_1 + (1-\lambda)b_2$ with $b_1, b_2 \in \mathcal{B}$ and $0 < \lambda < 1$. Since $0 < \lambda b_1 \leq b$ and $0 < (1-\lambda)b_2 \leq b$ are both true, there exist two scalars $\alpha, \beta > 0$ such that $\lambda b_1 = \alpha b$ and $(1-\lambda)b_2 = \beta b$. This implies that $b = \frac{\lambda}{\alpha} b_1 = \frac{1-\lambda}{\beta} b_2$. Now a glance at the definition of a base guarantees that $b_1 = b_2 = b$. Therefore, b is an extreme point of $\mathcal{B}$.



For the converse, assume that $b \in \mathcal{B}$ is an extreme point of the convex set $\mathcal{B}$. Let $0 < x < b$. We must show that x is a scalar multiple of b. To see this, start by observing that since $b - x > 0$ and $x > 0$ there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $b_1, b_2 \in \mathcal{B}$ such that $b + x = \lambda_1 b_1$ and $b - x = \lambda_2 b_2$. From

$$2b = (b+x) + (b-x) = \lambda_1 b_1 + \lambda_2 b_2 = (\lambda_1 + \lambda_2)\big[\tfrac{\lambda_1}{\lambda_1+\lambda_2} b_1 + \tfrac{\lambda_2}{\lambda_1+\lambda_2} b_2\big]$$

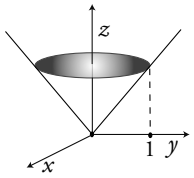
and the fact that $\mathcal{B}$ is a base of the cone $\mathcal{K}$, we get $b = \frac{\lambda_1}{\lambda_1+\lambda_2} b_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} b_2$. Since b is an extreme point of $\mathcal{B}$, it follows that $b_1 = b_2 = b$. This implies $x = (b+x) - b = \lambda_1 b_1 - b = (\lambda_1 - 1)b$. (A cone having a base with four extreme points is shown in the above figure.) ■

Corollary 1.49. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases of a cone, then the sets of extreme points of $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same cardinality.*

We shall close the section by presenting two examples of cones with bases. The first one is an "**ice cream cone**" in $\mathbb{R}^3$. One of them is defined by

$$\begin{aligned} \mathcal{K} &= \left\{(x,y,z) \in \mathbb{R}^3 \colon\ z \geq \sqrt{x^2+y^2}\right\} \\ &= \left\{\lambda(x,y,1) \in \mathbb{R}^3 \colon\ \lambda \geq 0 \text{ and } x^2+y^2 \leq 1\right\}. \end{aligned}$$

Clearly, the compact convex set $\mathcal{B} = \left\{(x,y,1) \colon\ x^2+y^2 \leq 1\right\}$ is a base for $\mathcal{K}$; see the shaded circle in the accompanying figure. In particular, notice that the linear functional $f \colon \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = z$ is $\mathcal{K}$-strictly positive.



The corresponding infinite dimensional analogue ice cream cone in ℓ_2 of the preceding example is the cone given by

$$\begin{aligned} \mathcal{K}_1 &= \left\{(x_1,x_2,x_3,\ldots) \in \ell_2 \colon\ x_1 \geq \sqrt{\textstyle\sum_{n=2}^{\infty} x_n^2}\right\} \\ &= \left\{\lambda(1,x_2,x_3,\ldots) \in \ell_2 \colon\ \lambda \geq 0 \text{ and } \textstyle\sum_{n=2}^{\infty} x_n^2 \leq 1\right\}. \end{aligned}$$

The linear functional $g\colon \ell_2 \to \mathbb{R}$ defined via the formula $g(x_1, x_2, \ldots) = x_1$ is again $\mathcal{K}_1$-strictly positive—and so $\mathcal{B}_1 = \big\{(x_1, x_2, \ldots) \in \ell_2 \colon\ x_1 = 1\big\}$ is a base for $\mathcal{K}_1$.

For an example of a Banach lattice whose cone does not have a base see Exercise 6 at the end of the chapter. For examples of general "ice cream cones" in normed spaces see Section 2.6. A general discussion of polyhedral cones will be presented in Section 3.3.

Exercises

1. Give an example of two bases of a cone in a topological vector space one of which is bounded and the other unbounded. [*Hint*: Consider an "open ice cream" cone in $\mathbb{R}^2$ or $\mathbb{R}^3$.]

2. Show that a subset $\mathcal{B}$ of a cone $\mathcal{K}$ of a vector space X is a base for the cone $\mathcal{K}$ if and only if there exists a $\mathcal{K}$-strictly positive linear functional $f\colon X \to \mathbb{R}$ such that $\mathcal{B} = \mathcal{K} \cap f^{-1}(\{1\})$.

3. Show that standard cones of the ordered vector spaces $C[0,1]$ and $L_p[0,1]$ $(1 \leq p \leq \infty)$ all have bases.

4. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two cones of a vector space X such that the cone $\mathcal{K}_1 \cap \mathcal{K}_2$ is nontrivial. Assume also that a linear functional $f \in X^*$ is both $\mathcal{K}_1$- and $\mathcal{K}_2$-strictly positive. If we consider the bases

$$\mathcal{B}_1 = \{x \in \mathcal{K}_1 \colon\ f(x) = 1\} \quad \text{and} \quad \mathcal{B}_2 = \{x \in \mathcal{K}_2 \colon\ f(x) = 1\}$$

of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, then show that

$$\mathcal{B}_1 \cap \mathcal{B}_2 = \{x \in \mathcal{K}_1 \cap \mathcal{K}_2 \colon\ f(x) = 1\}\,.$$

(Conclude from this that $\mathcal{B}_1 \cap \mathcal{B}_2$ is a base for the cone $\mathcal{K}_1 \cap \mathcal{K}_2$.)

5. Let $\mathcal{B}$ be a base of a cone K. If $\alpha, \beta \in \mathbb{R}_+$ and $b_1, b_2 \in \mathcal{B}$ satisfy $\alpha b_1 < \beta b_2$, then show that $\alpha < \beta$. [*Hint*: Assume $\alpha > 0$ and that $\alpha b_1 < \beta b_2$. Pick $\lambda > 0$ and $b_3 \in \mathcal{B}$ (both uniquely determined) such that $\beta b_2 - \alpha b_1 = \lambda b_3$. This implies $\beta b_2 = (\lambda + \alpha)\big(\frac{\lambda}{\alpha+\lambda} b_3 + \frac{\alpha}{\alpha+\lambda} b_1\big)$, and from this it follows that $\beta = \lambda + \alpha > \alpha$.]

6. Let Ω be an uncountable set and let $L = B(\Omega)$ be the Banach lattice of all bounded real-valued functions on Ω equipped with the standard cone $L_+ = \big\{f \in B(\Omega) \colon\ f(\omega) \geq 0 \text{ for all } \omega \in \Omega\big\}$. Show that the closed cone L_+ does not have a base. [*Hint*: According to Theorem 1.47 we must show that L does not admit any strictly positive linear functional. If $\phi\colon L \to \mathbb{R}$ is a positive linear functional, then for each $\epsilon > 0$ the set $\big\{\omega \in \Omega \colon\ \phi(\chi_{\{\omega\}}) \geq \epsilon\big\}$ is a finite set. This implies that the set $\big\{\omega \in \Omega \colon\ \phi(\chi_{\{\omega\}}) \geq 0\big\}$ is at most countable. Since Ω is uncountable, it is easy to see that ϕ cannot be strictly positive.]

7. Let X be a finite dimensional vector space and let $\{x_1, \dots, x_n\}$ be a (Hamel) basis of X. Consider the set

$$\mathcal{K} = \Big\{ \sum_{i=1}^{n} \lambda_i x_i \colon \ \lambda_i \geq 0 \ \text{ for each } i = 1, \dots, n \Big\} .$$

Establish the following properties.

(a) The set $\mathcal{K}$ is a cone of X.

(b) If $x = \sum_{i=1}^{n} \lambda_i x_i$ and $y = \sum_{i=1}^{n} \mu_i x_i$, then $x \geq_{\mathcal{K}} y$ if and only if $\lambda_i \geq \mu_i$ holds in $\mathbb{R}$ for each $i = 1, \dots, n$.

(c) The cone $\mathcal{K}$ is an Archimedean lattice cone whose lattice operations for each $x = \sum_{i=1}^{n} \lambda_i x_i$ and $y = \sum_{i=1}^{n} \mu_i x_i$ are given by

$$x \vee y = \sum_{i=1}^{n} \max\{\lambda_i, \mu_i\} x_i \quad \text{ and } \quad x \wedge y = \sum_{i=1}^{n} \min\{\lambda_i, \mu_i\} x_i \, .$$

(For this part see also Theorem 3.17.)

(d) The linear functional $f \colon X \to \mathbb{R}$, defined for each $x = \sum_{i=1}^{n} \lambda_i x_i$ by $f(x) = \sum_{i=1}^{n} \lambda_i$, is $\mathcal{K}$-strictly positive—and conclude from this that the cone $\mathcal{K}$ has a compact base.

(e) The extreme rays of $\mathcal{K}$ are precisely the half-rays $\mathcal{A}_{x_i}$, $i = 1, \dots, n$.

(f) Consider $\mathbb{R}^n$ as a Riesz space with its natural order and define the mapping $T \colon \mathbb{R}^n \to X$ via the formula $T(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i x_i$. Then T is a surjective lattice isomorphism—and so the Riesz space $(X, \mathcal{K})$ can be identified with $\mathbb{R}^n$.

8. Consider the $\mathbb{R}^3$ ice cream cone $\mathcal{K} = \big\{(x, y, z) \in \mathbb{R}^3 \colon \ z \geq \sqrt{x^2 + y^2}\big\}$. Find its extremal vectors.

1.8. Decomposability in ordered vector spaces

The purpose of this section is to discuss a very important property known as the decomposition property.[17]

Definition 1.50. *An ordered vector space L is said to have the* ***Riesz decomposition property*** (*or simply the* ***decomposition property***) *if it follows from $x, u, v \in L_+$ and $x \leq u + v$ that there exist $x_1, x_2 \in L$ satisfying $0 \leq x_1 \leq u$, $0 \leq x_2 \leq v$ and $x = x_1 + x_2$.*

A cone $\mathcal{K}$ of a vector space is said to have the ***Riesz decomposition property*** (*or simply the* ***decomposition property***) *if the ordered vector space $(L, \mathcal{K})$ has the Riesz decomposition property.*

The reader should notice at once the following simple result.

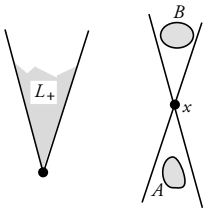
[17] In economics, the decomposition property is associated with the important economic notion of the *divisibility of commodities.*

Lemma 1.51. *An ordered vector space L has the Riesz decomposition property if and only if*

$$[0, x] + [0, y] = [0, x + y]$$

holds for all $x, y \in L_+$.

Now let us introduce the following notation. Let L be an ordered vector space. If A and B are subsets of L, let us write $A \leq B$ to mean that $a \leq b$ for all $a \in A$ and all $b \in B$. Also, the symbol $A \leq x$ means that $a \leq x$ for all $a \in A$. We shall say that a vector x in an ordered vector space is **intermediate** between two nonempty sets A and B if either $A \leq x \leq B$ or $B \leq x \leq A$ is true. The geometric interpretation of the intermediate vector x between the sets A and B is shown in the accompanying figure.



With this terminology at hand, the interpolation property in ordered vector spaces is introduced as follows.

Definition 1.52. *An ordered vector space L is said to have the **interpolation property** if for every pair of finite nonempty subsets A and B of L satisfying $A \leq B$ there exists an intermediate vector between A and B, i.e., there exists some $x \in L$ with $A \leq x \leq B$.*

*A cone $\mathcal{K}$ of a vector space L is said to have the **interpolation property** if the ordered vector space $(L, \mathcal{K})$ has the interpolation property.*

In the definition of the interpolation property we can assume that each of the sets A and B has two vectors.

Lemma 1.53. *An ordered vector space has the interpolation property if and only if every pair of subsets A and B consisting of two vectors each that satisfies $A \leq B$ has an intermediate point between them.*

Proof. Let L be an ordered vector space such that every pair of subsets A and B of L consisting of two vectors each that satisfies $A \leq B$ has an intermediate point between them. To establish that L has the interpolation property, it suffices to show that for any two nonempty finite subsets V and W with the same number of vectors satisfying $V \leq W$ there is an intermediate vector between them. We shall prove this using induction on the number n of elements of the sets V and W.

For $n = 1$ the conclusion is obvious and for $n = 2$ it is true by assumption. For the induction step, assume that any two subsets V and W with $n \geq 2$ vectors satisfying $V \leq W$ have an intermediate vector. It easily follows that any two nonempty finite subsets C and D each having at most n vectors and satisfying $C \leq D$ have an intermediate vector.

Let two subsets $V = \{v_1, \ldots, v_n, v_{n+1}\}$ and $W = \{w_1, \ldots, w_n, w_{n+1}\}$ of L satisfy $V \leq W$. By the induction hypothesis, there exists a vectors x_1 such that $\{v_1, \ldots, v_n\} \leq x_1 \leq \{w_1, \ldots, w_n\}$. Since $\{v_1, \ldots, v_n\} \leq \{w_{n+1}, x_1\}$ and $n \geq 2$, the induction hypothesis guarantees the existence of a vector y_1 such that $\{v_1, \ldots, v_n\} \leq y_1 \leq \{w_{n+1}, x_1\}$. In particular, note that

$$\{v_1, \ldots, v_n\} \leq y_1 \leq W\,. \tag{$\star$}$$

Similarly, there exist vectors x_2 and y_2 satisfying

$$\{v_2, \ldots, v_{n+1}\} \leq x_2 \leq \{w_1, \ldots, w_n\} \text{ and } \{v_2, \ldots, v_{n+1}\} \leq y_2 \leq \{w_{n+1}, x_2\}\,.$$

It follows that $\{y_1, y_2\} \leq W$ and $\{v_2, \ldots, v_{n+1}\} \leq y_2 \leq W$.

Next, it is not difficult to see that there exist vectors $y_3, \ldots, y_{n+1}$ such that $\{y_1, y_2, y_3, \ldots, y_n, y_{n+1}\} \leq W$. By the conclusion shown in ($\star$), there exists a vector z such that $\{y_1, y_2, y_3, \ldots, y_n\} \leq z \leq W$. Since $n \geq 2$, it follows that $\{y_1, y_2\} \leq z$ and from this we see that $V \leq z \leq W$. This completes the induction and the proof. ■

The interpolation and the decomposition properties are equivalent. This was proven by F. Riesz [**81**].

Theorem 1.54 (F. Riesz)**.** *For an ordered vector space L the following statements are equivalent.*

(a) *L has the interpolation property.*

(b) *L has the Riesz decomposition property.*

(c) *For any two collections $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ of positive vectors satisfying $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$ there exists another collection of positive vectors $\{z_{ij}\colon\ i=1, \ldots, n;\ j=1, \ldots, m\}$ such that*

$$x_i = \sum_{j=1}^{m} z_{ij} \text{ for } i=1, \ldots, n \text{ and } y_j = \sum_{i=1}^{n} z_{ij} \text{ for } j=1, \ldots, m\,.$$

(d) *If $x_1, x_2, y_1, y_2 \in L_+$ satisfy $x_1 + x_2 = y_1 + y_2$, then there exists a collection of positive vectors $\{z_{ij}\colon\ i, j = 1, 2\}$ such that*

$$x_i = z_{i1} + z_{i2} \text{ for } i=1, 2 \text{ and } y_j = z_{1j} + z_{2j} \text{ for } j=1, 2\,.$$

Proof. (a) $\Longrightarrow$ (b) Assume that $0 \leq x \leq u + v$ holds in L with $u, v \in L_+$. It should be clear that $\{0, x - u\} \leq \{v, x\}$, and so by our hypothesis there exists some $x_1 \in L$ such that $\{0, x - u\} \leq x_1 \leq \{v, x\}$. It follows that $0 \leq x_1 \leq v$ and if we let $x_2 = x - x_1$, then from $x - u \leq x_1 \leq x$ we get $0 \leq x_2 = x - x_1 \leq u$. Clearly, $x_1 + x_2 = x$, and so L has the Riesz decomposition property.

(b) $\Longrightarrow$ (c) We shall use induction on m. For $m = 1$ the desired conclusion is obvious. Therefore, assume the result to be true for some m and all $n = 1, 2, \ldots$. Let

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{m+1} y_j \,,$$

where the vectors x_i and y_j are positive vectors. Since $\sum_{j=1}^{m} y_j \leq \sum_{i=1}^{n} x_i$ holds true, it follows from (b) that there exist vectors $u_1, \ldots, u_n$ satisfying $0 \leq u_i \leq x_i$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^{n} u_i = \sum_{j=1}^{m} y_j$. Therefore, from our induction hypothesis, there exists a collection of positive vectors $\{z_{ij} \colon\ i = 1, \ldots, n;\ j = 1, \ldots, m\}$ such that

$$u_i = \sum_{j=1}^{m} z_{ij} \ \text{ for } \ i = 1, \ldots, n \quad \text{and} \quad y_j = \sum_{i=1}^{n} z_{ij} \ \text{ for } \ j = 1, \ldots, m \,.$$

For each $i = 1, \ldots, n$ put $z_{i,m+1} = x_i - u_i \geq 0$ and note that the collection of positive vectors $\{z_{ij} \colon\ i = 1, \ldots, n;\ j = 1, \ldots, m+1\}$ satisfies

$$x_i = \sum_{j=1}^{m+1} z_{ij} \ \text{ for } \ i = 1, \ldots, n \quad \text{and} \quad y_j = \sum_{i=1}^{n} z_{ij} \ \text{ for } \ j = 1, \ldots, m+1 \,.$$

Thus, the conclusion holds true for $m + 1$ and all $n = 1, 2, \ldots$, and the validity of (c) has been established.

(c) $\Longrightarrow$ (d) Obvious.

(d) $\Longrightarrow$ (a) Assume that two subsets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ of L satisfy $A \leq B$. From the identity

$$(b_1 - a_1) + (b_2 - a_2) = (b_1 - a_2) + (b_2 - a_1)$$

and the fact that each of the above four vectors in the parentheses is positive, it follows from our hypothesis that there exist positive vectors $z_{11}, z_{12}, z_{21}, z_{22}$ such that

$$b_1 - a_1 = z_{11} + z_{12} \,, \ \ b_2 - a_2 = z_{21} + z_{22} \,, \ \ b_1 - a_2 = z_{11} + z_{21} \,, \ \text{ and}$$

$$b_2 - a_1 = z_{12} + z_{22} \,.$$

To finish the proof, let $x = a_1 + z_{12}$ and note that $A \leq x \leq B$ holds. ■

Corollary 1.55. *Every Riesz space has the decomposition property. That is, every lattice cone in a vector space has the decomposition property.*

Proof. Assume that two nonempty finite sets A and B in a Riesz space L satisfy $A \leq B$. It follows that $\sup A \leq \inf B$, and from this we see immediately that L has the interpolation property, and hence the decomposition property. ■

An Archimedean ordered vector space with the decomposition property need not be a Riesz space. The next three examples will clarify the situation. The first one is due to F. Riesz [**81**].

Example 1.56. (F. Riesz) Let us say that a function $x\colon [0,1] \to \mathbb{R}$ is a **rational function** if there exist two polynomials p and q satisfying $q(t) \neq 0$ and $x(t) = \frac{p(t)}{q(t)}$ for all $t \in [0,1]$. Let L denote the collection of all rational functions. Under the usual pointwise algebraic operations L is a vector space—in fact, L is a vector subspace of $C[0,1]$. Thus, under the pointwise ordering, L is also an ordered vector space. Clearly, the constant function one is an order unit—and so the cone L_+ is generating. It is not difficult to see that L is not a Riesz space; see Exercise 9 at the end of the section. However, we claim that L satisfies the Riesz decomposition property.

To see this, assume that the three rational functions x, y_1 and y_2 satisfy $0 \leq x \leq y_1 + y_2 = z$ with $z > 0$. A moment's thought reveals that we can suppose without loss of generality that the rational functions x, y_1 and y_2 (and so z) are all polynomials. From this and the fact that z is a nonzero polynomial, we see that the set $\mathcal{Z} = \{t \in [0,1]\colon\ z(t) = 0\}$ is a finite set. Also, let $D = [0,1] \setminus \mathcal{Z} = \{t \in [0,1]\colon\ z(t) > 0\}$. Next for each $i = 1, 2$ define $x_i\colon D \to \mathbb{R}$ via the formula

$$x_i(t) = \frac{x(t)y_i(t)}{z(t)}\,.$$

Clearly, the x_i are positive continuous functions on D and they satisfy

$$0 \leq x_i \leq y_i \ \text{ for } \ i = 1, 2 \ \text{ and } \ x_1 + x_2 = x\,. \tag{$\star$}$$

We claim that the functions x_i can be extended to rational functions on all of $[0,1]$. If this is done, then a continuity argument shows that $(\star)$ holds true on $[0,1]$ and this will establish that L has the Riesz decomposition property.

If $\mathcal{Z} = \emptyset$, then there is nothing to prove. So, let $\mathcal{Z} = \{t_1, \ldots, t_k\}$. In this case, we can write $z(t) = (t - t_1)^{r_1} \cdots (t - t_k)^{r_k} p(t)$, where the r_i are natural numbers and p is a polynomial satisfying $p(t) \neq 0$ for all $t \in [0,1]$. Next, fix i and note that x and y_i vanish on $\mathcal{Z}$. This implies we can write $x(t)y_i(t) = (t - t_1)^{m_1} \cdots (t - t_k)^{m_k} q(t)$, where q is a polynomial satisfying $q(t_\ell) \neq 0$ for all $\ell = 1, \ldots, k$. In particular, for each $t \in D$ we have

$$x_i(t) = \frac{x(t)y_i(t)}{z(t)} = (t - t_1)^{m_1 - r_1} \cdots (t - t_k)^{m_k - r_k} \frac{q(t)}{p(t)}\,. \tag{$\star\star$}$$

This implies that if for some ℓ we have $m_\ell - r_\ell < 0$, then $\lim_{t \to t_\ell} x_i(t) = \infty$, where the limit is one-sided if t_ℓ equals 0 or 1. However, since for each $t \in D$ (in view of $0 \leq \frac{y_i(t)}{z(t)} \leq 1$ and $0 \leq x(t) \leq z(t)$) we have $0 \leq x_i(t) \leq z(t)$, the latter conclusion is impossible. Hence, $m_\ell - r_\ell \geq 0$ holds true for each

$\ell = 1, \dots, k$, and from $(\star\star)$ we see that each x_i is a rational function, as desired. (For another proof see Exercise 10 at the end of the section.) ■

The second example of an Archimedean ordered vector space with the decomposition property that is not a Riesz space is due to I. Namioka [**71**].

Example 1.57 (Namioka)**.** We consider the Riesz space $C[0,2]$, where as usual the lattice operations $f \vee g$ and $f \wedge g$ are defined pointwise. The example is the vector subspace L of $C[0,2]$ defined by

$$L = \big\{ f \in C[0,2] \colon \ f(0) + f(2) = f(1) \big\} .$$

The vector ordering on L is the one induced by $C[0,2]$, i.e., the pointwise ordering. Since the function u whose graph consists of the polygonal path joining the points $(0,1)$, $(1,2)$, and $(2,1)$ belongs to L and is an order unit, it follows that the cone L_+ is generating.

To see that L is not a Riesz space, consider the function $f \in L$ defined by $f(x) = x - 1$. We claim that the supremum of f and 0 does not exist in L. To establish this, assume by way of contradiction that the supremum $g = \sup\{f, 0\}$ exists in L. Then for each $0 \leq \alpha < 1 < \beta \leq 2$ the function $h_{\alpha,\beta}$ whose graph is the polygonal line joining the points $(0,0)$, $(\alpha,0)$, $(1,1)$, $(\beta, \beta - 1)$ and $(2,1)$ belongs to L and satisfies $h_{\alpha,\beta} \geq 0$ and $h_{\alpha,\beta} \geq f$. This implies $g(x) = 0$ for all $0 \leq x \leq \alpha$ and $g(x) = f(x)$ for all $1 < \beta \leq 2$. Since α and β are arbitrary, we infer that $g(x) = 0$ for all $0 \leq x \leq 1$ and $g(x) = f(x)$ for all $1 < x \leq 2$. Now notice that from $g(0) + g(2) = 1 \neq 0 = g(1)$ we get $g \notin L$, which is impossible. Hence, $\sup\{f, 0\}$ does not exist in L.

We shall show next that L has the Riesz decomposition property. For this, according to Theorem 1.54, it suffices to show that L satisfies the interpolation property. To this end, assume that two subsets $A = \{f_1, f_2\}$ and $B = \{g_1, g_2\}$ of L satisfy $A \leq B$. We must prove that there exists a function of L between A and B.

Without loss of generality, we can assume that

$$f_2(1) \leq f_1(1) \leq g_1(1) \leq g_2(1) . \tag{$\dagger$}$$

We claim that

$$f_1(1) \leq \max\big\{ f_1(0), f_2(0) \big\} + \max\big\{ f_1(2), f_2(2) \big\} \leq g_1(1) . \tag{$\dagger\dagger$}$$

The left inequality follows immediately from $f_1(1) = f_1(0) + f_1(2)$. For the right inequality, assume by way of contradiction that

$$\max\big\{ f_1(0), f_2(0) \big\} + \max\big\{ f_1(2), f_2(2) \big\} > g_1(1) .$$

If $\max\big\{ f_1(0), f_2(0) \big\} = f_1(0)$ and $\max\big\{ f_1(2), f_2(2) \big\} = f_1(2)$, then we see that $f_1(1) = f_1(0) + f_1(2) > g_1(1)$, which contradicts $(\dagger)$. This means that,

without loss of generality, we can assume

$$\max\{f_1(0), f_2(0)\} = f_1(0) \quad \text{and} \quad \max\{f_1(2), f_2(2)\} = f_2(2)\,.$$

This in turn yields $g_1(1) = g_1(0) + g_1(2) \geq f_1(0) + f_2(2) > g_1(1)$, which is also absurd, and the validity of (††) has been established.

Now, for simplicity put

$$a = \max\{f_1(0), f_2(0)\} \quad \text{and} \quad b = \max\{f_1(2), f_2(2)\}\,,$$

and note that (††) becomes $f_1(1) \leq a + b \leq g_1(1)$.

Next, let h be the piecewise linear function defined on $[0, 2]$ by joining the points $(0, a)$, $(1, a + b)$, and $(2, b)$ by line segments, and then consider the function

$$f = (h \wedge g_1 \wedge g_2) \vee (f_1 \vee f_2)\,,$$

where $\vee$ and $\wedge$ are the (pointwise) lattice operations in $C[0, 2]$. Clearly, $f \in C[0, 2]$ and from (††), it follows that $f(0) + f(2) = a + b = f(1)$, so that $f \in L$. Finally, note that

$$f_1 \vee f_2 \leq f \leq (h \wedge g_1 \wedge g_2) \vee (g_1 \wedge g_2) = g_1 \wedge g_2\,.$$

Consequently, f is a desired intermediate function between A and B. ■

Example 1.58. The third example of an ordered vector space with the Riesz decomposition property that fails to be a Riesz space is $L = C^1[0, 1]$, the real vector space of all continuously differentiable functions on $[0, 1]$, equipped with the cone $L_+ = \{x \in C^1[0, 1]\colon\ x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ of all nonnegative functions of L. As in Example 1.57, we can show that if $f \in L$ is defined by $f(t) = t - \frac{1}{2}$, then f^+ does not exist in L, and so L is not a Riesz space. Since $\mathbf{1}$ (the constant function one) is an order unit, it follows that the cone L_+ is generating. We claim that L has the Riesz decomposition property.[18]

To see this, assume that three functions $x, y_1, y_2 \in L_+$ satisfy the inequality $0 \leq x \leq y_1 + y_2$. We must show that there exist two functions $x_1, x_2 \in L$ satisfying $0 \leq x_1 \leq y_1$, $0 \leq x_2 \leq y_2$ and $x = x_1 + x_2$. We shall establish that the functions $x_1, x_2\colon [0, 1] \to \mathbb{R}$, defined by

$$x_i(t) = \begin{cases} \frac{y_i(t)x(t)}{y_1(t)+y_2(t)} & \text{if } \ y_1(t) + y_2(t) > 0, \\ 0 & \text{if } \ y_1(t) + y_2(t) = 0\,, \end{cases}$$

satisfy the desired properties. To this end, notice first that $0 \leq x_i \leq y_i$ for $i = 1, 2$ and $x_1 + x_2 = x$ are trivially true. So what remains to be shown is that $x_1, x_2 \in L$.

[18] The ordered vector space $C^1[0, 1]$ enjoys many other interesting properties and will be invoked several times in this work; see Examples 2.41, 2.72 and Exercises 8 and 9 of Section 2.8.

We shall prove next that $x_1 \in L$; it will then follow by symmetry that $x_2 \in L$ is also true. This will be done by establishing that x_1 is differentiable on $[0,1]$ and that

$$x_1'(t) = \begin{cases} \left[\frac{y_1'x+y_1x'}{y_1+y_2} - \frac{y_1x[y_1'+y_2']}{[y_1+y_2]^2}\right](t) & \text{if } t\in[0,1] \text{ and } y_1(t)+y_2(t)>0, \\ 0 & \text{if } t\in(0,1) \text{ and } y_1(t)+y_2(t)=0, \\ (-1)^t \frac{y_1'(t)x'(t)}{y_1'(t)+y_2'(t)} & \text{if } t\in\{0,1\}, [y_1+y_2](t)=0 \,\&\, [y_1'+y_2'](t)\neq 0, \\ 0 & \text{if } t\in\{0,1\}, [y_1+y_2](t)=0 \,\&\, [y_1'+y_2'](t)=0. \end{cases}$$

We shall check each case separately. In what follows, keep in mind that if $y_1(t)+y_2(t)>0$, then $0 \le \frac{y_i(t)}{y_1(t)+y_2(t)} \le 1$ and $0 \le \frac{x(t)}{y_1(t)+y_2(t)} \le 1$.

CASE I: $t \in [0,1]$ and $y_1(t)+y_2(t)>0$.

In this case, the formula for the derivative follows immediately from the quotient rule. The formula also shows that in this case the derivative x_1' is continuous.

CASE II: $t \in (0,1)$ and $y_1(t)+y_2(t)=0$.

Fix some $t_0 \in (0,1)$ with $y_1(t_0)+y_2(t_0)=0$. It easily follows that $y_1(t_0)=y_2(t_0)=x(t_0)=0$ and so y_1, y_2, and x attain their minimum at t_0. Since t_0 is an interior point of $[0,1]$, we see that $y_1'(t_0)=y_2'(t_0)=x'(t_0)=0$. Now note that for sufficiently small $h \neq 0$ we have

$$\left|\tfrac{x_1(t_0+h)-x_1(t_0)}{h}\right| = \left|\tfrac{x_1(t_0+h)}{h}\right| \le \left|\tfrac{x(t_0+h)}{h}\right| \xrightarrow[h\to 0+]{} 0\,.$$

This implies that x_1 is differentiable at t_0 and that $x_1'(t_0)=0$.

Next, notice that if $y_1(t_0)+y_2(t_0)>0$, then from $0 \le \frac{y_i(t_0)x(t_0)}{[y_1(t_0)+y_2(t_0)]^2} \le 1$, $0 \le \frac{y_i(t_0)y_j(t_0)}{[y_1(t_0)+y_2(t_0)]^2} \le 1$ and the formula for $x_1'(t_0)$, it follows that for each $t \in (0,1)$ sufficiently close to t_0 we have

$$|x_1'(t)-x_1'(t_0)| = |x_1'(t)| \le |y_1'(t)| + 2|x'(t)| + |y_2'(t)| \xrightarrow[t\to t_0]{} 0\,. \qquad (\star)$$

This shows that x' is continuous at t_0.

CASE III: $t \in \{0,1\}$, $y_1(t)+y_2(t)=0$ and $y_1'(t)+y_2'(t)\neq 0$.

Consider the case $t=0$ and note that $y_1'(t)+y_2'(t)>0$ must be true. (The case $t=1$ can be obtained from this case by replacing x, y_1, and y_2 by the functions v, u_1, and u_2 defined by $v(t)=x(1-t)$ and $u_i(t)=y_i(1-t)$.) In this case we have $y_1(0)=y_2(0)=x(0)=0$ and moreover $y_1(h)+y_2(h)>0$ for all $h>0$ sufficiently close to 0. The latter implies that for all $h>0$ sufficiently close to 0 we have

$$x_1'(h) = \frac{y_1'(h)x(h)+y_1(h)x'(h)}{y_1(h)+y_2(h)} - \frac{y_1(h)x(h)[y_1'(h)+y_2'(h)]}{[y_1(h)+y_2(h)]^2}\,. \qquad (\star\star)$$

Now note that

$$\frac{x_1(0+h)-x_1(0)}{h} = \frac{x_1(h)}{h} = \frac{[y_1(h)/h][x(h)/h]}{[y_1(h)/h]+[y_2(h)/h]} \xrightarrow[h\to 0+]{} \frac{y_1'(0)x'(0)}{y_1'(0)+y_2'(0)}\,.$$

So, in this case we have $x'(0) = \frac{y_1'(0)x'(0)}{y_1'(0)+y_2'(0)}$. Next, according to Taylor's formula, for sufficiently small $h > 0$ we can write

$$y_i(h) = y_i'(0)h + o(h) \quad \text{and} \quad x(h) = x'(0)h + o(h)\,.$$

It follows that

$$\begin{aligned}\frac{y_1'(h)x(h)+y_1(h)x'(h)}{y_1(h)+y_2(h)} &= \frac{y_1'(h)[x'(0)+\frac{o(h)}{h}]+[y_1'(0)+\frac{o(h)}{h}]x'(h)}{y_1(h)/h+y_2(h)/h}\\ &\xrightarrow[h\to 0+]{} \frac{2y_1'(0)x'(0)}{y_1'(0)+y_2'(0)}\end{aligned}$$

and that

$$\begin{aligned}\frac{y_1(h)x(h)[y_1'(h)+y_2'(h)]}{[y_1(h)+y_2(h)]^2} &= \frac{[y_1'(0)+\frac{o(h)}{h}][x'(0)+\frac{o(h)}{h}][y_1'(h)+y_2'(h)]}{[y_1(h)/h+y_2(h)/h]^2}\\ &\xrightarrow[h\to 0+]{} \frac{y_1'(0)x'(0)}{y_1'(0)+y_2'(0)}\,.\end{aligned}$$

So, from $(\star\star)$, we see that $\lim_{h\to 0+} x_1'(h) = \frac{y_1'(0)x'(0)}{y_1'(0)+y_2'(0)} = x_1'(0)$, which proves that in this case x_1' is continuous at 0.

CASE IV: $t \in \{0,1\}$, $y_1(t)+y_2(t) = 0$ and $y_1'(t)+y_2'(t) = 0$.

Again, we let $t = 0$ and notice that in this case $y_1(0) = y_2(0) = x(0) = 0$. So, for each $0 < h < 1$ we have

$$0 \le \tfrac{x(h)}{h} = \tfrac{x(h)-x(0)}{h} \le \tfrac{y_1(h)}{h} + \tfrac{y_2(h)}{h} = \tfrac{y_1(h)-y_1(0)}{h} + \tfrac{y_2(h)-y_2(0)}{h} \xrightarrow[h\to 0+]{} 0\,,$$

and from this we see that x is differentiable at 0 and that $x'(0) = 0$. Now use $(\star)$ to conclude that $\lim_{t\to 0+} x'(t) = x'(0)$ is also true in this case and the proof of the Riesz decomposition property of L has been completed. ■

For more about the Riesz decomposition property see Theorems 2.47 and 3.9. See also the works of L. Fuchs [**44, 45, 46**].

Exercises

1. For every nonempty set S of a partially ordered set, let $\mathcal{U}(S)$ denote the (possibly empty) collection of all upper bounds of the set S. For an ordered vector space L show that the following statements are equivalent.
   (a) L has the Riesz decomposition property.
   (b) For every nonempty finite subset F of L the collection of all upper bounds $\mathcal{U}(F)$ of F is directed downward.

(c) For every pair of nonempty finite subsets F and G of L we have $\mathcal{U}(F)+\mathcal{U}(G)=\mathcal{U}(F+G)$.

2. Consider the cone $\mathcal{K}=\{(x,y)\in\mathbb{R}^2\colon\ x>0 \text{ and } y>0\}\cup\{(0,0)\}$ of $\mathbb{R}^2$. Show that $\mathcal{K}$ is not a lattice cone and that it satisfies the Riesz decomposition property. [*Hint*: Show that the cone $\mathcal{K}$ has the interpolation property.]

3. If in an ordered vector space with the Riesz decomposition property three vectors x,y,z satisfy $x\wedge y=x\wedge z=0$, then show that $x\wedge(y+z)=0$. [*Hint*: If $u\le\{x,y+z\}$, then $\{0,u-z\}\le\{x,y\}$.]

4. Establish that a cone $\mathcal{K}$ of a vector space X has the Riesz decomposition property (in X) if and only if the cone $\mathcal{K}$ as a cone of the vector subspace $Y=\mathcal{K}-\mathcal{K}$ (=the linear span of $\mathcal{K}$) also has the Riesz decomposition property in Y.

In particular, conclude from this that if $\mathcal{K}-\mathcal{K}$ is a lattice-subspace of $(X,\mathcal{K})$, then $\mathcal{K}$ has the Riesz decomposition property.

5. For the product ordered vector space $\prod_{i\in I}L_i$ of a family $\{L_i\}_{i\in I}$ of ordered vector spaces establish the following.

(a) L is a Riesz space if and only if each factor L_i is a Riesz space.

(b) L has the Riesz decomposition property if and only if each factor L_i has the Riesz decomposition property.

6. Establish the following property that was used in the proof of Lemma 1.53. Let A and B be two nonempty subsets of an ordered vector space L such that A has m vectors and B has n vectors. If $n\neq m$, then there exist two subsets A' and B' of L each of which has $\max\{m,n\}$ vectors such that $A\subseteq A'$, $B\subseteq B'$ and $A'\le B'$.

7. Show that a generating cone $\mathcal{K}$ of a vector space X satisfies the interpolation property if and only if $\{a_1,a_2\}\ge\{b_1,b_2\}>0$ implies the existence of some $x\in X$ satisfying $a_i\ge x\ge b_j$ for all $i,j=1,2$. [*Hint*: For the "only if" part assume $\{a_1,a_2\}\ge\{b_1,b_2\}$. Since $\mathcal{K}$ is generating, for each $i=1,2$ there exists some $u_i>0$ such that $u_i\ge -b_i$. If $u=u_1+u_2$, then $\{a_1+u,a_2+u\}\ge\{b_1+u,b_2+u\}>0$. If $x\in X$ satisfies $a_i+u\ge x\ge b_j+u$ for all i and j, then $a_i\ge x-u\ge b_j$ for all $i,j=1,2$.]

8. Let L be an ordered vector space with a generating cone and the Riesz decomposition property. Show that if $y\le\sum_{i=1}^n x_i$ holds in L, then there exist vectors $y_1,\dots,y_n$ in L satisfying $y_i\le x_i$ for each i and $y=\sum_{i=1}^n y_i$. [*Hint*: Pick a vector $z\in L_+$ satisfying $z+y\ge 0$ and $z+x_i\ge 0$ for each i and consider the inequality $0\le y+nz\le\sum_{i=1}^n(x_i+z)$.]

9. Show that the ordered vector spaces of all polynomials and all rational functions on $[0,1]$ are not Riesz spaces. [*Hint*: We provide a hint for the rational functions—the same arguments work for the polynomials. Assume that f,g,h are rational functions satisfying $h>f$ and $h>g$. Pick $c>0$ so that the functions $\mathbf{1}-c(h-f)$ and $\mathbf{1}-c(h-g)$ are strictly positive on $[0,1]$. Now consider the rational function $\pi=h-c(h-f)(h-g)$ and note that $\pi\ge f$, $\pi\ge g$ and $\pi<h$.]

10. (Fuchs [**46**]) This exercise presents another proof of the fact that the ordered vector space L of all rational functions on $[0,1]$ satisfies the Riesz decomposition property (see Example 1.56) by establishing that L has the interpolation property. Prove the following.

(a) Assume that four real numbers a_1, a_2, b_1, b_2 that are not all equal satisfy $\{a_1, a_2\} \leq \{b_1, b_2\}$. If $c = \frac{b_1 b_2 - a_1 a_2}{(b_1+b_2)-(a_1+a_2)}$, then for all i and j we have

$$c - a_i = \tfrac{(b_1-a_i)(b_2-a_i)}{(b_1+b_2)-(a_1+a_2)} \quad \text{and} \quad b_j - c = \tfrac{(b_j-a_1)(b_j-a_2)}{(b_1+b_2)-(a_1+a_2)}\,.$$

In particular, $\{a_1, a_2\} \leq c \leq \{b_1, b_2\}$ and if $\{a_1, a_2\} < \{b_1, b_2\}$, then $\{a_1, a_2\} < c < \{b_1, b_2\}$.

(b) L has the interpolation property if and only if whenever four polynomials p_1, p_2, q_1, q_2 satisfy $\{p_1, p_2\} \leq \{q_1, q_2\}$ (on $[0,1]$), there exists a rational function f such that $\{p_1, p_2\} \leq f \leq \{q_1, q_2\}$.

(c) If four distinct polynomials p_1, p_2, q_1, q_2 satisfy $\{p_1, p_2\} \leq \{q_1, q_2\}$ (on $[0,1]$), then the function $c\colon [0,1] \to \mathbb{R}$, defined by

$$c(t) = \begin{cases} \frac{q_1(t)q_2(t) - p_1(t)p_2(t)}{[q_1(t)+q_2(t)]-[p_1(t)+p_2(t)]} & \text{if } \ q_1(t)+q_2(t) \neq p_1(t)+p_2(t), \\ p_1(t) & \text{if } \ q_1(t)+q_2(t) = p_1(t)+p_2(t)\,, \end{cases}$$

is a rational function satisfying $\{p_1, p_2\} \leq c \leq \{q_1, q_2\}$.

[*Hint*: For part (b), notice that $\{p_1(t), p_2(t)\} \leq c(t) \leq \{q_1(t), q_2(t)\}$ for all $t \in [0,1]$. Moreover, observe that $(q_1 + q_2)(t) - (p_1 + p_2)(t) = 0$ if and only if $q_1(t) = q_2(t) = p_1(t) = p_2(t)$. Let

$$N = \{t_1, \ldots, t_k\} = \{t \in [0,1]\colon \ (q_1 + q_2)(t) - (p_1 + p_2)(t) = 0\}$$

and write $(q_1 + q_2)(t) - (p_1 + p_2)(t) = (t - t_1)^{r_1} \cdots (t - t_k)^{r_k} \pi(t)$, where the r_i are natural numbers and π is a polynomial satisfying $\pi(t) \neq 0$ for all $t \in [0,1]$.]

11. Consider the positive rational functions x, y_1, y_2 defined on $[0,1]$ by

$$x(t) = \tfrac{1+2t}{4}\,, \quad y_1(t) = \tfrac{t}{1+t^2} \ \text{ and } \ y_2(t) = \tfrac{1}{2-t}\,,$$

and note that $0 \leq x \leq y_1 + y_2$. Find two rational functions x_1 and x_2 on $[0,1]$ satisfying $x = x_1 + x_2$ and $0 \leq x_i \leq y_i$ for $i = 1, 2$.

12. Let us say that a function $f\colon [0,1] \to \mathbb{R}$ is **strictly positive** if $f(t) > 0$ holds true for all $t \in [0,1]$. Now let L denote the partially ordered vector space of all polynomials on $[0,1]$ and let $\mathcal{K}$ be the collection of all strictly positive polynomials of L together with the zero polynomial. Establish the following.

(a) The set $\mathcal{K}$ is a non-Archimedean cone of L.

(b) $\mathcal{K}$ is not a lattice cone of L.

(c) $\mathcal{K}$ satisfies the Riesz decomposition property.

[*Hint*: For (c) use Exercise 10 in conjunction with the classical Weierstrass approximation theorem.]

13. Let P_n denote the ordered vector space of all polynomials on $[0,1]$ of degree less than or equal to a given natural number n. (The vector ordering is the pointwise ordering.) Establish the following.

(a) P_1 is a Riesz space (and so it has the Riesz decomposition property).

(b) If $n \geq 2$, then P_n does not have the Riesz decomposition property.

14. Give a direct proof of Corollary 1.55. That is, show that if in a vector lattice we have $0 \leq x \leq u + v$ with $u \geq 0$ and $v \geq 0$, then there exist $0 \leq x_1 \leq u$ and $0 \leq x_2 \leq v$ such that $x = x_1 + x_2$. [*Hint*: Let $x_1 = x \wedge u$ and show that the vector $x_2 = x - x_1$ satisfies $0 \leq x_2 \leq v$.]

15. Show that every Riesz space satisfies the following general Riesz Decomposition Property: *If* $|x| \leq |y_1 + \cdots + y_n|$ *holds true in a Riesz space, then there exist vectors* $x_1, \ldots, x_n$ *satisfying* $x = x_1 + \cdots + x_n$ *and* $|x_i| \leq |y_i|$ *for each* $i = 1, \ldots, n$. [*Hint*: Consider the case $n = 2$ and put $x_1 = [x \vee (-|y_1|)] \wedge |y_1|$ and $x_2 = x - x_1$.]

16. Let us say that an ordered vector space has the **infinite interpolation property** if for every pair of nonempty subsets A and B satisfying $A \leq B$ there exists some x such that $A \leq x \leq B$.

Show that a vector lattice L is order complete (i.e., every nonempty subset of L that is order bounded from above has a supremum) if and only if L has the infinite interpolation property. [*Hint*: If A is an arbitrary nonempty subset of the vector lattice L that is order bounded from above, let $B = \{b \in L\colon\ a \leq b \text{ for all } a \in A\}$.]

17. Let us say that an ordered vector space has the **strong interpolation property** if for every pair of nonempty subsets A and B one of which is finite satisfying $A \leq B$ there exists some x such that $A \leq x \leq B$.

Show that an ordered vector space with a generating cone has the strong interpolation property if and only if it is a vector lattice.

1.9. The Riesz–Kantorovich formulas

Although (as we have seen in Section 1.8) an ordered vector space with the Riesz decomposition property need not be a Riesz space, we have the following basic result that is due to the founders of Riesz space theory, F. Riesz [**80**] and L. V. Kantorovich [**57, 58**].

Theorem 1.59 (Riesz–Kantorovich)**.** *If an ordered vector space L with a generating cone has the Riesz decomposition property, then for every Dedekind complete Riesz space M the ordered vector space $\mathcal{L}_{\mathrm{b}}(L, M)$ is also a Dedekind complete Riesz space—and therefore, in this case, we have $\mathcal{L}_{\mathrm{r}}(L, M) = \mathcal{L}_{\mathrm{b}}(L, M)$.*

Moreover, for any $S, T \in \mathcal{L}_{\mathrm{b}}(L, M)$ and any $x \in L_+$ the lattice operations of $\mathcal{L}_{\mathrm{b}}(L, M)$ satisfy:

$$\begin{aligned} [S \vee T](x) &= \sup\bigl\{S(y) + T(z)\colon\ y, z \in L_+ \text{ and } y + z = x\bigr\}, \\ [S \wedge T](x) &= \inf\bigl\{S(y) + T(z)\colon\ y, z \in L_+ \text{ and } y + z = x\bigr\}, \\ |S|(x) &= \sup\bigl\{S(y)\colon\ -x \leq y \leq x\bigr\}. \end{aligned}$$

In addition, $T_\alpha \uparrow T$ holds in $\mathcal{L}_{\rm b}(L,M)$ if and only if $T_\alpha(x) \uparrow T(x)$ holds in M for each $x \in L_+$.

Proof. Assume that M is a Dedekind complete Riesz space. Start by observing that since L_+ is generating, it follows that the collection of all positive operators from L to M is a cone and so $\mathcal{L}_{\rm b}(L,M)$ is an ordered vector space. From the lattice identities

$$S \vee T = (S-T)^+ + T\,, \quad S \wedge T = -[(-S) \vee (-T)]\,, \quad \text{and} \quad |S| = S \vee (-S)\,,$$

it suffices to prove the validity of the first formula, when $T = 0$. That is, it is enough to show that $S^+(x) = [S \vee 0](x) = \sup\big\{S(y)\colon\ 0 \le y \le x\big\}$ holds true for all $S \in \mathcal{L}_{\rm b}(L,M)$ and all $x \in L_+$.

To this end, fix an operator $S \in \mathcal{L}_{\rm b}(L,M)$, and then define the mapping $R\colon L_+ \to M_+$ for each $x \in L_+$ by

$$R(x) = \sup\big\{S(y)\colon\ 0 \le y \le x\big\} = \sup S([0,x])\,.$$

Notice that since $S \in \mathcal{L}_{\rm b}(L,M)$, we indeed have $R(x) \in M_+$ for each $x \in L_+$. We claim that S is additive. To establish this claim, let $u, v \in L_+$. If $0 \le x \le u$ and $0 \le y \le v$, then $0 \le x+y \le u+v$ and from

$$S(x) + S(y) = S(x+y) \le R(u+v)\,,$$

it follows that $R(u)+R(v) \le R(u+v)$. On the other hand, if $0 \le x \le u+v$, then by the Riesz decomposition property there exist vectors $0 \le x_1 \le u$ and $0 \le x_2 \le v$ satisfying $x_1 + x_2 = x$, and so we have

$$S(x) = S(x_1+x_2) = S(x_1) + S(x_2) \le R(u) + R(v)\,.$$

This implies $R(u+v) \le R(u) + R(v)$, and thus $R(u+v) = R(u) + R(v)$, i.e., the mapping $R\colon L_+ \to M_+$ is additive.

Since M is Archimedean (see Exercise 6 of Section 1.3), it follows from part (3) of Lemma 1.26 that the additive mapping R has a unique positive linear extension to all of L (which we shall denote by R again). Clearly, if $x \in L_+$, then $R(x) \ge S(x)$ and $R(x) \ge 0$, i.e., $R \ge S$ and $R \ge 0$. On the other hand, if some $\Phi \in \mathcal{L}_{\rm b}(L,M)$ satisfies $\Phi \ge S$ and $\Phi \ge 0$, then for each $0 \le y \le x$ we have $\Phi(x) \ge \Phi(y) \ge S(y)$, from which it follows that $\Phi(x) \ge R(x)$ for all $x \in L_+$. Consequently, $R = S \vee 0$ holds in $\mathcal{L}_{\rm b}(L,M)$, as desired.

The Dedekind completeness of $\mathcal{L}_{\rm b}(L,M)$ (as well as the last part of the theorem) is left as an exercise for the reader; see also Exercise 2 at the end of this section. ■

For a glimpse of the usefulness of the Riesz–Kantorovich theorem we establish that a positive operator whose domain is a Riesz subspace extends to a positive operator if and only if it is dominated by a monotone sublinear

mapping. As usual, a mapping $f\colon L \to M$ between two ordered vector spaces is called **monotone** whenever $x \le y$ in L implies $f(x) \le f(y)$ in M.

Theorem 1.60. *Let L and M be Riesz spaces with M Dedekind complete. If G is a Riesz subspace of L and $T\colon G \to M$ is a positive operator, then the following statements are equivalent.*

(1) *T extends to a positive operator from L to M.*

(2) *T extends to an order bounded operator from L to M.*

(3) *There exists a monotone sublinear mapping $p\colon L \to M$ satisfying $T(x) \le p(x)$ for all $x \in G$.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $S \in \mathcal{L}_{\mathrm{b}}(E,F)$ satisfy $S(x) = T(x)$ for all $x \in G$. Then the mapping $p\colon L \to M$ defined by $p(x) = |S|(x^+)$ is monotone, sublinear and satisfies

$$T(x) \le T(x^+) = S(x^+) \le |S|(x^+) = p(x)$$

for all $x \in G$.

(3) $\Longrightarrow$ (1) Let $p\colon L \to M$ be a monotone sublinear mapping satisfying $T(x) \le p(x)$ for all $x \in G$. Then the formula $q(x) = p(x^+)$ defines a sublinear mapping from L to M such that

$$T(x) \le T(x^+) \le p(x^+) = q(x)$$

holds for all $x \in G$. Thus, by the Hahn–Banach Extension Theorem 1.27 there exists an extension $R \in \mathcal{L}(E,F)$ of T satisfying $R(x) \le q(x)$ for all $x \in L$. In particular, if $x \in L^+$, then the relation

$$-R(x) = R(-x) \le q(-x) = p\big((-x)^+\big) = p(0) = 0$$

implies $R(x) \ge 0$. That is, R is a positive linear extension of T to all of L, and the proof is finished. ∎

When $M = \mathbb{R}$, Theorem 1.59 specializes to the following important case that will be used extensively in this monograph.

Theorem 1.61 (Riesz–Kantorovich)**.** *If an ordered vector space L with a generating cone has the Riesz decomposition property, then its order dual $L^\sim$ is a Dedekind complete Riesz space and coincides with its regular dual, that is, $L^\sim = L^r$. Moreover, for any $f, g \in L^\sim$ and any $x \in L_+$ the lattice operations satisfy*

$$\begin{aligned}
[f \vee g](x) &= \sup\big\{f(y) + g(z)\colon\ y, z \in L_+ \ \text{ and } \ y + z = x\big\}, \\
[f \wedge g](x) &= \inf\big\{f(y) + g(z)\colon\ y, z \in L_+ \ \text{ and } \ y + z = x\big\}, \\
|f|(x) &= \sup\big\{f(y)\colon\ -x \le y \le x\big\}.
\end{aligned}$$

The formulas in Theorem 1.59 describing the lattice operations in $\mathcal{L}_{\rm b}(L,M)$ are known as the **Riesz–Kantorovich formulas**. The formula

$$[S\vee T](x)=\sup\big\{S(y)+T(z)\colon\ y,z\in L_+\ \text{ and }\ y+z=x\big\}$$

will especially be referred to as the **Riesz–Kantorovich formula** of the linear functionals f and g and it will play a fundamental role in this monograph. Its general formulation is as follows: If $S_1,\dots,S_m$ belong to $\mathcal{L}_{\rm b}(L,M)$, then their supremum for each $x\in L_+$ is given by the Riesz–Kantorovich formula

$$\Big[\bigvee_{i=1}^m S_i\Big](x)=\sup\Big\{\sum_{i=1}^m S_i(x_i)\colon\ x_i\in L_+\ \text{ for each }\ i\ \text{ and }\ \sum_{i=1}^m x_i=x\Big\}\,.$$

The Riesz–Kantorovich formula will be the subject of an extensive investigation in Sections 5.2, 6.2, and 6.3. The absolute value $|T|$ of an operator $T\in\mathcal{L}_{\rm b}(L,M)$ is also known as the **modulus of the operator** T.

In terms of directed sets the lattice operations of $\mathcal{L}_{\rm b}(L,M)$ are described as follows.

Theorem 1.62. *Assume that L is an ordered vector space with a generating cone and the Riesz decomposition property and let M be a Dedekind complete Riesz space. Then for all $S,T\in\mathcal{L}_{\rm b}(L,M)$ and each $x\in L^+$ we have*

$$(1)\quad \Big\{\sum_{i=1}^n S(x_i)\vee T(x_i)\colon\ x_i\in L^+\ \text{ and }\ \sum_{i=1}^n x_i=x\Big\}\uparrow[S\vee T](x)\,,$$

$$(2)\quad \Big\{\sum_{i=1}^n S(x_i)\wedge T(x_i)\colon\ x_i\in L^+\ \text{ and }\ \sum_{i=1}^n x_i=x\Big\}\downarrow[S\wedge T](x)\,,$$

$$(3)\quad \Big\{\sum_{i=1}^n |T(x_i)|\colon\ x_i\in L^+\ \text{ and }\ \sum_{i=1}^n x_i=x\Big\}\uparrow|T|(x)\,.$$

Proof. (1) Consider the set

$$D=\Big\{\sum_{i=1}^n S(x_i)\vee T(x_i)\colon\ x_i\in L^+\ \text{ for each } i \text{ and }\ \sum_{i=1}^n x_i=x\Big\}\,.$$

Since $\sum_{i=1}^n x_i=x$ with each $x_i\in L^+$ implies

$$\sum_{i=1}^n S(x_i)\vee T(x_i)\le\sum_{i=1}^n\big[(S\vee T)x_i\big]\vee\big[(S\vee T)x_i\big]=[S\vee T](x)\,,$$

we see that $D\le[S\vee T](x)$. On the other hand, if $D\le u$ holds, then for each $y,z\in L^+$ with $y+z=x$ we have

$$S(y)+T(z)\le S(y)\vee T(y)+S(z)\vee T(z)\le u\,,$$

and consequently

$$[S\vee T](x)=\sup\big\{S(y)+T(z)\colon\ y,z\in L^+\ \text{ and }\ y+z=x\big\}\le u\,.$$

Thus, $\sup D = [S \vee T](x)$, and it remains to be shown that D is directed upward.

To this end, let $x = \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$ with all the x_i and y_j in L^+. By Theorem 1.54 there exists a finite collection $\{z_{ij}\colon\ i = 1, \dots, n;\ j = 1, \dots, m\}$ of positive vectors such that

$$x_i = \sum_{j=1}^{m} z_{ij}\,, \quad \text{for each } \ i = 1, \dots, n\,,$$

and

$$y_j = \sum_{i=1}^{n} z_{ij}\,, \quad \text{for each } \ j = 1, \dots, m\,.$$

In particular, we have $\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} = x$. On the other hand, using the lattice identity $x \vee y = \frac{1}{2}(x + y + |x - y|)$, we see that

$$\begin{aligned}
&\sum_{i=1}^{n} S(x_i) \vee T(x_i) \\
&\qquad = \tfrac{1}{2} \sum_{i=1}^{n} \big[S(x_i) + T(x_i) + |S(x_i) - T(x_i)|\big] \\
&\qquad = \tfrac{1}{2} \sum_{i=1}^{n} \Big[\sum_{j=1}^{m} S(z_{ij}) + \sum_{j=1}^{m} T(z_{ij}) + \Big|\sum_{j=1}^{m} \big\{S(z_{ij}) - T(z_{ij})\big\}\Big|\Big] \\
&\qquad \leq \tfrac{1}{2} \sum_{i=1}^{n} \Big[\sum_{j=1}^{m} \big\{S(z_{ij}) + T(z_{ij}) + |S(z_{ij}) - T(z_{ij})|\big\}\Big] \\
&\qquad = \sum_{i=1}^{n} \sum_{j=1}^{m} S(z_{ij}) \vee T(z_{ij})\,.
\end{aligned}$$

Similarly,

$$\sum_{j=1}^{m} S(y_j) \vee T(y_j) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} S(z_{ij}) \vee T(z_{ij})$$

holds, and so D is directed upward.

(2) Use (1) in conjunction with the identity $T \wedge S = -\big[(-S) \vee (-T)\big]$.

(3) Use (1) and the identity $|T| = T \vee (-T)$. ■

Regarding the Riesz decomposition property and the Riesz–Kantorovich formula we have the following open problems.

(1) *Is the converse of the Riesz–Kantorovich Theorem* 1.61 *true? That is, does an ordered vector space whose cone is generating and whose order dual is a Riesz space have the Riesz decomposition property?*

(2) *Assume that L is an ordered vector space with a generating cone and that two linear functions $f, g \in L^{\sim}$ have a supremum (in the lattice sense) in $L^{\sim}$. Is $f \vee g$ given by the Riesz–Kantorovich formula? That is, if $f \vee g$ exists in $L^{\sim}$, is it then true that*

$$[f \vee g](x) = \sup\{f(y) + g(x-y)\colon\ 0 \leq y \leq x\}$$

for each $x \in L_+$?

There are several papers in the literature that deal with these open problems; see, for instance, [**2, 7, 8, 9, 10, 21, 83**]. These questions will be discussed in detail in Chapter 5 (and in particular in Sections 6.2 and 6.3).

Exercises

1. If L is an ordered vector space with the Riesz decomposition property and a generating cone and M is a Dedekind complete Riesz space, then show that for all $S \in \mathcal{L}_{\rm b}(L,M)$ and all $x \in L_+$ we have $|S(x)| \leq |S|(x)$.

2. Establish the Dedekind completeness of $\mathcal{L}_{\rm b}(L,M)$ in Theorem 1.59. [*Hint*: Assume $0 \leq T_\alpha \uparrow \leq T$ in $\mathcal{L}_{\rm b}(L,M)$. For each vector $x \in L_+$ let $S(x) = \sup\{T_\alpha(x)\}$ in M. Show that the mapping $S\colon L_+ \to M_+$ is additive and conclude (by using part (3) of Lemma 1.26) that it extends uniquely to a positive operator from L to M, which we denote by S again. Now verify that $S = \sup\{T_\alpha\}$ holds in $\mathcal{L}_{\rm b}(L,M)$.]

3. Verify that the Riesz space $L_1[0,1]$ is Dedekind complete. Now assume that $L = M = L_1[0,1]$ and consider the positive operators $S, T\colon L \to M$ defined by

$$S(f) = f \quad \text{and} \quad T(f) = \Big[\int_0^1 f(x)\,dx\Big] \cdot \mathbf{1}\,,$$

where $\mathbf{1}$ is the constant function one. Show that $S \wedge T = 0$ holds in $\mathcal{L}_{\rm b}(L,M)$.

4. Let L be an ordered vector space with the Riesz decomposition property and a generating cone and let M be a Dedekind complete Riesz space. Use the formula

$$[S \vee T](x) = \sup\big\{S(y) + T(z)\colon\ y,z \in L_+ \text{ and } y+z = x\big\}$$

for all $S, T \in \mathcal{L}_{\rm b}(L,M)$ and all $x \in L_+$ to prove that for all $S \in \mathcal{L}_{\rm b}(L,M)$ and all $x \in L_+$ we have

$$|S|(x) = \sup\big\{S(y)\colon\ -x \leq y \leq x\big\}\,.$$

[*Hint*: Recall the definition of the absolute value: $|S| = S \vee (-S)$. Now note that if $0 \leq y \leq x$, then the vector $z = 2y - x$ satisfies $-x \leq z \leq x$. On the other hand, if $-x \leq z \leq x$, then the vector $y = \frac{z+x}{2}$ satisfies $0 \leq y \leq x$ and $2y - x = z$.]

5. Let L be an ordered vector space with a generating cone and let M be an arbitrary ordered vector space. Assume that for two operators $S, T \in \mathcal{L}_{\mathrm{b}}(L, M)$ their supremum $S \vee T$ exists in $\mathcal{L}_{\mathrm{b}}(L, M)$ (in the lattice sense). Show that for each $x \in L_+$ we have

$$\sup\{S(y) + T(x - y)\colon\ 0 \leq y \leq x\} \leq [S \vee T](x)\,.$$

6. Let $T\colon L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. If $x, y \in L$, then show that

(a) $T(x \vee y) = \max\bigl\{Rx + Sy\colon\ R, S \in \mathcal{L}_{\mathrm{b}}^{+}(L, M)\ \text{ and }\ R + S = T\bigr\}$,

(b) $T(x \wedge y) = \min\bigl\{Rx + Sy\colon\ R, S \in \mathcal{L}_{\mathrm{b}}^{+}(L, M)\ \text{ and }\ R + S = T\bigr\}$.

7. Consider the continuous function $g\colon [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} x & \text{if}\quad 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} & \text{if}\quad \frac{1}{2} < x \leq 1\,. \end{cases}$$

Next consider the operator $T\colon C[0,1] \to C[0,1]$ defined via the formula $[Tf](x) = f\bigl(g(x)\bigr) - f\bigl(\frac{1}{2}\bigr)$.

Show that T is a regular operator whose modulus does not exist in $\mathcal{L}_{\mathrm{b}}(C[0,1], C[0,1])$.

8. Let $T\colon C[0,1] \to C[0,1]$ be the regular operator defined by

$$[Tf](x) = f(\sin x) - f(\cos x)\,.$$

Show that T^+ and T^- both exist and that

$$[T^+ f](x) = f(\sin x) \quad \text{and} \quad [T^- f](x) = f(\cos x)\,.$$

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