

Option Pricing and Portfolio Optimization

Modern Methods of Financial Mathematics

Ralf Korn
Elke Korn

Graduate Studies
in Mathematics

Volume 31



American Mathematical Society
Providence, Rhode Island

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Preface

There are only a few things in daily life which are regarded as a better synonym for uncertainty than security prices. No one seems to be able to predict their exact future values. There are just too many factors and unpredictable events which influence security prices, e.g. the economic situation, political events, company influences, behavior of buyers and sellers, technical innovations,.... So it is natural to describe financial markets where equities and other securities are traded by stochastic models.

The starting point of the history of such stochastic modeling was the dissertation of L. Bachelier [**BACH 00**]. However, the event marking a new period in mathematical finance was the Black-Scholes formula for pricing European options developed some seventy years later; see [**BL/SC 73**]. Modeling financial markets with stochastic models got a major boost from this as the formula became widely accepted by both academics and practitioners. The Black-Scholes formula proved to be so useful in real-life applications that trading in options flourished. Thus, it was natural that R. Merton and M. Scholes were awarded the Nobel Prize in economics for their work contributing to the Black-Scholes formula.

Subsequently, the financial markets introduced (and still are introducing!) more and more types of derivatives often having very complex structures. For their quantitative valuation it is essential to have a sound knowledge of mathematical models for financial markets and to be able to handle the corresponding mathematical toolbox. Here, the most important tool has become the Itô calculus. The impact of the applications of Itô calculus to the finance sector was tremendous. Banks and finance houses all over the world have realized this and have recruited an enormous number of mathematicians, physicists, and economists with the relevant knowledge.

One aim of this book is a fast and at the same time rigorous introduction to Itô calculus. This introduction is tailored to applications in financial mathematics. It therefore forsakes generality which is not needed for the applications. Based on this introduction we build up the standard diffusion type security market model in Chapter 2. The first major problem in finance, the pricing of options, will be treated in detail in Chapter 3. We shall start by introducing the method of option pricing via replication and no arbitrage. This approach is based on the principle that the price of an option should exactly equal the amount of money needed to create the option's payments synthetically. We also present the method to price options with partial differential equations which is the original approach taken by Black and Scholes. In recent years many new types of options, so-called exotic options, appeared at the market and also inspired the research. We shall present numerous examples of such options in this book, some of them with explicit pricing formulas. For obtaining prices for exotic options where an explicit pricing formula cannot be found, numerical methods are needed. Therefore, in Chapter 3 we also describe the basics of Monte Carlo methods, tree methods, and finite difference methods. Finally, another problem in finance is to find optimal investment and consumption strategies, the so-called portfolio problem. This subject will be dealt with in Chapter 5 where we shall describe the martingale method and the stochastic control method for portfolio optimization. As a very recent application we have also included a portfolio problem where only trading in options is allowed.

Although the main parts of this book are based on probabilistic methods and results, we have to emphasize that modern financial mathematics is related to various mathematical fields. After having a quick look through the book you will realize that methods of numerical analysis, partial differential equations, optimization, and functional analysis are also needed.

Required knowledge. For understanding most parts of this book a basic course in probability theory is sufficient. All other tools that are needed will be presented in this book. Of course, knowledge of stochastic processes would be desirable, but only the concept of conditional expectation is definitely needed. As we also introduce the main economic concepts there is no need for having a preknowledge in this area. For a more detailed background on option trading and the economic theory we refer to the books by Hull [**HULL 93**] or Jarrow and Turnbull [**JA/TU 96**].

The concept. With the exception of a short introduction to the Markowitz mean variance approach, we concentrate on the presentation of so-called continuous-time models in this book. Our aim is to give a sound introduction to the mathematical methods of continuous-time finance and thus we present

them in detail. We do not simply cite the major results, we also develop concepts like “stochastic integration”, “change of measure”, and “stochastic control” just at that moment when they are applied for the first time in financial mathematics. So this book consists of a mathematical description of continuous-time finance with extra sections, which we call excursions, which supply the essential mathematical tools in a compact form.

As we meant this book to be a basis of a lecture course, we made some compromises. As we decided not to present stochastic integration in a very general form, we restricted ourselves to Itô processes as integrators. For the applications this is not a severe restriction and it allows us to present the theory without using the Doob-Meyer decomposition. Its presentation would have increased substantially both the number of pages and the degree of difficulty of this book.

How to read this book. Apart from reading the book chapter by chapter – this is what we recommend – there are many other reasonable ways to read it. It is possible to skip the option chapters and jump to portfolio optimization after Chapter 2. A more traditional approach would be to work first through all excursions and then read the mathematical finance parts. On the other hand, a reader with a theoretical background could even skip the excursions.

An introductory book such as ours can only give a first impression of the large area of financial mathematics. Good sources for more details about stochastic calculus are [KA/SH 91], [OKS 92], [R/YOR 91], [RO/WI 87] or [WE/WI 90]. For recent aspects of portfolio optimization we refer to [KORN 97]. In [KA/SH 98] you can find recent aspects of various topics of mathematical finance.

Typos and errors. Even good books on graduate mathematical topics contain typing errors and sometimes even real errors. This book is probably no exception. So if you get stuck on a formula, do not despair, it might simply be a typing error. And do not invest millions of dollars relying on a formula or description you found here! Mistakes can be hidden anywhere. We absolutely do not accept any responsibility or liability for losses or damages occasioned through this book.