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# Introduction

In his autobiography [12], Winston Churchill remembered his struggles with Latin at school. ‘... even as a schoolboy I questioned the aptness of the Classics for the prime structure of our education. So they told me how Mr Gladstone read Homer for fun, which I thought served him right.’ ‘Naturally’ he says ‘I am in favour of boys learning English. I would make them all learn English; and then I would let the clever ones learn Latin as an honour, and Greek as a treat.’

This book is intended for those students who might find rigorous analysis a treat. The content of this book is summarised in Appendix J and corresponds more or less (more rather than less) to a recap at a higher level of the first course in analysis followed by the second course in analysis at Cambridge in 2004 together with some material from various methods courses (and thus corresponds to about 60 to 70 hours of lectures). Like those courses, it aims to provide a foundation for later courses in functional analysis, differential geometry and measure theory. Like those courses also, it assumes complementary courses such as those in mathematical methods and in elementary probability to show the practical uses of calculus and strengthen computational and manipulative skills. In theory, it starts more or less from scratch, but the reader who finds the discussion of Section 1.1 baffling or the  $\epsilon$ ,  $\delta$  arguments of Section 1.2 novel will probably find this book unrewarding. I assume a fair degree of algebraic fluency and, from Chapter 4 onwards, some exposure to linear algebra.

This book is about mathematics for its own sake. It is a guided tour of a great but empty Opera House. The guide is enthusiastic but interested only in sight-lines, acoustics, lighting and stage machinery. If you wish to see the

stage filled with spectacle and the air filled with music you must come at another time and with a different guide.

Although I hope this book may be useful to others, I wrote it for students to read either before or after attending the appropriate lectures. For this reason, I have tried to move as rapidly as possible to the points of difficulty, show why they are points of difficulty and explain clearly how they are overcome. If you understand the hardest part of a course then, almost automatically, you will understand the easiest. The converse is not true.

In order to concentrate on the main matter in hand, some of the simpler arguments have been relegated to exercises. The student reading this book *before* taking the appropriate course may take these results on trust and concentrate on the central arguments which are given in detail. The student reading this book *after* taking the appropriate course should have no difficulty with these minor matters and can also concentrate on the central arguments. I think that doing at least some of the exercises will help students to ‘internalise’ the material, but I hope that even students who skip most of the exercises can profit from the rest of the book.

I have included further exercises in Appendix K. Some are standard, some form commentaries on the main text and others have been taken or adapted from the Cambridge mathematics exams. None are ‘makeweights’, they are all intended to have some point of interest. Sketches of some solutions are available from the home pages given on page xiii. I have tried to keep to standard notations, but a couple of notational points are mentioned in the index under the heading ‘notation’.

I have not tried to strip the subject down to its bare bones. A skeleton is meaningless unless one has some idea of the being it supports, and that being in turn gains much of its significance from its interaction with other beings, both of its own species and of other species. For this reason, I have included several sections marked by a ♡. These contain material which is not necessary to the main argument but which sheds light on it. Ideally, the student should read them but not study them with anything like the same attention which she devotes to the unmarked sections. There are two sections marked ♡♡ which contain some, very simple, philosophical discussion. It is entirely intentional that removing the appendices and the sections marked with a ♡ more than halves the length of the book.

It is an honour to publish with the AMS; I am grateful to the editorial staff for making it a pleasure. I thank Dr Gunther Leobacher for computer generating the figures for this book. I owe particular thanks to Jorge Aarao, Brian Blank, Johan Grundberg, Jonathan Partington, Ralph Sizer, Thomas Ward and an anonymous referee, but I am deeply grateful to the many other people who pointed out errors in and suggested improvements

to earlier versions of this book. My e-mail address is `twk@dpmms.cam.ac.uk` and I shall try to keep a list of corrections accessible from an AMS page at [www.ams.org/bookpages/gsm-62](http://www.ams.org/bookpages/gsm-62) as well as from my home page page at <http://www.dpmms.cam.ac.uk/~twk/>.

I learned calculus from the excellent *Calculus* [13] of D. R. Dickinson and its inspiring author. My first glimpse of analysis was in Hardy's *Pure Mathematics* [24] read when I was too young to really understand it. I learned elementary analysis from Ferrar's *A Textbook of Convergence* [18] (an excellent book for those making the transition from school to university, now, unfortunately, out of print) and Burkill's *A First Course in Mathematical Analysis* [10]. The books of Kolmogorov and Fomin [31] and, particularly, Dieudonné [14] showed me that analysis is not a collection of theorems but a single coherent theory. Stromberg's book *An Introduction to Classical Real Analysis* [48] lies permanently on my desk for browsing. The expert will easily be able to trace the influence of these books on the pages that follow. If, in turn, my book gives any student half the pleasure that the ones just cited gave me, I will feel well repaid.

Cauchy began the journey that led to the modern analysis course in his lectures at the *École Polytechnique* in the 1820's. The times were not propitious. A reactionary government was determined to keep close control over the school. The faculty was divided along fault lines of politics, religion and age whilst physicists, engineers and mathematicians fought over the contents of the courses. The student body arrived insufficiently prepared and then divided its time between radical politics and worrying about the job market (grim for both staff and students). Cauchy's course was not popular<sup>1</sup>.

Everybody can sympathise with Cauchy's students who just wanted to pass their exams and with his colleagues who just wanted the standard material taught in the standard way. Most people neither need nor want to know about rigorous analysis. But there remains a small group for whom the ideas and methods of rigorous analysis represent one of the most splendid triumphs of the human intellect. We echo Cauchy's defiant preface to his printed lecture notes.

As to the methods [used here], I have sought to endow them with all the rigour that is required in geometry and in such a way that I have not had to have recourse to formal manipulations. Such arguments, although commonly accepted ... cannot be considered, it seems to me, as anything other than [suggestive] to be used sometimes in guessing the truth.

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<sup>1</sup>Belhoste's splendid biography [4] gives the fascinating details.

Such reasons [moreover] ill agree with the mathematical sciences' much vaunted claims of exactitude. It should also be observed that they tend to attribute an indefinite extent to algebraic formulas when, in fact, these formulas hold under certain conditions and for only certain values of the variables involved. In determining these conditions and these values and in settling in a precise manner the sense of the notation and the symbols I use, I eliminate all uncertainty. . . . It is true that in order to remain faithful to these principles, I sometimes find myself forced to depend on several propositions that perhaps seem a little hard on first encounter . . . . But, those who will read them will find, I hope, that such propositions, implying the pleasant necessity of endowing the theorems with a greater degree of precision and restricting statements which have become too broadly extended, will actually benefit analysis and will also provide a number of topics for research, which are surely not without importance.

I dedicate this book to the memory of my parents in gratitude for many years of love and laughter.