Preface

The algebraic theory of quadratic forms over fields originated from a classical (1937) paper of Witt [Wi]. However, while quadratic form theory over local fields and global fields flourished steadily through the middle of the 20th century, Witt's theory of quadratic forms over general fields seemed to have gone into dormancy. This situation changed dramatically with the appearance of Pfister's work [Pf_1, Pf_2, Pf_3] in 1965-66. In these seminal papers, Pfister introduced the powerful method of multiplicative forms in quadratic form theory, proved the first significant structural results on the Witt ring, and established a fundamental local-global principle for quadratic forms over a general field. Pfister's contributions in 1965-66 not only revived the general quadratic form theory so ingeniously conceived by Witt in 1937, but also brought it into fruitful contact with the algebraic theory of formally real fields and real-closed fields invented in a different context by Artin and Schreier [AS] in 1927. In the early 1970s, Arason and Pfister succeeded in proving the "Hauptsatz" in quadratic form theory (a Krull intersection theorem for the ideal of even-dimensional forms in the Witt ring). Aside from its intrinsic importance, the Hauptsatz of Arason and Pfister in [AP_1] also turned out to be a harbinger for the powerful use of the method of function fields of quadratic forms in the algebraic theory of Witt rings.

Against the above historical backdrop, I wrote my Benjamin book "The Algebraic Theory of Quadratic Forms" (hereafter referred to as "ATQF") in 1972. Only five years beyond my Ph.D. and looking for something new to do in research, I was quickly caught up in the atmosphere of excitement then prevailing in the new field of the algebraic theory of quadratic forms. My decision to write "ATQF" was perhaps based in part on the ill-advised excuse that "to learn a new subject you write a book about it". Looking back, I certainly don't see anything that could have qualified me to author such a book other than a copious dosage of youthful invincibility. Anyway, my debut as a mathematical author took hardly more than a year. I still remember churning out chapter after chapter of the book in my mother's small apartment in West Vancouver when I had a prolonged visit with her in the summer of 1972; somehow, one tends to write very fast when one is very young!

In "ATQF" (which came out in 1973), I started with two chapters introducing Witt's algebraic theory of quadratic forms and the basic facts about Witt rings of fields (of characteristic not 2). This was followed by a self-contained exposition on quaternion algebras, Brauer-Wall groups, simple graded algebras, and Clifford algebras. A chapter on the rich quadratic form theory over local and global fields served as a reminder of (as well as an introduction to) the classical origins of the subject. The book then progressed into the treatments of quadratic forms under algebraic and transcendental field extensions, with an intermittent coverage of the quadratic form theory over formally real fields and pythagorean fields (highlighting Pfister's local-global principle and key structural results on the Witt ring). A penultimate chapter featured Pfister's theory of multiplicative forms and the Arason-Pfister Hauptsatz, culminating in a final chapter dealing with the quadratic form theoretic invariants of fields, such as the level, the Pythagoras number, and the u-invariant, etc. The book closed with a list of eight open problems.

The wonderful reception given to "ATQF" by the mathematical community came to me as a total surprise. A rather informal introduction to quadratic form theory based on my lecture courses at Berkeley turned out to be a welcome entry text for students learning the theory for the first time, and in the meantime, the research community in the theory of quadratic forms quickly accepted the book as a convenient reference for the basic results in the area. In retrospect, the success of the Benjamin book obviously owed little to its author, but was solely a result of the fortuitous circumstance that it just happened to have been the right book written at the right time.

Starting from the early 1970s, the algebraic theory of quadratic forms experienced a tremendous growth. A dramatic illustration of this growth is given in a special chart prepared by W. Scharlau in the bibliography section of his book [Sc_4] (c. 1985), which showed a spectacular jump of hundreds of pages of published research in the theory of quadratic forms in the period 1970-1980. Throughout this decade (and thereafter), the theory of quadratic forms made contact with a large number of other research topics in algebra, including the theory of central simple algebras and their involutions, linear algebraic groups, algebraic geometry (especially Chow groups), algebraic K-theory, Galois theory and Galois cohomology, the theory of ordered fields and valuated fields, axiomatic geometry, real commutative algebra, and semialgebraic geometry. These interactions with other fields have greatly enriched the scope of the research in the algebraic theory of quadratic forms, and have permanently established this theory as a significant and vibrant branch of modern abstract algebra.

While "ATQF" served its function well in the 1970s, it went out of print by the end of the decade. A second printing with revisions issued by Benjamin in 1980 kept the book in the market for a few more years, but "ATQF" finally succumbed to the fate of orphanhood as Benjamin later became defunct! No author can completely escape the unspeakable feeling that his/her book might have killed its publisher, but the practical effect of the demise of my publisher was clearly that "ATQF" would thereafter survive only in beaten up copies in private collections and university libraries.

This situation would have been tolerable if "ATQF" had outlived its use. However, although at least a few other books on the subject of quadratic forms have been written in the intermittent years, "ATQF" continued to be a textbook of choice for students in quadratic form theory, and a frequent reference for researchers in the field. This trend finally firmed up my resolve to make the book available again to the mathematical community. My primary choice of a publisher was the American Mathematical Society, since the AMS has graciously honored the book with the award of a Leroy P. Steele Prize in Mathematical Exposition in 1982. Plans for reissuing the book went underway in 1998.

From the very start of the republication process, it was clear that "ATQF" could not, and should not, just reappear in its original form. Many new results and new viewpoints have been obtained in quadratic form theory; the 1980 version of the book would have simply looked as outdated as its typography produced then by an IBM Selectric typewriter. But, short of writing a brand new book on the subject, how could one even begin to transform a 20-year past into a relatively satisfying present?

After much vacillation, I decided that the best course of action is to keep the main organization of "ATQF" intact, but rewrite as many chapters in it as possible to accommodate the new results and new viewpoints in the field. Furthermore, two chapters of new material are added to the original text, making the book into one with thirteen chapters. As the result of these expansions, the book has more than doubled in size. To tell it apart from the Benjamin 1973/1980 versions, I have renamed the book "Introduction to Quadratic Forms over Fields". The main focus of the book is still on the algebraic theory of quadratic forms over fields of characteristic not 2;

discussions on the interactions of this theory with other parts of algebra are deliberately kept to a minimum. By limiting the scope of the present book in this way, I hope to have preserved the suitability of this book both as a general reference work and as an introductory text to quadratic form theory. The many exciting ongoing mathematical developments at the interface of quadratic form theory and other branches of algebra (especially algebraic geometry and various cohomology theories) certainly will merit a detailed exposition in the near future, but such an ambitious project is better left to the pen of a more capable author.

Once the boundaries for this book were set, the rewriting of "ATQF" started in earnest in 1999. Needless to say, I fully recognized that being able to rewrite a book after a passage of thirty years is a rare privilege granted to very few authors. After all, how many can boast about publishing two books on the very same subject in two different centuries? And mustn't there have been some fateful "principle of symmetry" at work that I started the first book five years after my Ph.D., and now finish the second one about five years before my retirement? With all of these poignant thoughts on my mind, I returned to write about the subject of my youthful love! But sadly enough, age has taken its toll. The young author who so gallantly churned out chapter after chapter of "ATQF" in a Vancouver apartment has now metamorphosed into a foot-dragging writer who took weeks to draft a single section. With the century mark quietly slipping by, I bore hapless witness to the harsh reality that the resuscitation of my 12-month maiden project in 1972 slowly turned into a six-year arduous writing ordeal.

It was with a tremendous sense of satisfaction and joy that the rewriting of "ATQF" was brought to a conclusion in September of this year. What lies ahead is the finished product, under its new name. The two books spanned much of my professional career, and were in large part the result of my efforts in teaching, research, and mathematical exposition. I am delighted that my maiden work in 1972 has now a second lease in life, and humbly offer this new version of it for the use of the international mathematical community in the years to come.

Needless to say, a book of this nature owes much to the work of others. I hereby thank all the authors whose work I have used in my exposition, implicitly or explicitly. This includes, but is by no means limited to, all authors whose books and papers are cited in the bibliography at the end of this book. Special thanks are due to Richard Elman and Adrian Wadsworth, who were two of my principal collaborators in quadratic form theory. In preparing this book for publication, I have also greatly profited from the input and help from Detlev Hoffmann, David Leep, and Jan Mináč; for their kind and generous assistance, I remain grateful. Incredibly, it was

almost forty years ago when I first took an introductory course in quadratic form theory at Columbia University from Professor Hyman Bass. I hope he is pleased to see that his teaching four decades ago has continued to have a substantial impact up to this very day.

It gives me special pleasure to acknowledge the role of the American Mathematical Society in making this book possible. I could not have found a better or more suitable place than the Society's Graduate Studies in Mathematics series for my book to appear in. I thank Dr. Sergei Gelfand for acquiring my book for this series, and for having unflagging faith for six long years that I would one day finish my book. Without his frequent encouragement (and occasional prodding), I probably would have taken even more years. The production of the book at AMS Headquarters was handled in the utmost professional manner by Ralph Sizer and Mary Letourneau. To both of them, I express my heartfelt appreciation. Very sadly, however, the fact that Ralph did not live to see the completion of this book will remain the deepest regret of my authorship.

Last, first, and always, I owe the greatest debt to members of my family. Chee King has lovingly stood by my side for 34 years; I could not have asked for anything more. When "ATQF" was written, we were virtually newlyweds; as this book goes to press, we are the proud parents of four grown children. Juwen, Fumei, Juleen and Tsai Yu are the focus and joy of our lives; whatever I have accomplished (or can accomplish) is in large measure a result of their love, devotion, and unstinting support. I take this opportunity also to thank my mentee Daisy, whose cheerful presence I have always found inspirational.

T.Y.L.

Berkeley, California
October, 2004

Notes to the Reader

The main text of this book consists of thirteen chapters, each containing a number of sections. The chapters are referred to in roman numerals, such as I, II, III, etc. A cross-reference such as XI.6 refers to Section 6 in Chapter XI. Within a given chapter, a reference such as 6.21 refers to the result (lemma, theorem, example, or remark) so labelled in Section 6 of that chapter, while, globally, X.5.6 refers to the result 5.6 in Chapter X. The running heads offer the quickest and most convenient way to tell what chapter and what section a particular page belongs to. This should make it very easy to find any result, such as X.5.6.

Each chapter concludes with a set of exercises that are consecutively numbered. "Exercise 10" refers to the exercise so numbered in a given chapter, whereas a reference such as XI.Exercise 10 refers to Exercise 10 at the end of Chapter XI. Many exercises belong rightfully to the folklore of the subject, while a number of others are adapted from original results published in the quadratic form literature. Some (by no means all) of the harder exercises come with hints toward their solutions.

Throughout the text, a good grounding in graduate level algebra is assumed. In particular, facts in field theory and Galois theory will be used rather freely. For the chapter on local fields and global fields (Ch. VI), some familiarity with number theory will be helpful, although it is not absolutely essential. In a couple of places (in discussing the Brauer group), we also assume Wedderburn's classification theorem on finite-dimensional central simple algebras over a field. Results of this nature are usually well covered in a beginning course in graduate algebra.

The title of this book is a slight misnomer, in that we treat here only the theory of quadratic forms over fields *of characteristic not* 2. Ideally, a book on quadratic forms over fields should cover the case of characteristic 2 as well, so that the theory would apply truly to *all* fields. However, there is an all-too-well-known predicament to this, which was perhaps best expressed through the following humorous limerick by an anonymous Irish poet:

A mathematician said "Who
Can quote me a theorem that's true?
For the ones that I know
Are simply not so
When the characteristic is two!"

But, while this poet's lament was solidly grounded and frequently echoed, the real truth is actually somewhere in between. Theorems on quadratic forms over fields of characteristic not 2 usually become problematic (and sometimes meaningless) when they are transferred verbatim to the characteristic 2 case. However, experience has shown that *many* such theorems do have complete, suitably formulated analogues for fields of characteristic 2. What one needs to do is to *find* such analogues, and to *devise new proofs for them*! Thus, *each* theorem would require extra work. For a book of this size with hundreds of results, the total amount of extra work needed to cover the characteristic 2 case would have been truly staggering.

With the preservation of his sanity uppermost on his mind, this senior-aged author has made his clear and unequivocal choice. Unless explicitly stated to the contrary, *all fields over which quadratic forms are considered in this book will be assumed to have characteristic not equal to* 2. Readers interested in learning the theory of quadratic forms in a wider setting (including the case of quadratic forms over rings) may consult some of the existing literature on the subject, such as the books of Baeza [Bae], Knus [Knu], and Milnor-Husemoller [MH].