Chapter 1

# Perspectives on Manifolds

Manifolds are studied in a variety of categories. We will introduce several of these categories and highlight the advantages of the different points of view. We will focus on topological manifolds (TOP), differentiable (or smooth) manifolds (DIFF) and triangulated (or combinatorial) manifolds (TRIANG). To see these definitions in context, consult introductory Chapters in [**93**], [**100**], [**48**], [**148**], [**63**], [**67**], [**59**], or [**37**].

## 1.1. Topological Manifolds

**Definition 1.1.1.** A *topological $n$-manifold* is a second countable Hausdorff space $M$ for which there exists a family of pairs $\{(M_\alpha, \phi_\alpha)\}$ with the following properties:

- $\forall \alpha$, $M_\alpha$ is an open subset of $M$ and $M = \bigcup_\alpha M_\alpha$;
- $\forall \alpha$, $\phi_\alpha$ is a homeomorphism from $M_\alpha$ to an open subset of $\mathbb{R}^n$.

We often refer to a topological manifold simply as a *manifold*.

A pair $(M_\alpha, \phi_\alpha)$ is called a *chart* of $M$. The family of pairs $\{(M_\alpha, \phi_\alpha)\}$ is called an *atlas* for $M$. (We sometimes write only $\{M_\alpha\}$ rather than $\{(M_\alpha, \phi_\alpha)\}$ when the maps $\phi_\alpha$ are not part of the discussion.)

The *dimension* of an $n$-manifold is $n$. Two $n$-manifolds are considered equivalent if they are homeomorphic.

**Remark 1.1.2.** Every subset of $\mathbb{R}^N$ is second countable and Hausdorff. Thus to show that a subset $M$ of $\mathbb{R}^N$ is an $n$-manifold it suffices to show that every point in $M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Requiring these homeomorphisms to map onto $\mathbb{R}^n$ is equivalent to requiring them to map onto open subsets of $\mathbb{R}^n$. In exhibiting homeomorphisms we often opt for the latter, in the interest of simplicity.

**Example 1.1.3.** One immediate example of an $n$-manifold is $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.

**Example 1.1.4.** The set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is an $n$-dimensional manifold called the *$n$-sphere*. Stereographic projection provides a homeomorphism $h : \mathbb{S}^n \backslash \{(0, \ldots, 0, 1)\} \to \mathbb{R}^n$. Thus any point $x \in \mathbb{S}^n$ such that $x \neq (0, \ldots, 0, 1)$ has the neighborhood $\mathbb{S}^n \backslash \{(0, \ldots, 0, 1)\}$ that is homeomorphic to $\mathbb{R}^n$. To exhibit a neighborhood of $(0, \ldots, 0, 1)$ that is homeomorphic to $\mathbb{R}^n$ we compose the reflection in $\mathbb{R}^n \times \{0\}$ with $h$ to obtain $h' : \mathbb{S}^n \backslash \{(0, \ldots, 0, -1)\} \to \mathbb{R}^n$. Thus $\mathbb{S}^n \backslash \{(0, \ldots, 0, -1)\}$ is a neighborhood of $(0, \ldots, 0, 1)$ homeomorphic to $\mathbb{R}^n$. See Figure 1.1.

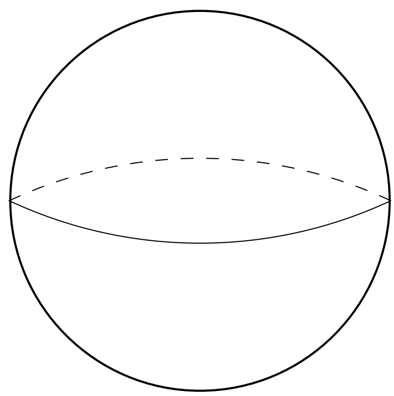

**Figure 1.1.** *The 2-dimensional sphere.*

**Example 1.1.5.** The set $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ ($n$ factors) is called the *$n$-torus*. The $n$-torus is a quotient space obtained as follows: In $\mathbb{R}^n$, consider the group $G$ generated by translations of distance 1 along the coordinate axes. Then identify two points $x, y \in \mathbb{R}^n$ if and only if there is a $g \in G$ such that $g(x) = y$. Denote this quotient map by $q : \mathbb{R}^n \to \mathbb{T}^n$.

To see that $\mathbb{T}^n$ is an $n$-manifold, let $[x] \in \mathbb{T}^n$ and let $U$ be the sphere of radius $\frac{1}{4}$ centered at $x \in \mathbb{R}^n$. Note that $g(U) \cap U = \emptyset \ \forall g \in G$. It follows that $q^{-1}|_{q(U)} : q(U) \to U$ is a homeomorphism. The set of all such homeomorphisms provides an atlas for $\mathbb{R}^n$. See Figures 1.2 and 1.3.

**Example 1.1.6.** The result of identifying antipodal points on $\mathbb{S}^n$ is an $n$-manifold. It is called *$n$-dimensional real projective space* and is denoted by $\mathbb{R}P^n$. To verify that $\mathbb{R}P^n$ is an $n$-manifold, consider a point $[x]$ in $\mathbb{R}P^n$. Since

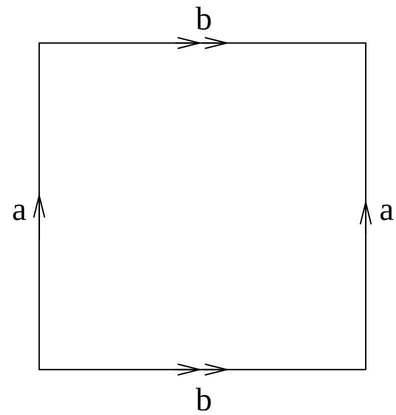


**Figure 1.2.** *The 2-dimensional torus is obtained from a square via identifications.*

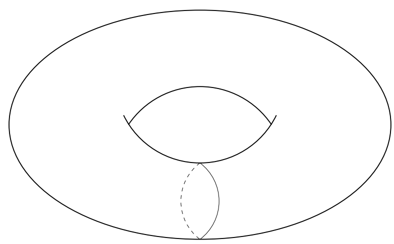

**Figure 1.3.** *The 2-dimensional torus.*

$\mathbb{S}^n$ is an $n$-manifold, there is a neighborhood $U$ of $x$ and a homeomorphism $h : U \to \mathbb{R}^n$. Set $-U = a(U)$, for $a : \mathbb{S}^n \to \mathbb{S}^n$ the antipodal map. Then $-U$ is a neighborhood of $-x$ and $-h = h \circ a$ is a homeomorphism between $-U$ and $\mathbb{R}^n$. After shrinking $U$, if necessary, $a(U) \cap U = \emptyset$. Thus we have a homeomorphism $[h] : [U] \to \mathbb{R}^n$. The set of all such homeomorphisms defines an atlas for $\mathbb{R}P^n$. See Figure 1.4.

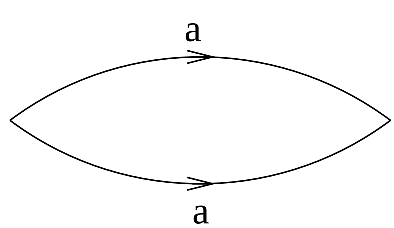


**Figure 1.4.** *The projective plane obtained from a bigon via identifications.*

**Definition 1.1.7.** Let $M$ be an $n$-manifold. A *$p$-dimensional submanifold* of $M$ is a closed subset $L$ of $M$ for which there exists an atlas $\{(M_\alpha, \phi_\alpha)\}$

of $M$ such that $\forall x \in L$ there is a chart in the atlas with $x \in M_\alpha$ and

$$\phi_\alpha(L \cap M_\alpha) = \{\mathbf{0}\} \times \mathbb{R}^p \subset \mathbb{R}^n.$$

**Remark 1.1.8.** A submanifold is itself a manifold.

**Example 1.1.9.** The equatorial circle in the 2-sphere indicated in Figure 1.1 is a submanifold of the 2-sphere.

**Definition 1.1.10.** Let $L, M$ be manifolds. A map $f : L \to M$ is an *embedding* if it is a homeomorphism onto its image $f(L)$ and $f(L)$ is a submanifold of $M$.

**Example 1.1.11.** If $L$ is a submanifold of $M$, then the inclusion map $i : \tilde{L} \to M$ of an abstract copy $\tilde{L}$ of $L$ to $L \subset M$ is an embedding.

We will also consider a slightly larger class of objects:

**Definition 1.1.12.** Set $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. An *$n$-manifold with boundary* is a second countable Hausdorff space $M$ with an atlas such that $\forall \alpha$, $\phi_\alpha$ is a homeomorphism from $M_\alpha$ to an open subset of $\mathbb{R}^n$ or $H^n$.

The *boundary* of $M$ is the set of all points in $M$ that have a neighborhood homeomorphic to $H^n$ but no neighborhood homeomorphic to $\mathbb{R}^n$. The boundary of $M$ is denoted by $\partial M$. Points not on the boundary are called interior points. Two $n$-manifolds with boundary are considered equivalent if they are homeomorphic.

**Example 1.1.13.** The set $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an $n$-dimensional manifold with boundary called the *$n$-ball*. For interior points, there is nothing to check (because the identity map on $\mathbb{R}^n$ provides the required homeomorphism). For boundary points, an extension of the map obtained by stereographic projection provides the required homeomorphism. See Figure 1.5.

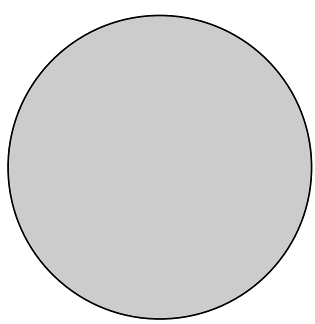

**Figure 1.5.** *The 2-ball is also called the disk.*

**Example 1.1.14.** The *pair of pants* is a 2-manifold with boundary. See Figure 1.6.

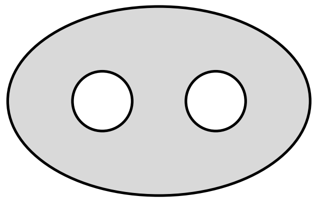

**Figure 1.6.** *The pair of pants.*

**Example 1.1.15.** The 1-holed torus is a 2-manifold with boundary. See Figure 1.7.

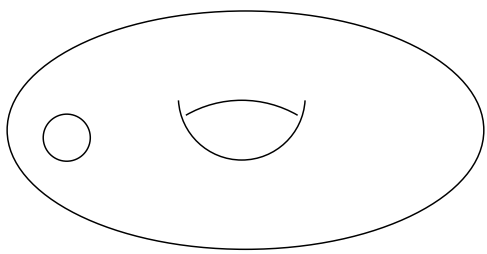

**Figure 1.7.** *The* 1*-holed torus.*

**Definition 1.1.16.** Let $M$ be an $n$-manifold with boundary. A *$p$-dimensional submanifold* of $M$ is a closed subset $L$ of $M$ for which there is an atlas $\{(M_\alpha, \phi_\alpha)\}$ of $M$ and $p \in \{0, \ldots, n\}$ such that $\forall x \in L$ in the interior of $M$ there is a chart in the atlas such that $x \in M_\alpha$ and

$$\phi_\alpha(L \cap M_\alpha) = \{\mathbf{0}\} \times \mathbb{R}^p \subset \mathbb{R}^n$$

and $\forall x \in L$ in the boundary of $M$ there is a chart in the atlas such that $x \in M_\alpha$ and

$$\phi_\alpha(L \cap M_\alpha) = \{\mathbf{0}\} \times H^p \subset H^n$$

and such that

$$\phi_\alpha(x) \in \{\mathbf{0}\} \times \partial H^p \subset \partial H^n.$$

**Remark 1.1.17.** The boundary, $\partial M$, of an $n$-manifold $M$ is not a submanifold of $M$, though it is an $(n-1)$-dimensional manifold that is contained in $M$.

**Example 1.1.18.** The diameter of the disk pictured in Figure 1.8 is a submanifold of a manifold with boundary.

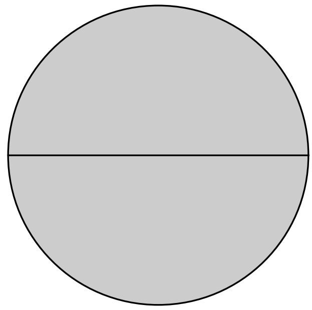

**Figure 1.8.** *A submanifold of the disk.*

**Definition 1.1.19.** We say that the $n$-manifold $M$ is *closed* if $M$ is compact and $\partial M = \emptyset$.

**Example 1.1.20.** Spheres and tori are examples of closed manifolds.

In the TOP category, that is, when we study manifolds from the point of view in this section, we are interested in continuous maps between manifolds.

**Example 1.1.21.** Projection from $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ onto the second factor is a continuous map between manifolds.

To catch a glimpse of intriguing topological manifolds browse [**78**].

## Exercises

**Exercise 1.** Convince yourself that the statements in Remark 1.1.2 are true.

**Exercise 2.** Prove that the product of two manifolds is a manifold. What can you say about its dimension? (This provides an alternate proof of the fact that $\mathbb{T}^n$ is a manifold.)

**Exercise 3.** Show that the boundary of an $n$-manifold with boundary is an $(n-1)$-manifold without boundary. See for instance the 1-holed torus in Figure 1.7.

**Exercise 4.** By drawing pictures, convince yourself that the surface with boundary called the pair of pants is aptly named.

**Exercise 5.** Argue that the 1-manifolds pictured in Figure 1.9 are equivalent. (You needn't give a formal proof.)

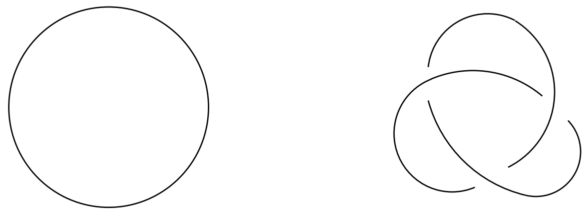

**Figure 1.9.** *Homeomorphic* 1*-manifolds.*

## 1.2. Differentiable Manifolds

Recall the following notion from calculus:

**Definition 1.2.1.** A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be $C^q$ if it has continuous partial derivatives of order $q$. A map is said to be *smooth* (or $C^\infty$) if it has partial derivatives of all orders.

**Definition 1.2.2.** A $C^q$*-manifold*, for $q \in [0, \infty]$, is a topological manifold $M$ with an atlas that satisfies the additional requirement of being $C^q$, meaning that for any pair of charts $(M_\alpha, \phi_\alpha)$, $(M_\beta, \phi_\beta)$ in this atlas, the map $\phi_\beta \circ \phi_\alpha^{-1}$ (where it is defined) is $C^q$. A $C^\infty$-manifold is also called a *differentiable, or smooth, manifold.*

Given an atlas for a manifold, a map of the form $\phi_\beta \circ \phi_\alpha^{-1}$ is called a *transition map* and is denoted by $\phi_{\alpha\beta}$.

**Example 1.2.3.** One immediate example of a smooth manifold is again $\mathbb{R}^n$ or open subsets of $\mathbb{R}^n$. These manifolds admit an atlas with a single chart. So the condition on transition maps is vacuously true.

**Example 1.2.4.** In Example 1.1.4 we saw that $\mathbb{S}^n$ is an $n$-manifold by exhibiting an atlas with two charts:

$$(\mathbb{S}^n \backslash (0, \dots, 0, 1), h), \quad (\mathbb{S}^n \backslash (0, \dots, 0, -1), h')$$

where

$$h(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n)$$

and

$$h'(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots, x_n).$$

Thus to show that $\mathbb{S}^n$ is a smooth manifold, we need to show that the transition maps $h' \circ h^{-1}$ and $h \circ (h')^{-1}$ are smooth. We will only check that

$h' \circ h^{-1}$ is smooth. (The case $h \circ (h')^{-1}$ is analogous.) Here

$$h^{-1}(y_1, \ldots, y_n) = \left( \frac{2y_1}{1 + y_1^2 + \cdots + y_n^2}, \ldots, \frac{2y_n}{1 + y_1^2 + \cdots + y_n^2}, \frac{-1 + y_1^2 + \cdots + y_n^2}{1 + y_1^2 + \cdots + y_n^2} \right),$$

$$h' \circ h^{-1}(y_1, \ldots, y_n) = \frac{1}{y_1^2 + \cdots + y_n^2}(y_1, \ldots, y_n).$$

It follows that $h' \circ h^{-1}$ is smooth except at the origin where the composition of maps is not defined. Thus $\mathbb{S}^n$ is a smooth manifold.

**Example 1.2.5.** In the exercises you proved that the product of manifolds is a manifold. Since the product of smooth maps is smooth, the product of smooth manifolds is a smooth manifold. It follows that $\mathbb{T}^n$ is a smooth manifold.

In calculus we learn about differentiable maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Some concepts extend to manifolds.

**Definition 1.2.6.** Let $M$ be a manifold with atlas $\{(M_\alpha, \phi_\alpha)\}$ and let $N$ be a manifold with atlas $\{(N_\beta, \psi_\beta)\}$. We say that the map $f : M \to N$ is $C^q$ if $\forall \alpha, \beta$, the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ (where it is defined) is $C^q$.

**Definition 1.2.7.** A $C^q$-map between $C^q$-manifolds with a $C^q$-inverse is called a $C^q$*-diffeomorphism.* A $C^\infty$-diffeomorphism is simply called a *diffeomorphism.*

**Remark 1.2.8.** The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a $C^\infty$-map but is not a diffeomorphism because its derivative is singular at 0. (In fact, it is not even a $C^1$-diffeomorphism.)

**Definition 1.2.9.** Two $C^q$-manifolds are considered *equivalent* if there is a $C^q$-diffeomorphism between them.

In the exercises, you will extend the notion of submanifold, manifold with boundary, and submanifold of a manifold with boundary to the DIFF category, that is, to the setting in which manifolds are considered in this section.

In the DIFF category we are interested in smooth maps between manifolds.

**Example 1.2.10.** Projection from $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ onto the second factor is a smooth map between manifolds.

In Appendix A, we introduce the notion of *transversality* in the category of DIFF manifolds. Another concept that is best described in this category is that of a Morse function. We discuss the concept in more detail in Appendix B but provide the basic definition here.

**Definition 1.2.11.** Let $M$ be a $C^q$-manifold for $q \geq 1$, $x \in M$, and $(M_\alpha, \phi_\alpha)$ a chart with $x \in M_\alpha$. We say that $x$ is a critical point of a function $f : M \to \mathbb{R}$ if it is a critical point of $f \circ \phi_\alpha^{-1}$.

**Definition 1.2.12.** A critical point of a function $g : \mathbb{R}^n \to \mathbb{R}$ is *non-degenerate* if the Hessian of $g$ is non-singular at $x$. For $M, x, (M_\alpha, \phi_\alpha), f$ as above, we say that $x$ is a *non-degenerate* critical point of $f$ if it is a non-degenerate critical point of $f \circ \phi_\alpha^{-1}$.

**Remark 1.2.13.** In the exercises, you will verify that these two definitions do not depend on the chart used.

**Definition 1.2.14.** A *Morse function* on a manifold $M$ is a smooth function $f : M \to \mathbb{R}$ that satisfies the following:

- $f$ has only non-degenerate critical points;
- distinct critical points of $f$ take on distinct values.

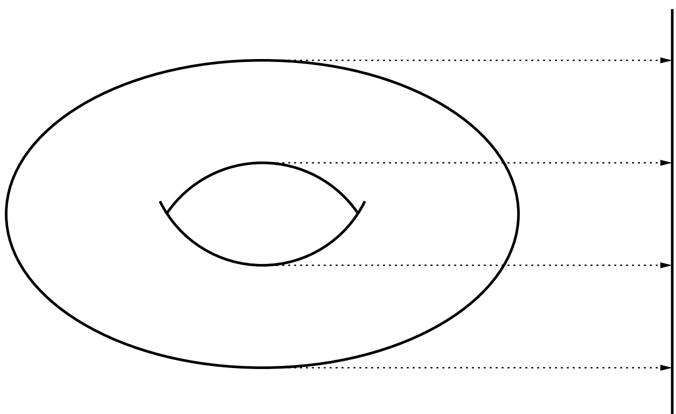

**Figure 1.10.** *Projection onto the $z$-axis.*

**Example 1.2.15.** The torus pictured in Figure 1.10 sits in $\mathbb{R}^3$ and projection onto the third coordinate defines a function. This function has four critical points: (1) a minimum; (2) two saddle points; (3) a maximum. You will verify in the exercises that all these critical points are non-degenerate. Since they occur at distinct levels, this function is a Morse function.

## Exercises

**Exercise 1.** Verify that the definitions of critical point and non-degenerate critical point for a function $f : M \to \mathbb{R}$ are independent of the chart used.

**Exercise 2.** Verify that the critical points of the function given in Example 1.2.15 are non-degenerate.

**Exercise 3.** Find other Morse functions on the torus by drawing other pictures representing the torus in $\mathbb{R}^3$.

**Exercise 4.** Compute the quantity #minima+#maxima−#saddles for the Morse function in Example 1.2.15 and the Morse functions from Exercise 3.

**Exercise 5.** Extend the notions of submanifold, manifold with boundary, and submanifold of a manifold with boundary to the DIFF category.

## 1.3. Oriented Manifolds

The notion of orientation allows us to partition manifolds into two types: orientable and non-orientable. We introduce the notion here in the DIFF category, though analogous notions exist in some other categories, most notably TOP (discussed above) and TRIANG (discussed below).

**Definition 1.3.1.** A $C^\infty$-manifold $M$ with boundary is *orientable* if it has an atlas such that the Jacobians of all transition maps have positive determinant. Otherwise $M$ is *non-orientable.* An *orientation* of $M$ is such an atlas. We often write $(M, \{\phi_\alpha\})$ to denote an oriented manifold.

**Example 1.3.2.** The annulus $\mathbb{A} = \mathbb{S}^1 \times (-1, 1)$ is orientable. It can be covered by two charts:

$$\left\{\left(\left\{(e^{ix}, t) : -\frac{\pi}{4} < x < \frac{5\pi}{4}, -1 < t < 1\right\}, \phi_1\right),\right.$$
$$\left.\left(\left\{(e^{ix}, t) : \frac{3\pi}{4} < x < \frac{9\pi}{4}, -1 < t < 1\right\}, \phi_2\right)\right\}$$

where $\phi_i((e^{ix}, t)) = (x, t)$. The transition maps are

$$\phi_2 \circ \phi_1^{-1}((x,t)) = \begin{cases} (x + 2\pi, t) & \text{if } -\frac{\pi}{4} < x < \frac{\pi}{4}, \\ (x, t) & \text{if } \frac{3\pi}{4} < x < \frac{5\pi}{4} \end{cases}$$

and

$$\phi_1 \circ \phi_2^{-1}((x,t)) = \begin{cases} (x - 2\pi, t) & \text{if } \frac{7\pi}{4} < x < \frac{9\pi}{4}, \\ (x, t) & \text{if } \frac{3\pi}{4} < x < \frac{5\pi}{4}. \end{cases}$$

Thus the Jacobians of all transition maps (where they are defined) have positive determinant.

**Example 1.3.3.** The Möbius band is the surface with boundary obtained by identifying two sides of a rectangle, as pictured in Figure 1.11. It is non-orientable. Intuitively speaking, this is because in an orientable surface, there is a well-defined notion of, for instance, “above”, or “being to the right". (You can make sense out of this in $\mathbb{R}^2$. The condition on Jacobians guarantees that the notion patches together correctly on overlapping charts.) Now consider the core curve of the Möbius band, as pictured in Figure 1.12, and imagine a curve "above" or "to the right" of the core curve. Such a curve does not exist, and so the Möbius band is non-orientable.


**Figure 1.11.** *The Möbius band.*

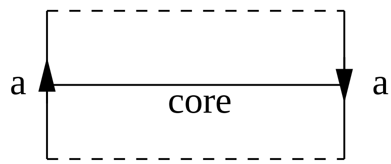


**Figure 1.12.** *The core of the Möbius band.*

Let $M$ be a differentiable manifold with two orientations $(M, \{\phi_\alpha\})$ and $(M, \{\psi_\beta\})$. The subset of $M$ on which $\phi_\alpha \circ \psi_\beta^{-1}$ is defined and has a Jacobian with a positive determinant is the subset where the orientations of $(M, \{\phi_\alpha\})$ and $(M, \{\psi_\beta\})$ are said to *coincide*. The subset where $\phi_\alpha \circ \psi_\beta^{-1}$ is defined and has a Jacobian with a negative determinant is the subset where the orientations of $(M, \{\phi_\alpha\})$ and $(M, \{\psi_\beta\})$ are said to *differ*. These two sets are open. Thus for a connected manifold $M$, two orientations either coincide on all of $M$ or differ on all of $M$.

Given an oriented manifold $(M, \{\phi_\alpha\})$, we can create an oriented manifold $(M, \{\psi_\alpha\})$ for which the orientations differ on all of $M$. Indeed, for each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, defined by $\phi_\alpha(x) = (x_1, x_2, \ldots, x_n)$, we substitute $\psi_\alpha : U_\alpha \to \mathbb{R}^n$, defined by $\psi_\alpha(x) = (-x_1, x_2, \ldots, x_n)$. The resulting orientation is called the *opposite* orientation of the original one. We denote the resulting oriented manifold by $-M$.

If $M$ is an oriented manifold with boundary, then there is a natural orientation on $\partial M$, called the *induced orientation of $\partial M$*. See for instance [**48**]. We will not discuss the technicalities here.

On a related note, given an oriented manifold $M$, we sometimes assign an orientation to an orientable submanifold. When the dimension of the submanifold is one less than the dimension of the manifold in which it lies, then choosing an orientation on the submanifold is equivalent to choosing a smoothly varying normal direction. For details see [**48**].

**Definition 1.3.4.** For oriented $C^\infty$-manifolds $(M, \{\phi_\alpha\}), (N, \{\psi_\beta\})$ of the same dimension, a smooth map $h : M \to N$ is *orientation-preserving* if the Jacobians of the maps $\psi_\beta \circ h \circ \phi_\alpha^{-1}$ (where they are defined) all have positive determinant. If they all have negative determinant it is *orientation-reversing.*

**Example 1.3.5.** The map $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $f(e^{2\pi ix}) = e^{4\pi ix}$ is orientation-preserving.

**Example 1.3.6.** The map $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $f(e^{2\pi ix}) = e^{-2\pi ix}$ is orientation-reversing.

In a non-orientable manifold $M$ each chart defines a local orientation. If $c$ is a closed 1-dimensional submanifold of $M$, then the transition maps for those charts of $M$ that meet $c$ may or may not have Jacobians with positive determinants. If there is an atlas for $M$ for which all charts that meet $c$ have transition maps whose Jacobians have positive determinants, then we say that $c$ is an *orientation-preserving* closed 1-dimensional submanifold of $M$. If there is no such atlas, then $c$ is an *orientation-reversing* closed 1-dimensional submanifold of $M$.

In the exercises, you will prove the following:

**Lemma 1.3.7.** *The manifold $M$ is non-orientable if and only if $M$ contains an orientation-reversing closed* 1*-dimensional submanifold.*

### Exercises

**Exercise 1.** Show that $\mathbb{S}^n$ is an orientable manifold.

**Exercise 2.** Show that the product of two orientable manifolds is an orientable manifold.

**Exercise 3.** Show that $\mathbb{T}^n$ is an orientable manifold.

**Exercise 4.** Prove Lemma 1.3.7.

## 1.4. Triangulated Manifolds

In this section we consider manifolds in the TRIANG category. The definitions that lay the foundation for this study (simplicial complexes and related notions) are discussed in greater detail in [**148**]. See also [**102**].

Note that in Definitions 1.4.12 and 1.4.13 we make a subtle departure from the traditional definitions. This is in line with a contemporary view of triangulations. For more detail, see Chapter 5, where we will reconcile the two notions.

**Definition 1.4.1.** Denote the $(k+1)$-tuple in $\mathbb{R}^{k+1}$ with $i$-th entry 1 and all other entries 0 by $v_i$. The set

$$\left\{ a_0 v_0 + \cdots + a_k v_k \ : \ a_0 \geq 0, \ldots, a_k \geq 0, \sum_{i=0}^{k} a_i = 1 \right\}$$

is called the *standard (closed) $k$-simplex* and is denoted by $[v_0, \ldots, v_k]$ or simply by $[s]$. The *dimension* of the standard $k$-simplex is $k$.

The set

$$\left\{ a_0 v_0 + \cdots + a_k v_k \ : \ a_0 > 0, \ldots, a_k > 0, \sum_{i=0}^{k} a_i = 1 \right\}$$

is called the *standard open $k$-simplex* and is denoted by $(v_0, \ldots, v_k)$ or simply by $(s)$. We also call $(s)$ the *interior* of $[s]$.

**Example 1.4.2.** The standard 0-simplex is the point $1 \in \mathbb{R}$. This is also the standard open 0-simplex.

**Definition 1.4.3.** A *$k$-simplex* in a topological space $X$ is a continuous map $f : [s] \to X$ such that $[s]$ is the standard $k$-simplex and the restriction $f|_{(s)}$ is a homeomorphism onto its image. An *open $k$-simplex* is the restriction of a $k$-simplex to the interior $(s)$ of $[s]$.

Abusing notation slightly, we will often refer to the image of $f$ as a $k$-simplex.

**Example 1.4.4.** The standard 1-simplex is homeomorphic to an interval. See Figures 1.13 and 1.14.

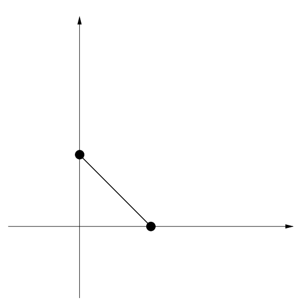

**Figure 1.13.** *The standard* 1*-simplex.*

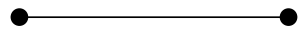

**Figure 1.14.** *A* 1*-simplex.*

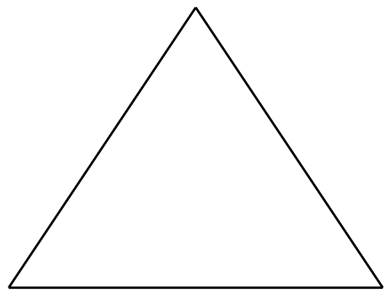

**Figure 1.15.** *A* 2*-simplex.*

**Example 1.4.5.** The standard 2-simplex is a "triangle" with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Figure 1.15 depicts a 2-simplex.

**Example 1.4.6.** The standard 3-simplex is a tetrahedron with vertices at $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Figure 1.16 depicts a 3-simplex.

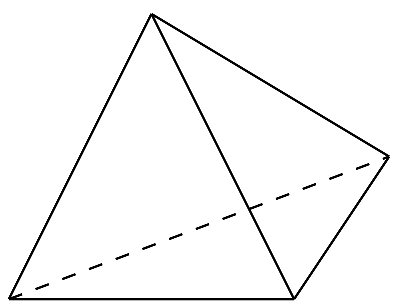

**Figure 1.16.** *A* 3*-simplex.*

**Definition 1.4.7** (Faces)**.** For $j = 0, \ldots, k$, a *face* of the standard $k$-simplex $[s]$ is a subset of $[s]$ of the form

$$\{a_0 v_0 + \cdots + a_k v_k : a_{i_1} = 0, \ldots, a_{i_j} = 0\}.$$

The *dimension* of the face is $k - j$. The faces of the $k$-simplex $f : [s] \to X$ are the restriction maps $f|_{[t]}$ for $[t]$ a face of the standard $k$-simplex $[s]$. A 0-dimensional face of a simplex is also called a *vertex*. A 1-dimensional face of a simplex is also called an *edge*.

**Example 1.4.8.** A 0-simplex has only itself as a face.

**Example 1.4.9.** The standard 1-simplex $[v_0, v_1]$ (and hence all 1-simplices) has itself as a 1-dimensional face. It also has two 0-dimensional faces, $[v_0]$ and $[v_1]$.

**Example 1.4.10.** A 2-simplex has one 2-dimensional, three 1-dimensional, and three 0-dimensional faces.

**Example 1.4.11.** A 3-simplex has one 3-dimensional, four 2-dimensional, six 1-dimensional, and four 0-dimensional faces.

**Definition 1.4.12** (Simplicial complex)**.** A *simplicial complex* based on the topological space $X$ is a set of simplices

$$K = \{f : [s] \to X\}$$

in the topological space $X$ such that

(1) $\forall$ simplices $f \in K$, all faces of $f$ are in $K$;

(2) $\forall$ simplices $f_1, f_2 \in K$, $\mathrm{im}(f_1|_{(s_1)}) \cap \mathrm{im}(f_2|_{(s_2)}) \neq \emptyset \implies \mathrm{im}(f_1|_{(s_1)}) = \mathrm{im}(f_2|_{(s_2)})$.

The *dimension* of a simplicial complex $K$ is the supremum of the dimensions of the simplices in $K$. We denote the union of the images of the simplices in $K$ by $|K|$ and call $|K|$ the *underlying space* of $K$.

For examples see Figures 1.17, 1.18, and 1.19. Traditionally, one also requires that each closed simplex be embedded and that two distinct closed simplices meet in at most one face. We do not make these assumptions here!

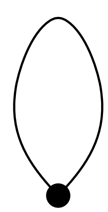

**Figure 1.17.** *A* 1*-dimensional simplicial complex.*

**Definition 1.4.13.** A *triangulated $n$-manifold* is a pair $(M, K)$, where $M$ is a topological $n$-manifold and $K$ is a simplicial complex based on $M$ such that

- $|K| = M$;
- $K$ is locally finite, i.e., for every compact subset $C$ of $M$, the set $\{f \in K : C \cap \mathrm{im}\, f \neq \emptyset\}$ is finite;

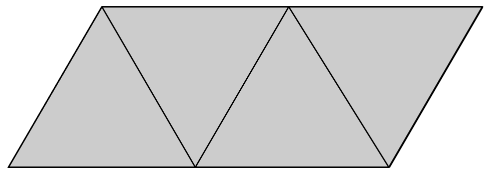

**Figure 1.18.** *A 2-dimensional simplicial complex.*

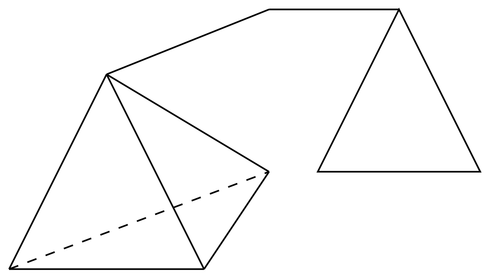

**Figure 1.19.** *A 3-dimensional simplicial complex.*

- for $f, g \in K$, restricted to open simplices, the map $g^{-1} \circ f$ is affine on its domain.

Here $K$ is called the *triangulation* of $M$. For $f : [s] \to M$ an $n$-simplex in $K$, the pair $(\operatorname{im} f, f^{-1})$ is called a *simplicial chart* of the triangulation. In this context, we often write $K = \{f_\alpha\}$ and denote the collection of all charts on $M$ by $\{(\operatorname{im} f_\alpha, f_\alpha^{-1})\}$. We also refer to $f_\alpha$ as a *simplex* of $M$.

There are related categories of manifolds (piecewise linear (PL), combinatorial) that we will not discuss here. Suffice it to say that some distinctions are subtle.

**Example 1.4.14.** Figure 1.20 depicts a triangulation of $\mathbb{S}^2$.

**Example 1.4.15.** Figure 1.21 depicts a triangulation of $\mathbb{T}^2$ with two 2-simplices.

**Example 1.4.16.** Figure 1.22 depicts a triangulated cube. It can be interpreted as a portion of a triangulation of $\mathbb{T}^3$. We obtain a triangulation of $\mathbb{T}^3$ by identifying eight appropriately chosen reflections of this cube. The resulting triangulation contains forty 3-simplices.

**Theorem 1.4.17.** *Every compact 1-manifold admits a triangulation.*

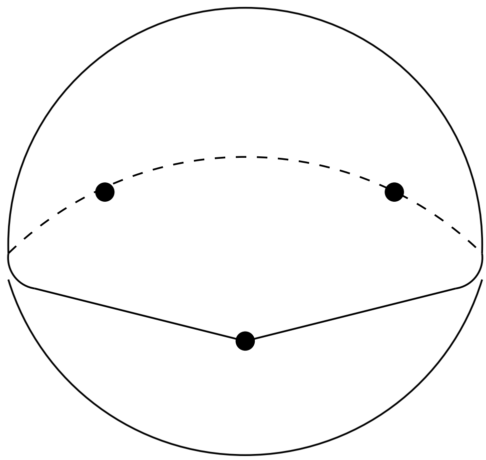

**Figure 1.20.** *A triangulation of* $\mathbb{S}^2$.

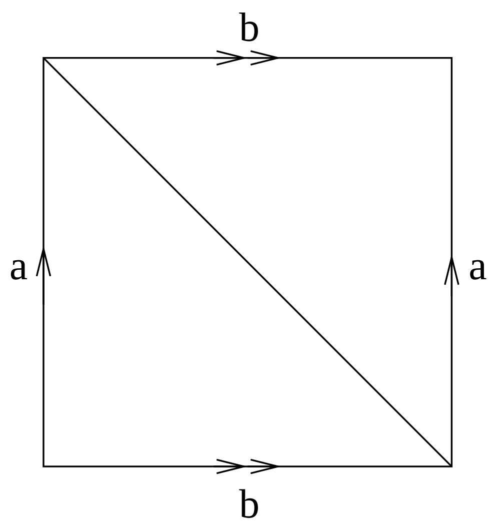


**Figure 1.21.** *A triangulation of* $\mathbb{T}^2$.

Proving the above theorem is an easy exercise in understanding compact 1-manifolds. Analogous theorems for 2- and 3-manifolds also hold but are much harder to prove.

**Theorem 1.4.18** (T. Radó, B. Kerekjarto)**.** *Every compact* 2*-manifold admits a triangulation.*

A proof can be found in [**4**].

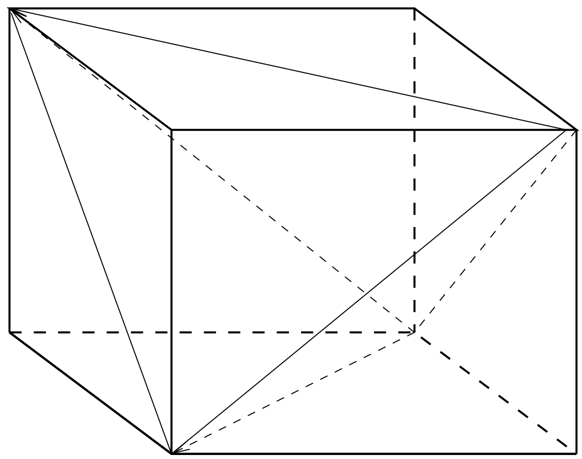

**Figure 1.22.** *A portion of a triangulation of* $\mathbb{T}^3$.

**Theorem 1.4.19** (R. H. Bing, E. Moise)**.** *Every compact* 3*-manifold admits a triangulation.*

A proof can be found in [**101**].

This theorem has been put to extensive use in the study of 3-manifolds. In view of Lemma 1.4.28 below, it allows us to build 3-manifolds by identifying 3-simplices along faces. In Chapter 5 we will see a brief introduction to normal surface theory (after Wolfgang Haken) where this point of view bears fruit.

**Definition 1.4.20.** Let $K$ and $L$ be simplicial complexes. We say that a continuous map $\phi : |K| \to |L|$ is a *simplicial map* if for every simplex $f$ in $K$, there is a simplex $g$ in $L$ such that $\phi \circ f = g$.

Informally speaking, a simplicial map maps simplices to simplices, but the latter might be of lower dimension than the former.

**Example 1.4.21.** Figure 1.23 depicts a simplicial map from the top simplicial complex (containing four triangles) to the bottom simplicial complex (containing two edges): The two vertices on the left go to the left vertex below, the two vertices in the middle go to the middle vertex below, and the two vertices on the right go to the right vertex below. Extend this map of vertices linearly over the 2-simplices to obtain a simplicial map.

**Non-Example.** A map that is not simplicial is depicted in Figure 1.24. It maps the top simplicial complex (containing four triangles) to the bottom simplicial complex (containing one edge) via nearest point projection (in $\mathbb{R}^2$).

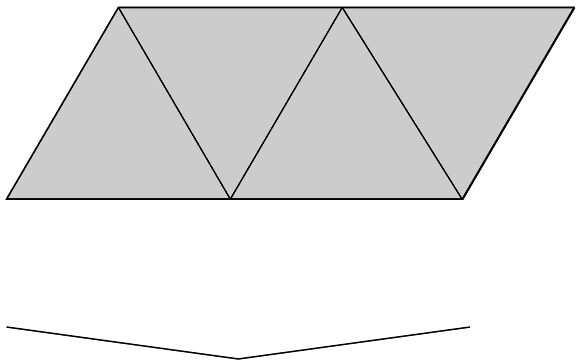

**Figure 1.23.** *A simplicial map.*

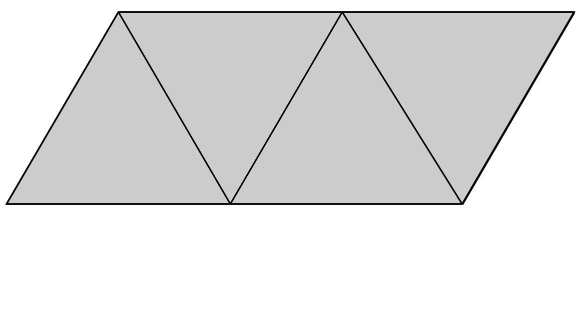

**Figure 1.24.** *A map between simplicial complexes that is not simplicial.*

**Definition 1.4.22.** Let $K_1, K_2$ be simplicial complexes and let $\phi : |K_1| \to |K_2|$ be a map. We say that $\phi$ is a *simplicial isomorphism* if it is simplicial and a homeomorphism. Two simplicial complexes $K_1, K_2$ are *isomorphic* if there is a simplicial isomorphism $\phi : |K_1| \to |K_2|$.

**Definition 1.4.23.** Let $(M_1, K_1)$ and $(M_2, K_2)$ be triangulated $n$-manifolds. The two triangulated $n$-manifolds are considered *equivalent* if there is a simplicial isomorphism $\phi : M_1 \to M_2$.

**Definition 1.4.24.** A *subcomplex* of a simplicial complex $K$ is a simplicial complex $L$ such that $f \in L$ implies $f \in K$. We write $L \subset K$.

**Example 1.4.25.** The unshaded area in Figure 1.25 depicts a subcomplex of the simplicial complex depicted in Figure 1.18.

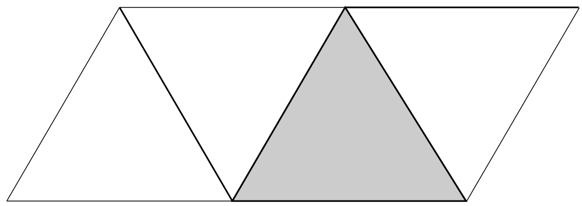

**Figure 1.25.** *A subcomplex of a simplicial complex.*

One concept that is easily defined in the TRIANG category is the following:

**Definition 1.4.26.** The *Euler characteristic* of a finite simplicial complex $K$ of dimension $k$ is computed via the following formula:

$$\chi(K) = \sum_{i=0}^{k} (-1)^i \#\{\text{simplices of dimension } i \text{ in } K\}.$$

**Remark 1.4.27.** In fact, Definition 1.4.26 is well-defined for the underlying space of a simplicial complex. This can be established via a computation involving the notion of homology from algebraic topology.

The standard $k$-simplex lies in $\mathbb{R}^{k+1}$ and is thus a topological space. Suppose that $(M, K)$ is a triangulated $k$-manifold and consider a collection $\mathcal{C}$ that is the disjoint union of copies of standard $k$-simplices, one copy for each $k$-simplex $f$ in $K$. Now construct a quotient space from $\mathcal{C}$ as follows: We identify points in the collection $\mathcal{C}$ if and only if the corresponding points $x \in [s]$ and $y \in [t]$ for $f : [s] \to M$ and $g : [t] \to M$ in $K$ satisfy $f(x) = g(y)$. We leave the proof of the following lemma as an exercise.

**Lemma 1.4.28.** *The quotient space $M'$ (in the quotient topology) is homeomorphic to the topological manifold $M$.*

The observation in Lemma 1.4.28 allows us to think of triangulated $n$-manifolds as objects that are constructed out of standard simplices by identifying faces.

**Example 1.4.29.** Figure 1.26 describes a triangulation of $\mathbb{S}^3$. The labeling gives the identification of the edges. 2-dimensional faces are identified if all three of their edges are identified.

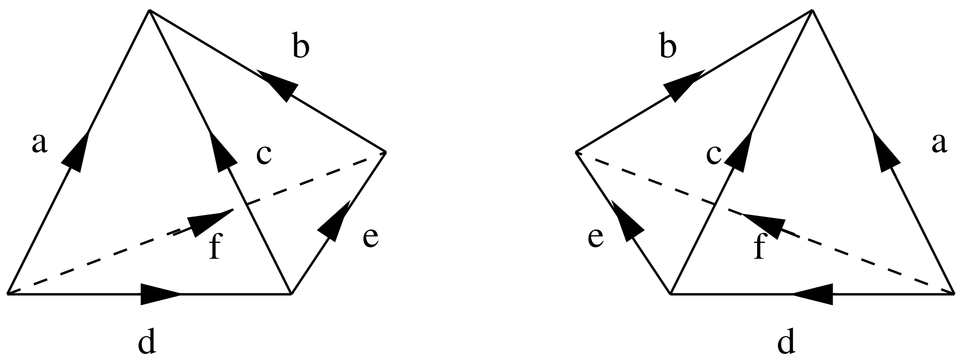


**Figure 1.26.** *A triangulation of* $\mathbb{S}^3$.

## Exercises

**Exercise 1.** Prove that a simplicial isomorphism has an inverse that is also a simplicial isomorphism.

**Exercise 2.** Prove Lemma 1.4.28. (Hint: The simplices in $M$ piece together to give a continuous bijection $h : M \to M'$. To prove that $h^{-1}$ is also continuous, use local finiteness of $K$ to establish that $h^{-1}$ is a closed map.)

**Exercise 3.** Find a triangulation of the projective plane.

**Exercise 4.** How many triangulations of the 2-torus can you find (e.g., 0, 1, $\infty$)?

**Exercise 5.** Prove that every compact 1-manifold admits a triangulation.

**Exercise 6.** Consider the triangulation of $\mathbb{S}^2$ in Figure 1.20. Try to add vertices and edges to create a triangulation such that any pair of 2-dimensional simplices meets in at most one edge.

**Exercise 7.** Compute the Euler characteristic of the sphere and of the torus by choosing triangulations.

## 1.5. Geometric Manifolds

Geometry has evolved since Euclid's time. Euclidean geometry and the other structures now also considered geometries provide insight into manifolds. We will assume a basic familiarity with Euclidean geometry in this section. In Chapter 7, we will develop the basics of hyperbolic geometry. While in this section we focus on Euclidean manifolds, the definitions carry over verbatim with the term "Euclidean" replaced by the term "hyperbolic" in order to define hyperbolic manifolds. For more on geometric 3-manifolds, see [**145**], [**154**], [**153**], or [**126**].

**Definition 1.5.1.** An *isometry* is an invertible map between metric spaces that preserves distances. A Euclidean isometry, i.e., an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$, is also called a *rigid motion*.

**Definition 1.5.2.** A *Euclidean manifold* is a topological manifold $M$ with an atlas that satisfies the additional requirement that the transition maps are restrictions of Euclidean isometries. We often write $(M, \{\phi_\alpha\})$ to denote a Euclidean manifold.

The more general term is that of a *geometric manifold*. There the additional requirement is that the transition maps are isometries in a given geometry.

**Example 1.5.3.** An immediate example of a Euclidean manifold is again $\mathbb{R}^n$. This manifold admits an atlas with a single chart. So the condition on transition maps is vacuously true.

**Example 1.5.4.** It follows from the description of $\mathbb{T}^n$ given in Example 1.1.5 that $\mathbb{T}^n$ is a Euclidean manifold.

**Definition 1.5.5.** Given two Euclidean manifolds $(M, \phi_\alpha), (N, \phi_\beta)$, a map $f : M \to N$ is an *isometry* if $\forall \alpha, \beta$ the maps $\phi_\alpha^{-1} \circ f \circ \psi_\beta$ are restrictions of Euclidean isometries.

**Definition 1.5.6.** Two Euclidean manifolds $M, N$ are considered *equivalent* if there is an invertible isometry $h : M \to N$.

**Remark 1.5.7.** On a connected Euclidean manifold there is a well-defined metric. Metric concepts such as completeness, diameter, volume, etc., are thus defined for Euclidean manifolds.

## Exercises

**Exercise 1.** Show that among Euclidean 2-manifolds there are infinitely many inequivalent 2-tori.

**Exercise 2.** The definition of Euclidean manifolds extends to manifolds with boundary. Show that among Euclidean manifolds with boundary there are infinitely many inequivalent annuli.

**Exercise 3.** Show that $\mathbb{S}^1$ is a Euclidean manifold.

**Exercise 4*.** Show that $\mathbb{S}^n$ is not a Euclidean manifold for $n > 1$.

## 1.6. Connected Sums

The following two notions are fundamental to the study of manifolds.

**Definition 1.6.1.** Two continuous maps $f_0, f_1 : M \to N$ are *homotopic* if there is a continuous map $H : M \times [0,1] \to N$ such that $H(x,0) = f_0$ and $H(x,1) = f_1(x)\ \forall x \in M$. The map $H$ is called a *homotopy* between $f_0$ and $f_1$.

**Definition 1.6.2.** Two embeddings $f_0, f_1 : M \to N$ are *isotopic* if there is a continuous map $H : M \times [0,1] \to N$ such that $H(x,0) = f_0$ and $H(x,1) = f_1(x)\ \forall x \in M$ and such that $\forall t \in [0,1]$, the map $f_t$ defined by $H(\ ,t)$ is an embedding. The map $H$ is called an *isotopy* between $f_0$ and $f_1$.

Two submanifolds $S_0, S_1$ of $M$ are *isotopic* if their inclusion maps are isotopic.

The following two theorems are due to Gugenheim (see [**47**]) in the TRIANG category and are also a consequence of the Isotopy Uniqueness of Regular Neighborhoods Theorem due to Rourke and Sanderson (see [**128**]). They are fundamental theorems in the study of $n$-manifolds. We will need these theorems in the discussion of Definition 1.6.5.

**Theorem 1.6.3.** *Every orientation-preserving homeomorphism of an $n$-ball or $n$-sphere is isotopic to the identity.*

We will prove a special case of Theorem 1.6.3, known as the Alexander Trick, in Section 2.5.

**Theorem 1.6.4.** *If $B_1, B_2$ are $n$-balls in the interior of a connected $n$-manifold $M$, then there is an isotopy $f : M \times I \to M$ such that $f(\ ,0)|_{B_1}$ is the identity and $f(\ ,1)|_{B_1}$ is a homeomorphism onto $B_2$.*

The proof of Theorem 1.6.4 is left as an exercise. Theorem 1.6.4 ensures that Definition 1.6.5 is well-defined. Orientations also play a subtle role in Definition 1.6.5.

**Definition 1.6.5.** Let $M_1, M_2$ be $n$-manifolds. Delete small open $n$-balls $B_1$ from $M_1$ and $B_2$ from $M_2$. Identify $M_1$ and $M_2$ along the resulting $(n-1)$-sphere boundary components. This results in an $n$-manifold called a *connected sum* of $M_1$ and $M_2$ and is denoted by $M_1 \# M_2$. In the case that $M_1$ and $M_2$ are oriented, we further require that the identification of the boundaries of $B_1$ and $B_2$ be via an orientation-reversing homeomorphism (with respect to the induced boundary orientations on $\partial B_1$, $\partial B_2$).

The connected sum $M_1 \# M_2$ is *trivial* if either $M_1$ or $M_2$ is $\mathbb{S}^n$. Otherwise, $M_1 \# M_2$ is *non-trivial*.

In deleting the small open $n$-balls, we are making a choice, but Theorem 1.6.4 ensures that this choice is inconsequential. If $M_1$, $M_2$ are oriented, then Theorem 1.6.3 tells us that any two choices of identification of $\partial B_1$ and $-\partial B_2$ are isotopic and it follows that the manifolds obtained via this identification are homeomorphic. Thus for oriented manifolds $M_1, M_2$, there is a unique connected sum (i.e., connected sum is well-defined in the oriented category).

For orientable (but not oriented) manifolds it is possible to have two non-homeomorphic connected sums of $M_1$ and $M_2$. Specifically, endow $M_1, M_2$ with orientations and consider $M_1\#M_2$ versus $M_1\#(-M_2)$. For examples of oriented 3-manifolds $M_1$, $M_2$ for which $M_1\#M_2 \neq M_1\#(-M_2)$, see [**63**].

If at least one of $M_1$, $M_2$, say $M_1$, is non-orientable, then there is, in fact, a unique connected sum $M_1\#M_2$. Indeed, this is because every non-orientable $n$-manifold $M$ contains an orientation-reversing closed 1-dimensional submanifold (by Lemma 1.3.7). Isotoping a small $n$-ball along such a 1-dimensional submanifold reverses the local orientation on an open neighborhood of the $n$-ball $B_1$ and hence also the induced boundary orientation of $(M_1\backslash B_1)\backslash\partial M_1$. This isotopy extends to an isotopy between the two possibilities for connected sums of $M_1$ and $M_2$, thereby demonstrating that they are homeomorphic.

In the case of surfaces, the situation is simpler. Every oriented surface $S$ admits an orientation-reversing homeomorphism $h : S \to S$. Thus for any pair $S_1$, $S_2$ of oriented surfaces, $S_1\#S_2 = S_1\#h(S_2) = S_1\# - S_2$. Hence for orientable surfaces, the choice of identifying homeomorphism is inconsequential and the connected sum is unique.

A word on notation: We denote repeated connected sums by $M = M_1\#\cdots\#M_n$. We leave it as an exercise to show that connected sum of manifolds is associative and commutative. (An example to consider is the following: The connected sum of four 3-manifolds $M_1, \ldots, M_4$ can be obtained in several ways. For instance, take the connected sum by removing three small 3-balls from $M_1$, one small 3-ball from $M_2, M_3, M_4$ and identifying the resulting boundary components of $M_1$ with those of $M_2, M_3, M_4$. On the other hand, take the connected sum by removing one small 3-ball from $M_1$ and $M_4$, two small 3-balls from $M_2$ and $M_3$ and identifying the resulting boundary component of $M_1$ with one of $M_2$'s, the resulting boundary component of $M_4$ with one of $M_3$'s, and the remaining resulting boundary components of $M_2$ and $M_3$ with each other. Figure 1.27 hints at why the repeated connected sum is nevertheless well-defined.)

**Definition 1.6.6** (Prime $n$-manifold)**.** An $n$-manifold $M$ is *prime* if $M = M_1\#M_2$ implies that either $M_1$ or $M_2$ is the $n$-sphere, i.e., that the connected sum $M = M_1\#M_2$ is trivial.

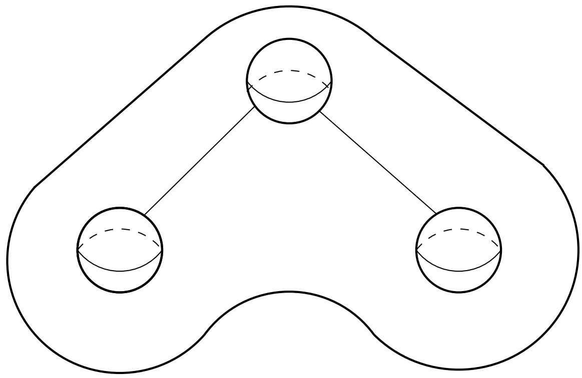

**Figure 1.27.** *The spheres in a prime decomposition.*

**Remark 1.6.7.** An $n$-manifold is prime if and only if it contains no separating essential $(n-1)$-sphere.

### Exercises

**Exercise 1.** Prove Theorem 1.6.4.

**Exercise 2.** Compute the Euler characteristic of $\mathbb{T}^2 \# \mathbb{R}P^2$.

**Exercise 3.** Prove that if $S_1, S_2$ are surfaces, then

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

**Exercise 4.** Prove that connected sum is associative and commutative.

**Exercise 5.** Is $\mathbb{S}^2 \times \mathbb{S}^1$ a non-trivial connected sum?

## 1.7. Equivalence of Categories

We have discussed manifolds from several points of view. The key idea, except in the TRIANG category, is the following: Choose a collection $G$ of invertible maps of $\mathbb{R}^n$ that is closed under group operations and require the transition maps to belong to $G$. The PL category is similar to the TRIANG category, but there are subtle differences that we do not wish to discuss here.

There are natural maps from all the categories considered here to TOP (in each case, it is the forgetful functor). Amazingly, for 2-manifolds and 3-manifolds, these maps are isomorphisms in the case of DIFF and TRIANG! This means that every topological 3-manifold admits a triangulation.

Furthermore, any two triangulations of a topological 3-manifold admit a common subdivision. Much of the work establishing the equivalence of the categories TOP and TRIANG is due to E. Moise and R. H. Bing. See [**102**] and [**101**].

It deserves to be mentioned that this equivalence of categories is not true for higher-dimensional manifolds. This was first shown by Milnor. He exhibited examples of 7-spheres with distinct differentiable structures, thus demonstrating that DIFF $\neq$ TOP for 7-dimensional manifolds. To date, it is unknown whether there are smooth structures on $S^4$ that are not diffeomorphic. A candidate for an "exotic" smooth structure on the 4-sphere was given by Scharlemann. See [**132**]. Decades later it was proved by Akbulut that this smooth structure is in fact diffeomorphic to the standard smooth structure on the 4-sphere. See [**5**].

The advantage of the equivalence of the categories TOP, DIFF, and TRIANG in the case of 3-manifolds is that we can introduce a concept or prove a theorem in the category that best suits the concept or theorem. For instance, we introduced the notion of a Morse function in the DIFF category and we introduced the notion of Euler characteristic in the TRIANG category. The notions can be introduced in the other categories as well, but the definitions are necessarily more cumbersome, so we will not do so here. We will be moving back and forth between categories when this is convenient. When categories are equivalent, it is acceptable, though not esthetically pleasing, to prove theorems using concepts introduced in distinct categories.

Recall, for instance, the definition of Euler characteristic in Section 1.4. We will use the notion of Euler characteristic in the other categories as well. For instance, given a smooth surface $S$, we can compute the Euler characteristic by triangulating the surface and taking the specified alternating sum. If $S$ contains a finite graph $\Gamma$ such that $S \backslash \Gamma$ consists of a finite number of disks, then the Euler characteristic can be calculated directly from $\Gamma$ and $S \backslash \Gamma$. You will prove this in the exercises.

We can compute the Euler characteristic not only for closed surfaces but also punctured surfaces. Specifically, since points have Euler characteristic 1, the result of removing $n$ points from a surface $S$ is $\chi(S) - n$. One theorem we will use repeatedly, stated here in the case of 2-manifolds, is the following:

**Theorem 1.7.1** (Poincaré-Hopf Index Theorem)**.** *Let $S$ be a surface and let $h : S \to \mathbb{R}$ be a Morse function. Then*

$$\chi(S) = \#(\textit{minima of } h) + \#(\textit{maxima of } h) - \#(\textit{saddles of } h).$$

For a proof of this therem, see [**48**].

## Exercises

**Exercise 1.** Show that, even in the case of closed 2-manifolds, the categories DIFF and GEOM are not equivalent.

**Exercise 2.** Prove that if a surface $S$ contains a finite graph $\Gamma$ with $v$ vertices and $e$ edges and $S \backslash \Gamma$ consists of $r$ disks, then

$$\chi(S) = v - e + r.$$

**Exercise 3.** Draw pictures to convince yourself that Theorem 1.7.1 is true.

**Exercise 4.** Explore how distinct structures on manifolds interact. For example, read [**126**].