Chapter 1

Simple Examples of Propagation

This chapter presents examples of wave propagation governed by hyperbolic equations. The ideas of propagation of singularities, group velocity, and short wavelength asymptotics are introduced in simple situations. The method of characteristics for problems in dimension $d = 1$ is presented as well as the method of nonstationary phase. The latter is a fundamental tool for estimating oscillatory integrals. The examples are elementary. They could each be part of an introductory course in partial differential equations, but often are not. This material can be skipped. If needed later, the reader may return to this chapter.[1] In sections 1.3, 1.5, and 1.6 we derive in simple situations the three basic laws of physical geometric optics.

Wave like solutions of partial differential equations have spatially localized structures whose evolution in time can be followed. The most common are solutions with propagating singularities and solutions that are modulated *wave trains* also called *wave packets*. They have the form

$$a(t,x)\, e^{i\phi(t,x)/\epsilon}$$

with smooth profile a, real valued smooth phase ϕ with $d\phi \neq 0$ on $\operatorname{supp} a$.

The parameter ϵ is a wavelength and is small compared to the scale on which a and ϕ vary. The classic example is light with a wavelength on the order of 5×10^{-5} centimeter. Singularities are often restricted to varieties of lower codimension, hence of width equal to zero, which is infinitely small compared to the scales of their other variations. Real world waves modeled

[1] Some ideas are used which are not formally presented until later, for example the Sobolev spaces $H^s(\mathbb{R}^d)$ and Gronwall's lemma.

by such solutions have their singular behavior spread over very small lengths, not exactly zero.

The path of a localized structure in space-time is curvelike, and such curves are often called *rays*. When phenomena are described by partial differential equations, linking the above ideas with the equation means finding solutions whose salient features are localized and in simple cases are described by transport equations along rays. For wave packets such results appear in an asymptotic analysis as $\epsilon \to 0$.

In this chapter some introductory examples are presented that illustrate propagation of singularities, propagation of energy, group velocity, and short wavelength asymptotics. That energy and singularities may behave very differently is a consequence of the dichotomy that, up to an error as small as one likes in energy, the data can be replaced by data with compactly supported Fourier transform. In contrast, up to an error as smooth as one likes, the data can be replaced by data with Fourier transform vanishing on $|\xi| \leq R$ with R as large as one likes. Propagation of singularities is about short wavelengths while propagation of energy is about wavelengths bounded away from zero. When most of the energy is carried in short wavelengths, for example the wave packets above, the two tend to propagate in the same way.

1.1. The method of characteristics

When the space dimension is equal to one, the method of characteristics reduces many questions concerning solutions of hyperbolic partial differential equations to the integration of ordinary differential equations. The central idea is the following. When $c(t,x)$ is a smooth real valued function, introduce the ordinary differential equation

$$\frac{dx}{dt} \;=\; c(t,x)\,. \tag{1.1.1}$$

Solutions $x(t)$ satisfy

$$\frac{dx(t)}{dt} \;=\; c(t,x(t)).$$

For a smooth function u,

$$\frac{d}{dt}\,u(t,x(t)) \;=\; \big(\partial_t u + c\,\partial_x u\big)\big|_{(t,x(t))}\,.$$

Therefore, solutions of the homogeneous linear equation

$$\partial_t u + c(t,x)\,\partial_x u \;=\; 0$$

are exactly the functions u that are constant on the integral curves $(t,x(t))$. These curves are called *characteristic curves* or simply *characteristics*.

Example 1.1.1. If $c \in \mathbb{R}$ is constant, then $u \in C^\infty([0,T] \times \mathbb{R})$ satisfies

$$\partial_t u + c\,\partial_x u \;=\; 0 \tag{1.1.2}$$

if and only if there is an $f \in C^\infty(\mathbb{R})$ so that $u = f(x - ct)$. The function f is uniquely determined.

Proof. For constant c the characteristics along which u is constant are the lines $(t, x + ct)$. Therefore, $u(t,x) = u(0, x - ct)$ proving the result with $f(x) := u(0,x)$. $\square$

This shows that the Cauchy problem consisting of (1.1.2) together with the initial condition $u|_{t=0} = f$ is uniquely solvable with solution $f(x - ct)$. The solutions are waves translating rigidly with velocity equal to c.

Exercise 1.1.1. *Find an explicit solution formula for the solution of the Cauchy problem*

$$\partial_t u + c\,\partial_x u + z(t,x)u \;=\; 0, \qquad u|_{t=0} = g,$$

where $z \in C^\infty$.

Example 1.1.2 (D'Alembert's formula)**.** If $c \in \mathbb{R} \setminus 0$, then $u \in C^\infty([0,T] \times \mathbb{R})$ satisfies

$$u_{tt} \;-c^2\; u_{xx} \;=\; 0 \tag{1.1.3}$$

if and only if there are smooth $f, g \in C^\infty(\mathbb{R})$ so that

$$u \;=\; f(x - ct) \;+\; g(x + ct)\,. \tag{1.1.4}$$

The set of all pairs $\tilde f, \tilde g$ so that this is so is of the form $\tilde f = f + b\,,\ \tilde g = g - b$ with $b \in \mathbb{C}$.

Proof. Factor

$$\partial_t^2 - c^2 \partial_x^2 \;=\; (\partial_t - c\partial_x)\,(\partial_t + c\partial_x) \;=\; (\partial_t + c\partial_x)\,(\partial_t - c\partial_x)\,.$$

Conclude that

$$u_+ \;:=\; \partial_t u - c\,\partial_x u \qquad \text{and} \qquad u_- \;:=\; \partial_t u + c\,\partial_x u$$

satisfy

$$\big(\partial_t \;\pm\; c\,\partial_x\big) u_\pm \;=\; 0\,. \tag{1.1.5}$$

Example 1.1.1 implies that there are smooth $F_\pm$ so that

$$u_\pm \;=\; F_\pm(x \mp ct)\,. \tag{1.1.6}$$

In order for (1.1.4) and (1.1.6) to hold, one must have

$$F_+ = (\partial_t - c\partial_x)u \;=\; (1 + c^2)f', \quad F_- = (\partial_t + c\partial_x)u \;=\; (1 + c^2)g'. \tag{1.1.7}$$

Thus if $G_\pm$ are primitives of $F_\pm$ that vanish at the origin, then one must have

$$f = \frac{G_+}{(1+c^2)} + C_+, \qquad g = \frac{G_+}{(1+c^2)} + C_-\,, \qquad C_+ + C_- = u(0,0)\,.$$

Reversing the process shows that if G_+, f, g are defined as above, then $\tilde{u} := f(x-ct) + g(x+ct)$ yields a solution of D'Alembert's equation with

$$(\partial_t \mp c\,\partial_x)\tilde{u} = F_\pm \qquad \text{so} \qquad (\partial_t \mp c\,\partial_x)(u-\tilde{u}) = 0\,.$$

Adding and subtracting this pair of equations shows that

$$\nabla_{t,x}(u - \tilde{u}) = 0.$$

Since $u(0,0) = \tilde{u}(0,0)$, it follows by connectedness of $[0,T]\times\mathbb{R}$ that $u = \tilde{u}$, and the proof is complete. $\square$

For speeds $c(t,x)$ that are not bounded, it is possible that characteristics escape to infinity with interesting consequences.

Example 1.1.3 (Nonuniqueness in the Cauchy problem)**.** Consider $c(t,x) := x^2$. The characteristic through $(0,x_0)$ is the solution of

$$x' = x^2, \qquad x(0) = x_0\,.$$

Then,

$$1 = \frac{x'}{x^2} = \frac{d}{dt}\left(\frac{-1}{x}\right).$$

Integrating from $t=0$ yields

$$\frac{-1}{x(t)} - \frac{-1}{x_0} = t \qquad \text{and, therefore,} \qquad x(t) = \frac{x_0}{1-x_0\,t}\,.$$

Through each point t,x with $t \geq 0$ there is a unique characteristic tracing backward to $t=0$. Therefore, given initial data $u(0,x) = g(x)$, the solution $u(t,x)$ is uniquely determined in $t \geq 0$ by requiring u to be constant on characteristics.

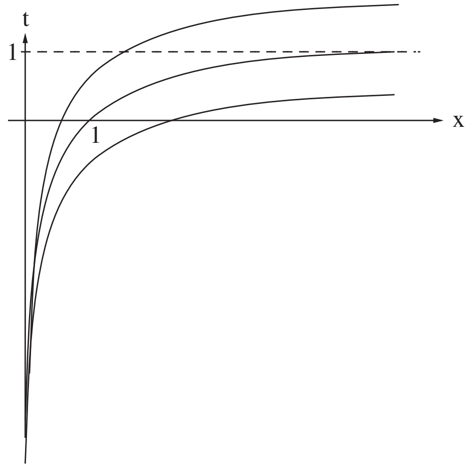

FIGURE 1.1.1. Characteristics diverge in finite time

As indicated in Figure 1.1.1, the characteristics through $(0, \pm 1)$ diverge to $\pm\infty$ at time $t = 1$. Thus all the backward characteristics starting in $t \geq 1$ meet $\{t = 0\}$ in the interval $]-1, 1[$. The data for $|x| \geq 1$ does not influence the solution in $t \geq 1$. There has been a loss of information. Another manifestation of this is that the initial values do not uniquely determine a solution in $t < 0$.

The characteristics starting at $t = 0$ meet $\{t = -1\}$ in the interval $]-1, 1[$. Outside that interval, the values of a solution are not determined, not even influenced by the initial data. There are many solutions in $t < 0$ which have the given Cauchy data. They are constant on characteristics which diverge to infinity, but their values on these characteristics is otherwise arbitrary.

To avoid this phenomenon we make the following assumption that not only prevents characteristics from diverging, but avoids some technical difficulties that occur for unbounded c with characteristics that do not diverge.

Hypothesis 1.1.1. *Suppose that for all $T > 0$*

$$\partial^\alpha_{t,x} c \ \in \ L^\infty([0, T] \times \mathbb{R})\,.$$

The coefficient $d(t, x)$ satisfies analogous bounds.

For arbitrary $f \in C^\infty(\mathbb{R}^2)$ and $g \in C^\infty(\mathbb{R})$, there is a unique solution of the Cauchy problem

$$\Big(\partial_t \ + \ c(t, x)\, \partial_x \ + \ d(t, x)\Big) u \ = \ f, \qquad u(0, x) \ = \ g.$$

Its values along the characteristic $(t, x(t))$ are determined by integrating the nonhomogeneous linear ordinary differential equation

$$\frac{d}{dt}\, u(t, x(t)) + d(t, x(t))\, u(t, x(t)) \;=\; f(t, x(t))\,. \tag{1.1.8}$$

There are finite regularity results too. If f, g are k times differentiable with $k \geq 1$, then so is u. Though the equation is first order, u is in general not smoother than f. This is in contrast to the elliptic case.

The method of characteristics also applies to systems of hyperbolic equations. Consider vector valued unknowns $u(t, x) \in \mathbb{C}^N$. The simplest generalization is diagonal real systems

$$u_t \;+\; \operatorname{diag}(c_1(t, x), \ldots, c_N(t, x))\, u \;=\; 0.$$

Here u_j is constant on characteristics with speed $c_j(t, x)$. This idea extends to some systems

$$L \;:=\; \partial_t \;+\; A(t, x)\, \partial_x \;+\; B(t, x),$$

where A and B are smooth matrix valued functions so that

$$\forall T,\ \forall \alpha, \quad \partial^\alpha_{t,x}\{A, B\} \;\in\; L^\infty([-T, T] \times \mathbb{R})\,.$$

The method of characteristics applies when the following hypothesis is satisfied. It says that the matrix A has real eigenvalues and is smoothly diagonalizable. The real spectrum as well as the diagonalizability are related to a part of the general theory of constant coefficient hyperbolic systems sketched in the Appendix 2.I to Chapter 2.

Hypothesis 1.1.2. *There is a smooth matrix valued function, $M(t, x)$, so that*

$$\forall T,\ \forall \alpha, \quad \partial^\alpha_{t,x} M \ \ \textit{and}\ \ \partial^\alpha_{t,x}(M^{-1}) \ \ \textit{belong to}\ \ L^\infty([0, T] \times \mathbb{R})$$

and

$$M^{-1}\, A\, M \;=\; \textit{diagonal and real.} \tag{1.1.9}$$

Examples 1.1.4. 1. The hypothesis is satisfied if for each t, x the matrix A has N distinct real eigenvalues $c_1(t, x) < c_2(t, x) < \cdots < c_N(t, x)$. Such systems are called *strictly hyperbolic.* To guarantee that the estimates on M, M^{-1} are uniform as $|x| \to \infty$, it suffices to make the additional assumption that

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}} \ \min_{2 \leq j \leq N} \ c_j(t, x) \;-\; c_{j-1}(t, x) \;>\; 0\,.$$

2. More generally the hypothesis is satisfied if for each (t, x), A has uniformly distinct real eigenvalues and is diagonalizable. It follows that the multiplicity of the eigenvalues is independent of t, x.

3. If A_1 and A_2 satisfy Hypothesis 1.1.2. then so does

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad \text{with} \qquad M \ := \ \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

In this way one constructs examples with variable multiplicity.

Exercise 1.1.2. *Prove assertions* **1** *and* **2**.

Since $A = MDM^{-1}$ with $D = \text{diag}\,(c_1, \dots, c_N)$,

$$L \ = \ \partial_t + MDM^{-1}\partial_x + B\,.$$

Define v by $u = Mv$ so

$$M^{-1}Lu \ = \ M^{-1}\Big(\partial_t \ + \ MDM^{-1}\partial_x \ + \ B\Big)M\,v\,.$$

When the derivatives on the right fall on v, the product $M^{-1}M = I$ simplifies. This shows that

$$M^{-1}Lu \ = \ \Big(\partial_t + D\,\partial_x + \widetilde{B}\Big)u \ := \ \widetilde{L}\,v,$$

where

$$\widetilde{B} \ := \ M^{-1}\,B\,M \ + \ M^{-1}\,M_t \ + \ M^{-1}\,A\,M_x\,.$$

This change of variable converts the equation $Lu = f$ to $\widetilde{L}v = M^{-1}f$ where $\widetilde{L}$ has the same form as L but with leading part that is a set of directional derivatives.

Theorem 1.1.1. *Suppose that Hypothesis* 1.1.2 *is satisfied,* $k \geq 1$*,* $f \in C^k([0,T] \times \mathbb{R})$*,* $g \in C^k(\mathbb{R})$*, and for all* α, β *with* $|\alpha| \leq k$ *and* $|\beta| \leq k$*,*

$$\partial^\alpha_{t.x} f \ \in \ L^\infty([0,T] \times \mathbb{R}) \qquad \textit{and} \qquad \partial^\beta_x g \ \in \ L^\infty([0,T] \times \mathbb{R})\,.$$

Then there is a unique solution $u \in C^k([0,T]\times\mathbb{R})$ *to the initial value problem* $Lu = f$*,* $u|_{t=0} = g$ *so that all partial derivatives of* u *of order* $\leq k$ *are in* $L^\infty([0,T] \times \mathbb{R})$*.*

The crux is the following estimate called *Haar's inequality.* For a vector valued function $w(x) = (w_1(x), \dots, w_N(x))$ on $\mathbb{R}$, the L^∞ norm is taken to be

$$\|w\|_{L^\infty(\mathbb{R})} \ := \ \max_{1 \leq j \leq N} \|w_j(x)\|_{L^\infty(\mathbb{R})}\,.$$

Haar's Lemma 1.1.2. *Suppose that Hypothesis* 1.1.2 *is satisfied.*

i. *There is a constant* $C = C(T, L)$ *so that if* u *and* Lu *are bounded continuous functions on* $[0,T] \times \mathbb{R}$*, then for* $t \in [0,T]$

$$\|u(t)\|_{L^\infty(\mathbb{R})} \ \leq \ C\left(\|u(0)\|_{L^\infty(\mathbb{R})} \ + \ \int_0^t \|Lu(\sigma)\|_{L^\infty(\mathbb{R})}\,d\sigma\right).$$

ii. *More generally, there is a constant* $C(k,T,L)$ *so that if for all* $|\alpha|\le k$, $\partial^\alpha_{t,x}u$ *and* $\partial^\alpha_{t,x}Lu$ *are bounded continuous functions on* $[0,T]\times\mathbb{R}$, *then*

$$m_k(u,t) \ := \ \sum_{|\alpha|\le k} \|\partial^\alpha_{t,x}u(t)\|_{L^\infty(\mathbb{R})}$$

satisfies for $t\in[0,T]$,

$$m_k(u,t) \ \le \ C\left(m_k(u,0) \ + \ \int_0^t m_k(Lu,\sigma)\,d\sigma\right).$$

Proof of Lemma 1.1.2. The change of variable shows that it suffices to consider the case of a real diagonal matrix $A = \operatorname{diag}(c_1(t,x),\dots,c_N(t,x))$.

i. For $\underline{t}\in[0,T]$ and $\epsilon>0$ choose j and $\underline{x}$ so that

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \ \le \ \|u_j(\underline{t},\underline{x})\|_{L^\infty(\mathbb{R})} \ + \ \epsilon\,.$$

Choose $(t,x(t))$ the integral curve of $x'=c_j(t,x)$ passing through $\underline{t},\underline{x}$. Then

$$u_j(\underline{t},\underline{x}) \ = \ u_j(0,x(0)) \ + \ \int_0^{\underline{t}} (\partial_t + c_j(t,x)\partial_x)u_j(\sigma,x(\sigma))\ d\sigma\,.$$

Therefore

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \ \le \ \|u(0)\|_{L^\infty(\mathbb{R})} \ + \ \int_0^{\underline{t}} \|(Lu \ - \ B\,u)(\sigma)\|_{L^\infty(\mathbb{R})}\ d\sigma \ + \ \epsilon.$$

Since this is true for all ϵ, one has

$$\|u(\underline{t})\|_{L^\infty(\mathbb{R})} \ \le \ \|u(0)\|_{L^\infty(\mathbb{R})} \ + \ \int_0^{\underline{t}} \|(Lu)(\sigma)\|_{L^\infty(\mathbb{R})} \ + \ C\,\|u(\sigma)\|_{L^\infty(\mathbb{R})}\ d\sigma\,,$$

and **i** follows using Gronwall's Lemma 2.1.3.

ii. Apply the inequality of **i** to $\partial^\alpha_{t,x}u$ with $|\alpha|\le k$.

$$\|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R})} \ \le \ C\Big(\|\partial^\alpha u(0)\|_{L^\infty(\mathbb{R})} \ + \ \int_0^t \|L\partial^\alpha u(\sigma)\|_{L^\infty(\mathbb{R})}\Big)\,d\sigma.$$

Compute

$$L\,\partial^\alpha u \ = \ \partial^\alpha Lu \ + \ [L,\partial^\alpha]\,u\,.$$

The commutator is a differential operator of order k with bounded coefficients, so

$$\|[L,\partial^\alpha]\,u(\sigma)\|_{L^\infty(\mathbb{R})} \ \le \ C\,m_k(u,\sigma)\,.$$

Therefore,

$$\|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R})} \le C\Big(\|\partial^\alpha u(0)\|_{L^\infty(\mathbb{R})} + \int_0^t \|\partial^\alpha Lu(\sigma)\|_{L^\infty(\mathbb{R})} \ + \ m_k(u,\sigma)\ d\sigma\Big).$$

Sum over $|\alpha|\le k$ to find

$$m_k(u,t) \le C\Big(m_k(u,0) + \int_0^t m_k(u,\sigma) \ + \ m_k(Lu,\sigma)\,d\sigma\Big)\,.$$

Gronwall's lemma implies **ii.** ◻

Proof of Theorem 1.1.1. The change of variable shows that it suffices to consider the case of $A = \mathrm{diag}(c_1, \dots, c_N)$.

The solution u is constructed as a limit of approximate solutions u^n. The solution u^0 is defined as the solution of the initial value problem

$$\partial_t u^0 \ + \ A\,\partial_x u^0 \ = \ f, \qquad u^0|_{t=0} = g\,.$$

The solution is explicit by the method of characteristics. It is C^k with bounded derivatives on $[0,T] \times \mathbb{R}$, so

$$\exists C_1, \quad \forall t \in [0,T], \quad m_k(u^0, t) \ \le \ C_1\,. \tag{1.1.10}$$

For $n > 0$ the solution u^n is again explicit by the method of characteristics in terms of u^{n-1},

$$\partial_t u^n \ + \ A\,\partial_x u^n \ + \ B\,u^{n-1} \ = \ f\,, \qquad u^{n-1}|_{t=0} = g\,. \tag{1.1.11}$$

Using (1.1.10) and Haar's inequality yields,

$$\exists C_2, \quad \forall t \in [0,T], \quad m_k(u^1, t) \ \le \ C_2\,. \tag{1.1.12}$$

For $n \ge 2$ estimate $u^n - u^{n-1}$ by applying Haar's inequality to

$$\tilde{L}\,(u^n - u^{n-1}) \ + \ B\,(u^{n-1} - u^{n-2}) \ = \ 0, \qquad (u^n - u^{n-1})\big|_{t=0} \ = \ 0\,,$$

to find

$$m_k(u^n - u^{n-1}, t) \ \le \ C \int_0^t m_k(u^{n-1} - u^{n-2}, \sigma)\; d\sigma\,. \tag{1.1.13}$$

For $n = 2$, this together with (1.1.10) and (1.1.12) yields

$$m_k(u^2 - u^1, t) \ \le \ (C_1 + C_2)Ct\,.$$

Injecting this in (1.1.13) yields

$$m_k(u^3 - u^2, t) \ \le \ (C_1 + C_2)C^2 t^2/2\,.$$

Continuing yields

$$m_k(u^n - u^{n-1}, t) \ \le \ (C_1 + C_2)C^{n-1} t^{n-1}/(n-1)!\,. \tag{1.1.14}$$

The summability of the right-hand side implies (Weierstrass's M-test) that u^n and all of its partials of order $\le k$ converge uniformly on $[0,T] \times \mathbb{R}$. The limit u is C^k with bounded partials. Passing to the limit in (1.1.11) shows that u solves the initial value problem.

To prove uniqueness, suppose that u and v are solutions. Haar's inequality applied to $u - v$ implies that $u - v = 0$. ◻

The proof also yields finite speed of propagation of signals. Define $\lambda_{\min}(t,x)$ and $\lambda_{\max}(t,x)$ to the smallest and largest eigenvalues of $A(t,x)$. Then the functions λ are uniformly Lipschitzean on $[0,T]\times\mathbb{R}$. To prove this, observe that equation (1.1.9) shows that the diagonal elements $c_j(t,x)$ of the right-hand side are the eigenvalues of A. Their expression by the left-hand side shows that their partial derivatives of first order (in fact any order) are bounded on $[0,T]\times\mathbb{R}$. Therefore the c_j are uniformly Lipschitzean on $[0,T]\times\mathbb{R}$. It follows that $\lambda_{\min} := \min_j \{c_j\}$ is uniformly Lipschitzean.

The characteristics have speeds bounded below by $\lambda_{\min}$ and above by $\lambda_{\max}$. The next result shows that these are lower and upper bounds, respectively, for the speeds of propagation of signals.

Corollary 1.1.3. *Suppose that $-\infty < x_l < x_r < \infty$ and γ_l (resp. γ_r) is the integral curve of $\partial_t + \lambda_{\min}(t,x)\partial_x$ (resp. $\partial_t + \lambda_{\max}(t,x)\partial_x$) passing through x_l (resp. x_r). Denote by Q the four sided region in $0 \le t \le T$ bounded on the left by γ_l and the right by γ_r. If g is supported in $[x_l, x_r]$ and $f\big|_{0\le t\le T}$ is supported in Q, then for $0 \le t \le T$ the solution u is supported in Q.*

Proof. The explicit formulas of the method of characteristics show that the approximate solutions u^n are supported in Q. Passing to the limit proves the result. $\square$

Consider next the case of $f = 0$ and $g \in C^1(\mathbb{R})$ whose restrictions to $]-\infty, \underline{x}[$ and $]\underline{x}, \infty[$ are each smooth with uniformly bounded derivatives of every order. Such a function is called *piecewise smooth*.

The simplest case is that of an operator $\partial_t + c(t,x)\partial_x$. Denote by γ the characteristic through $\underline{x}$. The values of u to the left of γ are determined by g to the left of $\underline{x}$. Choose a $\tilde g \in C^\infty(\mathbb{R})$ which agrees with g to the left and has bounded derivatives of all orders. The solution $\tilde u$ then agrees with u to the left of γ and $\tilde u$ has bounded partials of all orders for $0 \le t \le T$. An analogous argument works for the right-hand side, and one sees that u is piecewise C^∞ in the decomposition of $[0,T]\times\mathbb{R}$ into two pieces by γ.

Suppose now that A satisfies Hypothesis 1.1.2, and for all $(t,x) \in [0,T]\times\mathbb{R}$ has N distinct real eigenvalues ordered so that $c_j < c_{j+1}$. Denote by γ_j the corresponding characteristics through $\underline{x}$. Define open wedges,

$$W_1 \;:=\; \big\{(t,x)\;:\; 0<t<T, \quad -\infty < x < \gamma_1(t)\big\},$$

$$W_{N+1} \;:=\; \big\{(t,x)\;:\; 0<t<T, \quad \gamma_N(t) < x < \infty\big\},$$

and for $2 < j < N$,

$$W_j \;:=\; \big\{(t,x)\;:\; 0<t<T, \quad \gamma_{j-1}(t) < x < \gamma_j(t)\big\}.$$

They decompose $[0,T]\times\mathbb{R}$ into pie shaped wedges with vertex at $(0,\underline{x})$ and numbering from left to right.

Definition 1.1.4. *For $\mathbb{Z} \ni k \geq 1$, the set PC^k consists of functions which are piecewise C^k as the set of bounded continuous functions u on $[0,T] \times \mathbb{R}$ so that for $\alpha \leq k$ and $1 \leq j \leq N$, the restriction $u|_{W_j}$ belongs to $C^k(W_j)$ and for all $\alpha \leq k, \partial^\alpha(u|_{W_j})$ extends to a bounded continuous function on the closure $\overline{W}_j$. It is a Banach space with the norm*

$$\|u\|_{L^\infty([0,T]\times\mathbb{R})} \ + \sum_{|\alpha|\leq k} \ \sum_{1\leq j\leq N+1} \left\|\partial^\alpha_{t,x}\big(u\big|_{W_j}\big)\right\|_{L^\infty(W_j)}.$$

The next result asserts that for piecewise smooth data with singularity at $\underline{x}$, the solution is piecewise smooth with its singularities restricted to the characteristics through $\underline{x}$.

Theorem 1.1.5. *Suppose in addition to Hypothesis 1.1.2, that A has N distinct real eigenvalues for all (t,x). If $f \in PC^k$ and $g \in L^\infty(\mathbb{R})$ have bounded continuous derivatives up to order k on each side of $\underline{x}$, then the solution u belongs to PC^k.*

Sketch of Proof. The construction of u yielded an $L^\infty([0,T]\times\mathbb{R})$ estimate. In addition we need estimates for the derivatives of order $\leq k$ on the wedge W_j. Introduce

$$\mu_k(u,\sigma) \ := \ \|u(\sigma)\|_{L^\infty(\mathbb{R})} \ + \sum_{2\leq|\alpha|\leq k} \ \sum_{1\leq j\leq N+1} \left\|\partial^\alpha_{t,x}\big(u\big|_{W_j}\big)(\sigma)\right\|_{L^\infty(W_j\cap\{t=\sigma\})}.$$

To estimate $u^n - u^{n-1}$ use the following lemma. $\square$

Lemma 1.1.6. *Assume the hypotheses of the theorem and that $c_j(t,x)$ is one of the eigenvalues of $A(t,x)$. Then, there is a constant $C(j,T,L)$ so that if $f \in PC^k$ and*

$$\big(\partial_t \ + \ c_j(t,x)\,\partial_x\big)w \ = \ f, \qquad w\big|_{t=0} \ = \ 0,$$

then $w \in PC^k$ and

$$\mu_k(w,t) \ \leq \ C\left(\mu_k(w,0) \ + \ \int_0^t \mu_k(f,\sigma)\ d\sigma\right).$$

Exercise 1.1.3. *Prove the lemma. Then finish the proof of the theorem.*

Exercise 1.1.4. *Suppose that u is as in the theorem, $f = 0$, and that for some $\epsilon > 0$ and j, the derivatives of u of order $\leq k$ are continuous across $\gamma_j \cap \{0 \leq t < \epsilon\}$. Prove that they are continuous across $\gamma_j \cap \{0 \leq t \leq T\}$.*
Hints. Show that the set of times $\underline{t}$ for which the solution is C^k on $\gamma_j \cap \{0 \leq t \leq \underline{t}\}$ is both open and closed. Use finite speed.

Denote by $\Phi_j(t,x)$ the flow of the ordinary differential equation $x' = c_j(t,x)$. That is $x(t) = \Phi_j(t,\underline{x})$ is the solution with $x(0) = \underline{x}$. The solution operator for the pure transport equation $(\partial_t + c_j \partial_x)u = 0$ with initial value g is then

$$u(t) \ = \ g(\Phi_j(-t,x))\,.$$

The values at time t are the rearrangements by the diffeomorphism $\Phi(-t,\cdot)$ of the initial function. Because of the uniform boundedness of the derivatives of c_j on slabs $[0,T]\times\mathbb{R}$, one has

$$\partial^\alpha_{t,x}\Phi \ \in \ L^\infty([0,T]\times\mathbb{R})\,.$$

The derivative $\partial_x\Phi$ measures the expansion or contraction by the flow. It is the length of the image of an infinitesimal interval divided by the original length. In particular Φ can at most expand lengths by a bounded quantity. The inverse of $\Phi(t,\cdot)$ is the flow by the ordinary differential equation from time t to time 0, so the inverse also cannot expand by much. This is equivalent to a lower bound,

$$\big(\partial_x\Phi\big)^{-1} \ \in \ L^\infty([0,T]\times\mathbb{R})\,.$$

The diffeomorphism $\Phi(t,\cdot)$ can neither increase nor decrease length by much. Therefore the maps $u(0)\mapsto u(t)$ are uniformly bounded maps from $L^p(\mathbb{R})$ to itself for all $p\in[1,\infty]$. The case $p=\infty$ is equivalent to the Haar inequalities. There are analogous estimates

$$\|u(t)\|_{L^p(\mathbb{R})} \ \le \ C\left(\|u(t)\|_{L^p(\mathbb{R})} \ + \ \int_0^t \|Lu(\sigma)\|_{L^p(\mathbb{R})}\,d\sigma\right),$$

with constant independent of p. This in turn leads to an existence theory like that just recounted but $m_k(u,t)$ is replaced by $\sum_{|\alpha|\le k}\|\partial^\alpha_{t,x}u(t)\|_{L^p(\mathbb{R})}$. For these one dimensional hyperbolic Cauchy problems, there is a wide class of spaces for which the evolution is well posed. The case of $p=1$ is particularly important for the theory of shock waves in $d=1$. Brenner's Theorem 3.3.5 shows that only the case $p=2$ remains valid for typical hyperbolic equations in dimension $d>1$.

1.2. Examples of propagation of singularities using progressing waves

D'Alembert's solution (see Example 1.1.2) of the one-dimensional wave equation,

$$u_{tt} - u_{xx} = 0\,, \tag{1.2.1}$$

is the sum of progressing waves

$$f(x-t) + g(x+t)\,. \tag{1.2.2}$$

The rays are the integral curves of

$$\partial_t \pm \partial_x . \tag{1.2.3}$$

Structures are rigidly transported at speeds ± 1.

There is an energy law. If u is a smooth solution whose support intersects each time slab $a \le t \le b$ in a compact set, one has

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u_t^2 + u_x^2 \, dx \;&=\; \int \partial_t (u_t^2 + u_x^2)\, dx \\ &=\; \int 2u_t(u_{tt} - u_{xx}) + \partial_x(2u_t\, u_x)\; dx \;=\; 0\,, \end{aligned}$$

since the first summand vanishes and the second is the x derivative of a function vanishing outside a compact set.

The fundamental solution that solves (1.2.3) together with the initial values

$$u(0,x) = 0\,, \qquad u_t(0,x) = \delta(x)\,, \tag{1.2.4}$$

is given by the explicit formula

$$u(t,x) \;=\; \frac{\operatorname{sgn} t}{2}\, \chi_{[-t,t]} \;=\; \frac{1}{2}\, \Big(h(x-t) - h(x+t) \Big)\,, \tag{1.2.5}$$

where h denotes Heaviside's function, the characteristic function of $]0,\infty[$.

Exercise 1.2.1. **i.** *Derive (1.2.5) by solving the initial value problem using the Fourier transform in x.* **Hint.** You will likely decompose an expression regular at $\xi = 0$ into two that are not. Use a principal value to justify this step.

ii. *The proof of D'Alembert's formula (1.2.2) shows that every distribution solution of (1.2.1) is given by (1.2.2) for f, g distributions on $\mathbb{R}$. Derive (1.2.5) by finding the f, g that yield the solution of (1.2.4).* **Hint.** You will need to find the solutions of $du/dx = \delta(x)$.

The singularities of the solution (1.2.5) lie on the characteristic curves through $(0,0)$. This is a consequence of Theorem 1.1.4. In fact, define v as the solution of

$$v_{tt} - v_{xx} = 0, \qquad v(0,x) = 0, \qquad v_t(0,x) = x_+^2/2, \qquad x_+ := \max\{x, 0\}\,.$$

Introduce

$$V \;:=\; (v_1, v_2, v), \qquad v_1 := \partial_t v - \partial_x v, \quad v_2 := \partial_t v + \partial_x v$$

to find

$$\partial_t V \;+\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \partial_x V \;+\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V \;=\; 0\,.$$

The Cauchy data, $V(0,x)$ are continuous, piecewise smooth, and singular only at $x = 0$. Theorem 1.1.4 shows that V is piecewise smooth with singularities only on the characteristics through $(0,0)$. In addition $u = \partial_x^3 v$ (in the sense of distributions) since they both satisfy the same initial value problem. Thus v and $u = \partial_x^3 v$ have singular support only on the characteristics through $(0,0)$.

Interesting things happen if one adds a lower order term. For example, consider the Klein–Gordon equation

$$u_{tt} - u_{xx} + u = 0\,. \tag{1.2.6}$$

In sharp contrast with equation (1.2.2), there are hardly any undistorted progressing wave solutions.

Exercise 1.2.2. *Find all solutions of (1.2.6) of the form $f(x - ct)$ and all solutions of the form $e^{i(\tau t - x\xi)}$.* **Discussion.** The solutions $e^{i(\tau t - x\xi)}$ with $\xi \in \mathbb{R}$ are particularly important since the general solution is a Fourier superposition of these special *plane waves*. The equation $\tau = \tau(\xi)$ defining such solutions is called the *dispersion relation* of (3.1.6).

There is an energy conservation law. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing smooth functions. That is, functions such that for all α, β,

$$\sup_{x \in \mathbb{R}^d} \left| x^\beta \partial_x^\alpha \psi(x) \right| \;<\; \infty\,.$$

Exercise 1.2.3. *Prove that if $u \in C^\infty(\mathbb{R} : \mathcal{S}(\mathbb{R}))$ is a real valued solution of the Klein–Gordon equation, then*

$$\int u_t^2 + u_x^2 + u^2 \; dx$$

is independent of t. This quantity is called the **energy**. **Hint.** Justify carefully differentiation under the integral sign and integration by parts. If you find weaker hypotheses which suffice, that is good.

The fundamental solution of the Klein–Gordon equation with initial data (1.2.4), is not as simple as the fundamental solutions of the wave equation. Theorem 1.1.4 implies that the singular support lies on $\{x = \pm t\}$. The proof is as for the wave equation except that the zeroth order term in the equation for V is replaced by

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} V\,.$$

The singularities are computed by the method of progressing waves. For $n \in \mathbb{N}$, introduce

$$h_n(x) \ := \ \begin{cases} x^n/n! & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x \leq 0. \end{cases} \tag{1.2.7}$$

Then

$$\frac{d}{dx}\, h_{n+1} = h_n \,, \qquad \text{for} \quad n \geq 0\,. \tag{1.2.8}$$

Exercise 1.2.4. *Show that there are uniquely determined functions $a_n(t)$ satisfying*

$$a_0(0) = 1/2 \quad \text{and} \quad a_n(0) = 0 \quad \text{for} \quad n \geq 1\,,$$

and so that for all $N \geq 2$,

$$\Big(\partial_t^2 - \partial_x^2 + 1\Big) \sum_{n=0}^{N} a_n(t)\, h_n(x-t) \in C^{N-2}(\mathbb{R}^2)\,. \tag{1.2.9}$$

In this case, we say that the series

$$\sum_{n=0}^{\infty} a_n(t)\, h_n(t-x)$$

is a formal solution of $(\partial_t^2 - \partial_x^2 + 1)u \in C^\infty$. **Hints.** Pay special attention to the most singular term(s). In particular show that, $\partial_t a_0 = 0$.

Exercise 1.2.5. *Suppose that u is the fundamental solution of the Klein–Gordon equation and $M \geq 0$. Find a distribution w_M such that $u - w_M \in C^M(\mathbb{R}^2)$. Show that the fundamental solution of the wave equation and that of the Klein–Gordon equation differ by a Lipschitz continuous function. Show that the singular supports of the two fundamental solutions are equal.* **Hint.** Add (1.2.9) to its spatial reflection and choose initial values for the two solutions to match the initial data.

Exercise 1.2.6. *Study the fundamental solution for the dissipative wave equation*

$$u_{tt} - u_{xx} + 2u_t = 0\,. \tag{1.2.10}$$

Use Theorem 1.1.4 to show that the singular support is contained in the characteristics through $(0,0)$. Show that it is not a continuous perturbation of the fundamental solution of the wave equation. **Hint.** Find solutions of $(\partial_t^2 - \partial_x^2 + 2\partial_t)u \in C^\infty$ of the form $\sum_n b_n(t)\, h_n(t-x)$ as in Exercises 1.2.4 and 1.2.5. Use two such solutions as in Exercise 1.2.5.

The method of *progressing wave expansions* from these examples is discussed in more generality in chapter 6 of Courant and Hilbert Vol. 2, and in [Lax, 2006]. The higher dimensional analogue of these solutions are singular along codimension one characteristic hypersurfaces in space-time. The singularities propagate satisfying transport equations along rays generating the hypersurface. The general class goes under the name *conormal solutions.* M. Beals' book [Beals, 1989] is a good reference. They describe propagating wavefronts. Luneberg's book [Luneberg, 1944] recounts his discovery that the propagation laws for fronts of singularities coincide with the physical laws of geometric optics.

1.3. Group velocity and the method of nonstationary phase

The Klein–Gordon equation has constant coefficients, and so it can be solved explicitly using the Fourier transform. The computation of the singularities of the fundamental solution of the Klein–Gordon equation in Exercise 1.2.5 suggests that the main part of solutions travel with speed equal to 1. One might expect that the energy in a disk growing linearly in time at a speed < 1 would be small for $t \gg 1$. For compactly supported data, such a disk would contain no singularities for large time. Thus it is not unreasonable to guess that for each $\sigma < 1$ and $R > 0$,

$$\limsup_{t\to\infty} \int_{|x|<R+\sigma t} u_t^2 + u_x^2 + u^2\, dx \;=\; 0. \tag{1.3.1}$$

Either the method of characteristics or the energy method shows that speeds are no larger than one. The idea about the Klein–Gordon energy expressed in (1.3.1) is dead wrong. The main part of the energy travels strictly slower than speed 1, even though singularities travel with speed exactly equal to 1.

The solution of the Cauchy problem for the Klein–Gordon equation in dimension d,

$$u_{tt} \;-\; \Delta\, u \;+\; u \;=\; 0\,, \qquad (t,x) \;\in\; \mathbb{R}^{1+d}\,,$$

is given by

$$u \;=\; \sum_{\pm} (2\pi)^{-d/2} \int a_\pm(\xi)\; e^{i(\pm\langle\xi\rangle t + x\xi)}\; d\xi\,, \qquad \langle\xi\rangle := \big(1+|\xi|^2\big)^{1/2},$$

$$\hat u(0,\xi) \;=\; a_+(\xi) + a_-(\xi)\,, \qquad \hat u_t(0,\xi) \;=\; i\,\langle\xi\rangle\,\big(a_+(\xi) - a_-(\xi)\big)\,.$$

The conserved energy is equal to

$$\frac{1}{2}\int u_t^2 \;+\; |\nabla_x u|^2 \;+\; u^2\; dx \;=\; \int \langle\xi\rangle^2\, \big(\,|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\,\big)\; d\xi\,.$$

Exercise 1.3.1. *Verify these formulas. Verify conservation of energy by an integration by parts argument as in Exercise* 1.2.3. **Hint.** Follow the computation that starts §1.4.

Consider the behavior for large times. The phases $\phi_\pm(t,x,\xi) := \pm\langle\xi\rangle t + x\xi$ have gradients

$$\nabla_\xi \phi_\pm(t,x,\xi) \ := \ \nabla_\xi\Big(\pm \langle \xi\rangle t + x\xi\Big) \ = \ \frac{\pm t\xi}{\langle\xi\rangle} + x \ = \ t\Big(\frac{\pm\xi}{\langle\xi\rangle} + \frac{x}{t}\Big).$$

At space-time points (t,x) with $t \gg 1$ and

$$\frac{\pm\xi}{\langle\xi\rangle} \ + \ \frac{x}{t} \ \neq \ 0\,,$$

the phase oscillates rapidly and the contribution to the integral is expected to be small. The contribution to the $a_\pm$ integral from $\xi \sim \underline{\xi}$ is felt predominantly at points where $x/t \sim \mp\underline{\xi}/\langle\underline{\xi}\rangle$. Setting $\tau_\pm(\xi) := \pm\langle\xi\rangle$, one has

$$\frac{\mp\underline{\xi}}{\langle\underline{\xi}\rangle} \ = \ -\nabla_\xi \tau_\pm(\underline{\xi})\,.$$

This agrees with the formula for the group velocity (re)introduced on purely geometric grounds in §2.4.

For $t \to \infty$ the contributions of the plane waves $a_\pm(\xi) e^{i(\tau_\pm(\xi)t + x\xi)}$ with $\xi \sim \underline{\xi}$ are expected to be felt at points with $x/t \sim -\nabla_\xi\tau_\pm(\underline{\xi})$. A precise version is proved using the method of nonstationary phase.

Proposition 1.3.1. *Suppose that $a_\pm(\xi) \in \mathcal{S}(\mathbb{R}^d)$, and define*

$$\mathbf{V} \ := \ \bigcup_\pm \ \big\{\mathbf{v} \ : \ \mathbf{v} = -\nabla_\xi \tau_\pm(\xi) \ \ \textit{for some } \xi \in \operatorname{supp} a_\pm\big\}$$

to be the closed set of group velocities that appear in the plane wave decomposition of u. For $\mu > 0$, let $\mathbf{K}_\mu \subset \mathbb{R}^d$ denote the set of points at distance $\geq \mu$ from $\mathbf{V}$. Denote by Γ_μ the cone

$$\Gamma_\mu \ := \ \Big\{(t,x) \ : \ t > 0 \ \ \textit{and} \ \ x/t \in \mathbf{K}_\mu\Big\}\,.$$

Then for all $N > 0$ and α,

$$(1 + t + |x|)^N \ \partial^\alpha_{t,x} u(t,x) \in L^\infty(\Gamma_\mu)\,.$$

Proof. The solution u is smooth with values in $\mathcal{S}$ so one need only consider $\{t \geq 1\}$. We estimate the u_+ summand. The u_- summand is treated similarly. The subscript $+$ is suppressed in u_+, ϕ_+, a_+ and τ_+.

Introduce the first order differential operator

$$(1.3.2) \quad \ell(t,x,\partial) \ := \ \frac{1}{i|\nabla_\xi\phi|^2} \ \sum_j \frac{\partial\phi}{\partial\xi_j}\,\frac{\partial}{\partial\xi_j}\,, \quad \text{so} \quad \ell(t,x,\partial_\xi)\, e^{i\phi} \ = \ e^{i\phi}\,.$$

The operator is only defined where $\nabla_\xi\phi \neq 0$. The coefficients are smooth functions on a neighborhood of Γ_μ, and are homogeneous of degree minus one in (t,x) and satisfy

$$\frac{1}{|\nabla_\xi \phi|^2}\,\Big|\frac{\partial \phi}{\partial \xi_j}\Big| \ \le\ C(t+|x|)^{-1} \quad \text{for} \quad (t,x,\xi)\in \Gamma_\mu \times\ \operatorname{supp} a\,.$$

The identity $\ell\, e^{i\phi} = e^{i\phi}$ implies

$$\int a(\xi)\ e^{i\phi}\ d\xi = \int a(\xi)\ \ell^N e^{i\phi}\ d\xi\,.$$

Denote by $\ell^\dagger$ the transpose of ℓ and integrate by parts to find

$$\int a(\xi)\ e^{i\phi}\ d\xi = \int \big[(\ell^\dagger)^N a(\xi)\big]\ e^{i\phi}\ d\xi\,.$$

The operator

$$(\ell^\dagger)^N \ =\ \sum_{|\alpha|\le N} c_\alpha(t,x,\xi)\,\partial_\xi^\alpha$$

with coefficients c_α smooth on a neighborhood of Γ_μ and homogeneous of degree $-N$ in t,x. Therefore,

$$|c_\alpha(t,x)| \ \le\ C(\alpha)(1+t+|x|)^{-N} \quad \text{for} \quad (t,x,\xi)\in\Gamma_\mu\times\ \operatorname{supp} a\,.$$

It follows that

$$\Big|\int a(\xi)\ e^{i\phi}\ d\xi\Big| \ \le\ C\,(1+t+|x|)^{-N}\,.$$

Since the t,x derivatives of this integral are again integrals of the same form, this suffices to prove the proposition. □

Example 1.3.1. Introduce for $0<\mu\ll 1$, $\tilde{\mathbf{V}}_\mu := \mathbb{R}^d\setminus \mathbf{K}_\mu$ an open set slightly larger than $\mathbf{V}$. For $t\to\infty$ virtually all the energy of a solution is contained in the cone $\big\{(t,x)\ :\ x/t\in\tilde{\mathbf{V}}\big\}$. This is particularly interesting when $a_\pm$ are supported in a small neighborhood of a fixed $\underline{\xi}$. For large times virtually all the energy is localized in a small conic neighborhood of the pair of lines $x=-t\,\nabla_\xi\tau_\pm(\underline{\xi})$ that travel with the group velocities associated to $\underline{\xi}$.

The integration by parts method introduced in this proof is very important. The next estimate for nonstationary oscillatory integrals is a straightforward application. The fact that the estimate is uniform in the phases is useful.

Lemma of Nonstationary Phase 1.3.2. *Suppose that Ω is a bounded open subset of $\mathbb{R}^d$, $m\in\mathbb{N}$, and $C_1>1$. Then there is a constant $C_2>0$ so that for all $f\in C_0^m(\Omega)$ and $\phi\in C^m(\Omega\,;\mathbb{R})$ satisfying*

$$\forall|\alpha|\le m\,,\quad \|\partial^\alpha\phi\|_{L^\infty}\le C_1\,,\quad \textit{and}\quad \forall x\in\Omega\,,\quad C_1^{-1}\ \le|\nabla_x\phi|\ \le\ C_1\,,$$

one has the estimate

$$\Big| \int e^{i\phi/\epsilon}\, f(x)\, dx \Big| \;\le\; C_2\, \epsilon^m \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L^1}\,.$$

Exercise 1.3.2. *Prove the lemma.* **Hint.** Use (1.3.2).

Example 1.3.2. Applied to the phases $\phi = x\xi := \sum_j x_j \xi_j$ with ξ belonging to a compact subset of $\mathbb{R}^d \setminus 0$, the lemma implies the rapid decay of the Fourier transform of smooth compactly supported functions. Conversely, the lemma can be reduced to the special case of the Fourier transform. Since the gradient of ϕ in the lemma does not vanish, for each $\underline{x} \in \operatorname{supp} f$, there is a neighborhood and a nonlinear change of coordinates so that in the new coordinates ϕ is equal to x_1. Using a partition of unity, one can suppose that f is the sum of a finite number of functions each supported in one of the neighborhoods. For each such function, a change of coordinates yields an integral of the form

$$\int e^{ix_1/\epsilon}\, g(x)\, dx \;=\; c\,\hat g(1/\epsilon, 0, \dots, 0)\,,$$

which is rapidly decaying since it is the transform of an element of $C_0^\infty(\mathbb{R}^d)$. Care must be taken to obtain estimates uniform in the family of phases of the lemma.

Exercise 1.3.3. *Suppose that $f \in H^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ and that u is the unique solution of the Klein–Gordon equation with initial data*

$$u(0,x) = f(x)\,, \qquad u_t(0,x) = g(x)\,. \tag{1.3.3}$$

Prove that for any $\epsilon > 0$ and $R > 0$, there is a $\delta > 0$ such that

$$\limsup_{t\to\infty} \int_{|x|>(1-\delta)t-R} u_t^2 + u_x^2 + u^2\, dx \;<\; \epsilon\,. \tag{1.3.4}$$

Hint. Replace $\hat f, \hat g$ by compactly supported smooth functions making an error at most $\epsilon/2$ in energy. Then use Lemma 1.3.2 noting that the group velocities are uniformly smaller than those for ξ belonging to the supports of $a_\pm$. **Discussion.** Note that as $\xi \to \infty$, the group velocities approach ± 1. High frequencies will propagate at speeds nearly equal to one. In particular they travel at the same speed. High frequency signals stay together better than low frequency signals. Since singularities of solutions are made of only the high frequencies (modifying the Fourier transform of the data on a compact set modifies the solution by a smooth term), one expects singularities to propagate at speeds ± 1. That is proved for the fundamental solution in Exercise 1.2.5. Once this is known for the fundamental solution, it follows for all. The simple proof is an exercise in my book [Rauch, 1991, pp. 164–165].

The analysis of Exercise 1.3.3 does not apply to the fundamental solution since the latter does not have finite energy. However it belongs to $C^j(\mathbf{R} : H^{s-j}(\mathbb{R}))$ for all $s < 1/2$ and $j \in \mathbb{N}$. The next result provides a good replacement for (1.3.4).

Exercise 1.3.4. *Suppose that u is the fundamental solution of the Klein–Gordon equation (1.1.6) and that $s < 1/2$. If $0 \le \chi \in C^\infty(\mathbb{R})$ is a plateau cutoff supported on the positive half line, that is*

$$\chi(x) = 0 \quad \textit{for} \quad x \le 0 \qquad \textit{and} \quad \chi(x) = 1 \quad \textit{for} \quad x \ge 1\,,$$

then for all $\epsilon > 0$ and $R > 0$ there is a $\delta > 0$ so that

$$\limsup_{t-\infty} \| \, \chi(R + |x| - (1-\delta)t)\, u(t,x) \, \|_{H^s(\mathbb{R}_x)} < \epsilon\,. \tag{1.3.5}$$

Hints. Prove that

$$\| \, \chi\, u(t) \, \|_{H^s(\mathbb{R})} \le C\Big(\, \|u(0)\|_{H^s(\mathbb{R})} + \|u_t(0)\|_{H^{s-1}(\mathbb{R})} \, \Big)$$

with C independent of t and the initial data. Conclude that it suffices to prove (1.3.4) with initial data $u(0), u_t(0)$ dense in $H^s \times H^{s-1}$. Take the dense set to be data with Fourier transform in $C_0^\infty(\mathbb{R})$.

These examples illustrate the important observation that the propagation of singularities in solutions and the propagation of the majority of the energy may be governed by different rules. For the Klein–Gordon equation, both answers can be determined from considerations of group velocities.

1.4. Fourier synthesis and rectilinear propagation

For equations with constant coefficients, solutions of the initial value problem are expressed as Fourier integrals. Injecting short wavelength initial data and performing an asymptotic analysis yields the approximations of geometric optics. This is how such approximations were first justified in the nineteenth century. It is also the motivating example for the more general theory. The short wavelength approximations explain the *rectilinear propagation of waves* in homogeneous media. This is the first of the three basic physical laws of geometric optics. It explains, among other things, the formation of shadows. The short wavelength solutions are also the building blocks in the analysis of the laws of reflection and refraction presented in §1.6 and §1.7.

Consider the initial value problem

$$0 \;=\; \Box u \;:=\; \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}\,, \qquad u(0,x) = f, \quad u_t(0,x) = g. \tag{1.4.1}$$

Fourier transformation with respect to the x variables yields

$$\partial_t^2 \hat{u}(t,\xi) \;+\; |\xi|^2\, \hat{u}(t,\xi) \;=\; 0\,, \qquad \hat{u}(0,\xi) = \hat{f}\,, \quad \partial_t \hat{u}(0,\xi) = \hat{g}\,.$$

Solve the ordinary differential equations in t to find

$$\hat{u}(t,\xi) \;=\; \hat{f}(\xi)\,\cos t|\xi| + \hat{g}(\xi)\,\frac{\sin t|\xi|}{|\xi|}\,.$$

For $t \neq 0$ the function $\xi \mapsto \sin(t|\xi|)/|\xi|$ is a real analytic function on $\mathbb{R}^d$ with derivatives uniformly bounded on bounded time intervals. Write

$$\cos t|\xi| \;=\; \frac{e^{it|\xi|} + e^{-t|\xi|}}{2}, \qquad \sin t|\xi| \;=\; \frac{e^{it|\xi|} - e^{-t|\xi|}}{2i},$$

to find

$$\hat{u}(t,\xi)\; e^{ix\xi} \;=\; a_+(\xi)\, e^{i(x\xi - t|\xi|)} \;-\; a_-(\xi)\, e^{i(x\xi + t|\xi|)}\,, \tag{1.4.2}$$

with

$$a_+ \;:=\; \frac{1}{2}\Big(\hat{f} \;+\; \frac{\hat{g}}{i|\xi|}\Big), \qquad a_- \;:=\; \frac{1}{2}\Big(\hat{f} \;-\; \frac{\hat{g}}{i|\xi|}\Big). \tag{1.4.3}$$

For each ξ, the right-hand side of (1.4.2) is a linear combination of the plane wave solutions of the wave equation $e^{i(x\xi + t\tau(\xi))}$ with dispersion relation $\tau = \mp|\xi|$ and amplitude $a_\pm(\xi)$. The group velocities associated to $a_\pm$ are

$$\mathbf{v} \;=\; -\nabla_\xi \tau \;=\; -\nabla_\xi(\mp|\xi|) \;=\; \pm\frac{\xi}{|\xi|}\,.$$

The solution is the sum of two terms,

$$u_\pm(t,x) \;:=\; \frac{1}{(2\pi)^{d/2}} \int a_\pm(\xi)\; e^{i(x\xi \mp t|\xi|)}\, d\xi\,.$$

The conserved energy for the spring equation satisfied by $\hat{u}(t,\xi)$ shows that

$$\begin{aligned}&\frac{1}{2}\Big(|\hat{u}_t(t,\xi)|^2 \;+\; |\xi|^2|\hat{u}(t,\xi)|^2\Big)\\ &\qquad\qquad \;=\; |\xi|^2\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big) \;=\; \text{independent of } t\,.\end{aligned}$$

Integrate $d\xi$ and use $\mathcal{F}\big(\partial u/\partial x_j\big) \;=\; i\xi_j\hat{u}$, and Parseval's theorem to show that the quantity

$$\int |\xi|^2\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\; d\xi \;=\; \frac{1}{2}\int |u_t(t,x)|^2 \;+\; |\nabla_x u(t,x)|^2\; dx$$

is independent of time for the solutions of the wave equation.

The formula for $a_\pm$ are potentially singular at $\xi = 0$. The energy for the wave equation is expressed in terms of the pair of functions $|\xi| a_\pm(\xi)$. They are given by nonsingular expressions in terms of $|\xi|\hat{f}$ and $\hat{g}$.

There are conservations of all orders. Multiplying the spring energy by $\langle\xi\rangle^{2s}$ and integrating $d\xi$ shows that each of the following quantities is independent of time:

$$\frac{1}{2}\,\|\nabla_{t,x}u(t)\|^2_{H^s(\mathbb{R}^d)} \;=\; \int \langle\xi\rangle^{2s}\;|\xi|^2\;\big(|a_+(\xi)|^2 \;+\; |a_-(\xi)|^2\big)\;d\xi\,.$$

Consider initial data a wave packet with wavelength of order ϵ and phase equal to x_1/ϵ,

$$(1.4.4)\qquad u^\epsilon(0,x) \;=\; \gamma(x)\,e^{ix_1/\epsilon}\,,\qquad u^\epsilon_t(0,x)\;=\;0\,,\qquad \gamma\;\in\;\bigcap_s H^s(\mathbb{R}^d)\,.$$

The choice $u_t = 0$ postpones dealing with the factor $1/|\xi|$ in (1.4.3). When ϵ is small, the initial value is an envelope or profile γ multiplied by a rapidly oscillating exponential.

Applying (1.4.3) with $g = 0$ and

$$\hat f(\xi) \;=\; \hat u(0,\xi) \;=\; \mathcal{F}\big(\gamma(x)\,e^{ix_1/\epsilon}\big) \;=\; \hat\gamma(\xi - \mathbf{e}_1/\epsilon)$$

yields $u = u_+ + u_-$ with

$$u^\epsilon_\pm(t,x) \;:=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\xi-\mathbf{e}_1/\epsilon)\;e^{i(x\xi\mp t|\xi|)}\,d\xi\,.$$

Analyze u^ϵ_+. The other term is analogous. For ease of reading, the subscript plus is omitted. Introduce

$$\zeta \;:=\; \xi \;-\; \mathbf{e}_1/\epsilon,\qquad \text{so}\qquad \xi \;=\; \frac{\mathbf{e}_1 \;+\; \epsilon\zeta}{\epsilon}\,,\qquad \text{and}$$

$$(1.4.5)\qquad u^\epsilon(t,x) \;=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\;e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\;e^{-it|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\;d\zeta\,.$$

The approximation of geometric optics comes from injecting the first order Taylor approximation,

$$\big|\mathbf{e}_1 + \epsilon\zeta\big| \;\approx\; 1 + \epsilon\zeta_1\,,$$

yielding

$$u^\epsilon_{\text{approx}} \;:=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\;e^{ix(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\;e^{-it(1+\epsilon\zeta_1)/\epsilon}\;d\zeta\,.$$

The rapidly oscillating terms $e^{i(x_1-t)/\epsilon}$ do not depend on ζ, so
(1.4.6)

$$u_{\text{approx}} = e^{i(x_1-t)/\epsilon}\;A(t,x),\qquad A(t,x) := \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x\zeta - t\zeta_1)}\,d\zeta.$$

Write $x\zeta - t\zeta_1 = (x - t\mathbf{e}_1)\zeta$ to find

$$A(t,x) \;=\; \frac{1}{2}\,\frac{1}{(2\pi)^{d/2}}\int \hat\gamma(\zeta)\,e^{i(x-t\mathbf{e}_1)\zeta}\,d\zeta \;=\; \frac{1}{2}\,\gamma(x - t\mathbf{e}_1)\,.$$

The approximation is a wave packet with envelope A and wavelength ϵ. The wave packet translates rigidly with velocity equal to $\mathbf{e}_1$. The waveform γ is arbitrary. The approximate solution resembles the columnated light from a flashlight. If the support of γ is small, the approximate solution resembles a light ray.

The amplitude A satisfies the transport equation

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x_1} \;=\; 0,$$

so it is constant on the *rays* $x = \underline{x} + t\mathbf{e}_1$. The construction of a family of short wavelength approximate solutions of D'Alembert's wave equations requires only the solution of a simple transport equation.

The dispersion relation of the family of plane waves,

$$e^{i(x\xi+\tau t)} \;=\; e^{i(x\xi-|\xi|t)},$$

is $\tau = -|\xi|$. The velocity of transport, $\mathbf{v} = (1, 0, \ldots, 0)$, is the group velocity $\mathbf{v} = -\nabla_\xi \tau(\underline{\xi}) = \underline{\xi}/|\underline{\xi}|$ at $\underline{\xi} = (1, 0, \ldots, 0)$. For the opposite choice of sign, the dispersion relation is $\tau = |\xi|$, the group velocity is $-\mathbf{e}_1$, and the rays are the lines $x = \underline{x} - t\mathbf{e}_1$.

Had we taken data with oscillatory factor $e^{ix\xi/\epsilon}$, then the approximate solution would have been the sum of two wave packets with group velocities $\pm\xi/|\xi|$,

$$\frac{1}{2}\left(e^{i(x\xi-t|\xi|)/\epsilon}\,\gamma\Big(x - t\frac{\xi}{|\xi|}\Big) \;+\; e^{i(x\xi+t|\xi|)/\epsilon}\,\gamma\Big(x + t\frac{\xi}{|\xi|}\Big)\right).$$

The approximate solution (1.4.6) is a function $H(x - t\mathbf{e}_1)$ with $H(x) = e^{ix_1/\epsilon}\,h(x)$. When h has compact support, or more generally tends to zero as $|x| \to \infty$, the approximate solution is localized and has velocity equal to $\mathbf{e}_1$. The next result shows that when $d > 1$, no exact solution can have this form. In particular the distribution $\delta(x - \mathbf{e}_1 t)$ that is the most intuitive notion of a light ray is *not* a solution of the wave equation or Maxwell's equation.

Proposition 1.4.1. *If $d > 1$, $s \in \mathbb{R}$, $K \in H^s(\mathbb{R}^d)$ and $u = K(x - \mathbf{e}_1 t)$ satisfies $\Box u = 0$, then $K = 0$.*

Exercise 1.4.1. *Prove Proposition 1.4.1.* **Hint.** Prove the following lemma.

Lemma. *If $k \le d$, $s \in \mathbb{R}$, and $w \in H^s(\mathbb{R}^d)$ satisfies $0 = \sum_k^d \partial^2 w/\partial^2 x_j$, then $w = 0$.*

Next, analyze the error in (1.4.6). Reintroduce the subscripts. Then extract the rapidly oscillating factor in (1.4.5) to find exact amplitudes,

$$u^\epsilon_\pm(t,x) \;=\; e^{i(x_1\mp t)/\epsilon}\, A^\pm_{\rm exact}(\epsilon,t,x)\,,$$

$$A^\pm_{\rm exact}(\epsilon,t,x) \;:=\; \frac{1}{(2\pi)^{d/2}2}\int \hat\gamma(\zeta)\; e^{ix.\zeta}\; e^{\mp it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon}\; d\zeta\,. \tag{1.4.7}$$

Proposition 1.4.2. *The exact and approximate solutions of* $\Box u^\epsilon = 0$ *with Cauchy data (1.4.4) are given by*

$$u^\epsilon = \sum_\pm e^{i(x_1\mp t)/\epsilon}\, A^\pm_{\rm exact}(\epsilon,t,x)\,,\qquad u^\epsilon_{\rm approx} = \sum_\pm e^{i(x_1\mp t)/\epsilon}\,\frac{\gamma(x\mp\mathbf{e}_1 t)}{2}\,,$$

as in (1.4.7) and (1.4.6). The error is $O(\epsilon)$ *on bounded time intervals. Precisely, there is a constant* $C>0$ *so that for all* s,ϵ,t,

$$\Big\| A^\pm_{\rm exact}(\epsilon,t,x) \;-\; \frac{\gamma(x\mp\mathbf{e}_1 t)}{2}\Big\|_{H^s(\mathbb{R}^N)} \;\le\; C\,\epsilon\,|t|\,\big\|\gamma\big\|_{H^{s+2}(\mathbb{R}^d)}\,.$$

Proof. It suffices to estimate the error with the plus sign. The definitions yield

$$A^+_{\rm exact}(\epsilon,t,x) - \gamma(x-\mathbf{e}_1 t)/2 \;=\; C\int \hat\gamma(\zeta)\, e^{ix.\zeta}\,\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon)} - e^{-it\zeta_1}\big)\, d\zeta\,.$$

The definition of the $H^s(\mathbb{R}^d)$ norm yields

$$\begin{aligned}\Big\|A^+_{\rm exact}(\epsilon,t,x) \;-\; \gamma(x-\mathbf{e}_1 t)/2\Big\|_{H^s(\mathbb{R}^N)}\\ = \Big\|\langle\zeta\rangle^s\;\hat\gamma(\zeta)\;\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{1.4.8}$$

Taylor expansion yields for $|\beta|\le 1/2$,

$$|\,\mathbf{e}_1+\beta\,| \;=\; 1+\beta_1+r(\beta)\,,\qquad |r(\beta)|\le C\,|\beta|^2\,.$$

Increasing C if needed, the same inequality is true for $|\beta|\ge 1/2$ as well.

Applied to $\beta=\epsilon\zeta$ this yields

$$\Big| t\big(\big|\mathbf{e}_1+\epsilon\zeta\big|-1\big)/\epsilon - t\,\zeta_1\,x_1\Big| \;\le\; C\,\epsilon\,|t|\,|\zeta|^2$$

so

$$\Big|e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;-\; e^{-it\zeta_1}\Big| \;\le\; C\,\epsilon\,|t|\,|\zeta|^2\,.$$

Therefore,

$$\Big\|\langle\zeta\rangle^s\;\hat\gamma(\zeta)\;\big(e^{-it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} - e^{-it\zeta_1}\big)\Big\|_{L^2(\mathbb{R}^d)} \;\le\; C\,\epsilon\,|t|\,\Big\|\langle\zeta\rangle^s|\zeta|^2\;\hat\gamma(\zeta)\Big\|_{L^2}. \tag{1.4.9}$$

Combining (1.4.8) and (1.4.9) yields the estimate of the proposition. $\square$

The approximation retains some accuracy so long as $t = o(1/\epsilon)$.

The approximation has the following geometric interpretation. One has a superposition of plane waves $e^{i(x\xi - t|\xi|)}$ with $\xi = (1/\epsilon, 0, \dots, 0) + O(1)$. Replacing ξ by $(1/\epsilon, 0, \dots, 0)$ and $|\xi|$ by $1/\epsilon$ in the plane waves yields the approximation (1.4.6).

The wave vectors, ξ, make an angle $O(\epsilon)$ with $\mathbf{e}_1$. The corresponding rays have velocities which differ by $O(\epsilon)$ so the rays remain close for times small compared with $1/\epsilon$. For longer times the fact that the group velocities are not parallel is important. The wave begins to spread out. Parallel group velocities are a reasonable approximation for times $t = o(1/\epsilon)$.

The example reveals several scales of time. For times $t \ll \epsilon$, u and its gradient are well approximated by their initial values. For times $\epsilon \ll t \ll 1$ $u \approx e^{i(x-t)/\epsilon} a(0,x)$. The solution begins to oscillate in time. For $t = O(1)$ the approximation $u \approx A(t,x)\, e^{i(x-t)/\epsilon}$ is appropriate. For times $t = O(1/\epsilon)$ the approximation ceases to be accurate. Refined approximations valid on this longer time scale are called *diffractive geometric optics*. The reader is referred to [Donnat, Joly, Métiver, and Rauch, 1995–1996] for an introduction in the spirit of Chapters 7–8.

It is typical of the approximations of geometric optics that

$$\Box\big(u_{\rm approx} - u_{\rm exact}\big) \;=\; \Box\, u_{\rm approx} = O(1)$$

is not small. The error $u_{\rm approx} - u_{\rm exact} = O(\epsilon)$ is smaller by a factor of ϵ. The residual $\Box\, u_{\rm approx}$ oscillates on the scale ϵ, and after applying $\Box^{-1}$ it is smaller by a factor ϵ.

The analysis just performed can be carried out without fundamental change for initial oscillations with nonlinear phase. An excellent description, including the phase shift on crossing a focal point, can be found in [Hörmander 1983, §12.2].

Next the approximation is pushed to higher accuracy with the result that the residuals can be reduced to $O(\epsilon^N)$ for any N. Taylor expansion to higher order yields

$$|\mathbf{e}_1 + \eta| \;=\; 1 + \eta_1 + \sum_{|\alpha|\ge 2} c_\alpha \eta^\alpha\,, \qquad |\eta| < 1, \tag{1.4.10}$$

so

$$\big(|\mathbf{e}_1 + \epsilon\zeta| - 1\big)/\epsilon \;\sim\; \zeta_1 \;+\; \sum_{|\alpha|\ge 2} \epsilon^{|\alpha|-1}\, c_\alpha\, \zeta^\alpha,$$

$$e^{it(|\mathbf{e}_1+\epsilon\zeta|-1)/\epsilon} \;\sim\; e^{it\zeta_1}\, e^{\sum_{|\alpha|\ge 2} it\epsilon^{|\alpha|-1}\; c_\alpha \zeta^\alpha} \;\sim\; e^{it\zeta_1}\left(1 + \sum_{j\ge 1} \epsilon^j\, h_j(t,\zeta)\right).$$

Here, $h_j(t,\zeta)$ is a polynomial in t,ζ. Injecting in the formula for $A_{\rm exact}(\epsilon,t,x)$ yields an expansion

$$\begin{aligned}&A_{\rm exact}(\epsilon,t,x) \ \sim\ A_0(t,x)+\epsilon\, A_1(t,x)+\epsilon^2 A_2(t,x)+\cdots\,,\\ &A_0(t,x)=\gamma(x-\mathbf{e}_1 t)/2,\end{aligned} \tag{1.4.11}$$

$$\begin{aligned}A_j \ &=\ \frac{1}{(2\pi)^{-d/2}\,2}\int \hat\gamma(\zeta)\ e^{i(x\zeta-t\zeta_1)}\ h_j(t,\zeta)\ d\zeta\\ &=\ \frac12\Big(h_j(t,\partial/i)\gamma\Big)(x-\mathbf{e}_1 t)\,.\end{aligned} \tag{1.4.12}$$

The series is asymptotic as $\epsilon\to0$ in the sense of Taylor series. For any s,N, truncating the series after N terms yields an approximate amplitude which differs from $A_{\rm exact}$ by $O(\epsilon^{N+1})$ in H^s *uniformly on compact time intervals.*

Exercise 1.4.2. *Compute the precise form of the first corrector a_1.*

Formula (1.4.11) implies that if the Cauchy data are supported in a set $\mathcal{O}$, then the amplitudes A_j are all supported in the tube of rays

$$\mathcal{T}\ :=\ \Big\{(t,x)\ :\ x=\underline{x}+t\mathbf{e}_1,\quad \underline{x}\in\mathcal{O}\Big\}\,. \tag{1.4.13}$$

Warning 1. Though the A_j are supported in this tube, it is not true, when $d\ge2$, that $A^\epsilon_{\rm exact}$ is supported in the tube. If it were, then u^ϵ would be supported in the tube. When $d\ge2$, the function $u=0$ is the only solution of D'Alembert's equation with support in a tube of rays with compact cross section (see Exercise 5.2.9).

Warning 2. By a closer inspection of (1.4.11) or by the analysis after Exercise 5.2.9, one can show that $\|A_j(t,\cdot)\|_{L^\infty}\sim (j!)^{-1}\sum_{|\beta|\le2j}\|\partial^\beta\gamma\|_{L^\infty}$. So, for a typical analytic γ, the series $\sum\epsilon^j A_j$ have terms of size $\epsilon^j C^j(2j)!/j!$ so they diverge no matter how small is ϵ. For a nonanalytic γ, for example $\gamma\in C_0^\infty$, matters are worse still. *The series $\sum\epsilon^j A_j$ is a divergent series that gives an accurate asymptotic expansion as $\epsilon\to0$.* It is a nonconvergent Taylor expansion of $A_{\rm exact}(\epsilon,t,x)$.

To analyze the oscillatory initial value problem with $u(0)=0$, $u_t(0)=\beta(x)\,e^{ix_1/\epsilon}$ requires one more idea to handle the contributions from $\xi\approx0$ in the expression

$$u(t,x)\ =\ (2\pi)^{-d/2}\int\ \frac{\sin t|\xi|}{|\xi|}\ \hat\beta\Big(\xi-\frac{\mathbf{e}_1}{\epsilon}\Big)\ e^{ix\xi}\ d\xi\,.$$

Choose $\chi\in C_0^\infty(\mathbb{R}^d_\xi)$ with $\chi=1$ on a neighborhood of $\xi=0$. The cutoff integrand is equal to

$$\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle\xi-\mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi-\mathbf{e}_1/\epsilon)\ e^{ix\xi}\,,\quad k_s(\xi)\ :=\ \langle\xi\rangle^s\ \hat\beta(\xi)\ \in\ L^2(\mathbb{R}^d_\xi)\,.$$

The $\sin t|\xi|/|\xi|$ factor is $\le |t|$. For ϵ small, the distance of $\mathbf{e}_1/\epsilon$ to the support of χ is $\ge C/\epsilon$. Therefore,

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\Big\|_{L^\infty(\mathbb{R}^d_\xi)}\ \le\ C_s\,|t|\,\epsilon^s\,,\qquad 0<\epsilon\le 1\,.$$

It follows that

$$\Big\|\chi(\xi)\ \frac{\sin t|\xi|}{|\xi|}\ \frac{1}{\langle \xi - \mathbf{e}_1/\epsilon\rangle^s}\ k_s(\xi-\mathbf{e}_1/\epsilon)\ \Big\|_{L^2(\mathbb{R}^d)}\ \le\ C_s\,|t|\,\epsilon^s\,\big\|k_s\big\|_{L^2(\mathbb{R}^d)}\,,$$

with s arbitrarily large. The small frequency contribution is negligible in the limit $\epsilon\to 0$. It is removed with a cutoff as above, and then the analysis away from $\xi=0$ proceeds by decomposition into plane wave as in the case with $u_t(0)=0$. It yields left and right moving waves with the same phases as before.

Exercise 1.4.3. *Solve the Cauchy problem for the anisotropic wave equation, $u_{tt}=u_{xx}+4u_{yy}$ with initial data given by*

$$u^\epsilon(0,x)\ =\ \gamma(x)\,e^{ix\xi/\epsilon}\,,\qquad u^\epsilon_t(0,x)\ =\ 0\,,\qquad \gamma\ \in\ \bigcap_s H^s(\mathbb{R}^d)\,.$$

Find the leading term in the approximate solution to u_+. In particular, find the velocity of propagation as a function of ξ. **Discussion.** The velocity is equal to the group velocity from §1.3.

1.5. A cautionary example in geometric optics

A typical science text discussion of a mathematics problem involves simplifying the underlying equations. The usual criterion applied is to ignore terms which are small compared to other terms in the equation. It is striking that in many of the problems treated under the rubric of geometric optics, such an approach can lead to completely inaccurate results. It is an example of an area where more careful mathematical consideration is not only useful but necessary.

Consider the initial value problems

$$\partial_t u^\epsilon+\partial_x u^\epsilon+u^\epsilon\ =\ 0\,,\qquad u^\epsilon\big|_{t=0}=a(x)\cos(x/\epsilon)$$

in the limit $\epsilon\to 0$. The function a is assumed to be smooth and to vanish rapidly as $|x|\to\infty$, so the initial value has the form of a wave packet. The initial value problem is uniquely solvable, and the solution depends continuously on the data. The exact solution of the general problem

$$\partial_t u+\partial_x u+u\ =\ 0\,,\qquad u\big|_{t=0}=f(x)\,,$$

is $u(t,x)=e^{-t}f(x-t)$, so the exact solution u^ϵ is

$$u^\epsilon(t,x)\ =\ e^{-t}\,a(x-t)\ \cos((x-t)/\epsilon)\,.$$

In the limit as $\epsilon \to 0$, one finds that both $\partial_t u^\epsilon$ and $\partial_x u^\epsilon$ are $O(1/\epsilon)$ while $u^\epsilon = O(1)$ is negligibly small in comparison. Dropping this small term leads to the simplified equation for an approximation v^ϵ,

$$\partial_t v^\epsilon + \partial_x v^\epsilon \;=\; 0\,, \qquad v^\epsilon\big|_{t=0} = a(x)\cos(x/\epsilon)\,.$$

The solution is

$$v^\epsilon(t,x) \;=\; a(x-t)\cos\big((x-t)/\epsilon\big)\,,$$

which misses the exponential decay. It is *not* a good approximation except for $t \ll 1$. The two large terms compensate so that the small term is not negligible compared to their sum.

1.6. The law of reflection

Consider the wave equation $\Box u = 0$ in the half-space $\mathbb{R}^d_- := \{x_1 \le 0\}$. At $\{x_1 = 0\}$ a boundary condition is required. The condition encodes the physics of the interaction with the boundary.

Since the differential equation is of second order, one might guess that two boundary conditions are needed as for the Cauchy problem. An analogy with the Dirichlet problem for the Laplace equation suggests that one condition is required.

A more revealing analysis concerns the case of dimension $d = 1$. D'Alembert's formula shows that at all points of space-time, the solution consists of the sum of two waves, one moving toward the boundary and the other toward the interior. The waves approaching the boundary will propagate to the edge of the domain. At the boundary one does not know what values to give to the waves which move into the domain. The boundary condition must give the value of the incoming wave in terms of the outgoing wave. That is one boundary condition.

Factoring

$$\partial_t^2 - \partial_x^2 \;=\; (\partial_t - \partial_x)(\partial_t + \partial_x) \;=\; (\partial_t + \partial_x)(\partial_t - \partial_x)$$

shows that $(\partial_t - \partial_x)(u_t + u_x) = 0$, so $u_t + u_x$ is transported to the left. Similarly, $u_t - u_x$ moves to the right. Thus from the initial conditions, $u_t - u_x$ is determined everywhere in $x \le 0$, including the boundary $x = 0$. The boundary condition at $\{x = 0\}$ must determine $u_t + u_x$. The conclusion is that half of the information needed to find all the first derivatives is already available and one needs only one boundary condition.

Consider the Dirichlet condition,

$$u(t,x)\big|_{x_1=0} = 0\,. \tag{1.6.1}$$

Differentiating (1.6.1) with respect to t shows that $u_t(t,0) = 0$, so at $t = 0$ $(u_t + u_x) = -(u_t - u_x)$. The incoming wave at the boundary has amplitude equal to -1 times the amplitude of the outgoing wave.

We next analyze the mixed initial boundary value problem for a function $u(t,x)$ defined in $x_1 \le 0$,

$$\text{(1.6.2)} \qquad \Box u = 0, \qquad u\big|_{x_1=0} = 0, \qquad u(0,x) = f\,, \qquad u_t(0,x) = g\,.$$

If the data are supported in a compact subset of $\mathbb{R}^d_-$, then for small time the support of the solution does not meet the boundary. When waves hit the boundary, they are reflected. We analyze this reflection process.

Uniqueness of solutions and finite speed of propagation for (1.6.2) are both consequences of a local energy identity. A function is a solution if and only if the real and imaginary parts are solutions. Thus it suffices to treat the real case for which

$$u_t \,\Box\, u \;=\; \partial_t e - \sum_{j\ge 1} \partial_j (u_t \partial_j u)\,, \qquad e := \frac{u_t^2 + |\nabla_x u|^2}{2}\,.$$

Denote by Γ a backward light cone

$$\Gamma \;:=\; \Big\{(t,x) \;:\; |x - \underline{x}|^2 < \underline{t} - t\Big\}$$

and by $\tilde\Gamma$ the part in $\{x_1 < 0\}$,

$$\tilde\Gamma \;:=\; \Gamma \;\cap\; \big\{x_1 < 0\big\}\,.$$

For any $0 \le s < \underline{t}$, the section at time s is denoted

$$\tilde\Gamma(s) \;:=\; \tilde\Gamma \;\cap\; \big\{t = s\big\}\,.$$

Both uniqueness and finite speed are consequences of the following estimate.

Proposition 1.6.1. *If u is a smooth solution of (1.6.2), then for $0 < t < \underline{t}$,*

$$\phi(t) \;:=\; \int_{\tilde\Gamma(t)} e(t,x)\; dx$$

is a nonincreasing function of t.

Proof. Translating the time, if necessary, it suffices to show that for $s > 0$, $\phi(s) \le \phi(0)$.

In the identity

$$0 \;=\; \int_{\tilde\Gamma \cap \{0 \le t \le s\}} u_t \,\Box u \; dt\, dx\,,$$

integrate by parts to find integrals over four distinct parts of the boundary. The tops and bottoms contribute $\phi(s)$ and $-\phi(0)$, respectively. The intersection of $\tilde\Gamma(s)$ with $x_1 = 0$ yields

$$\int_{\tilde\Gamma(s)\cap\{x_1=0\}} u_t\ \partial_1 u\ dt\, dx_2\ \cdots\ dx_d\,.$$

The Dirichlet condition implies that $u_t = 0$ on this boundary, so the integral vanishes.

The contribution of the sides $|x - \underline{x}| = \underline{t} - t$ yield an integral of

$$n_0\, e\ +\ \sum_{j=1}^{d} n_j\, u_t\, \partial_j u\,,$$

where $(n_0, n_1, n_2, \ldots, n_d)$ is the outward unit normal. Since the cone has sides of slope one,

$$n_0\ =\ \Big(\sum_{j=1}^{d} n_j^2\Big)^{1/2}\ =\ \frac{1}{\sqrt{2}}\,.$$

The Cauchy–Schwarz inequality yields

$$\Big|\sum_{j=1}^{d} n_j\, u_t\, \partial_j u\Big|\ \le\ \frac{1}{\sqrt{2}}\ |u_t||\nabla_x u|\ \le\ \frac{1}{\sqrt{2}}\, e\,.$$

Thus the integrand from the contributions of sides is nonnegative, so the integral over the sides is nonnegative.

Combining yields

$$0\ =\ \int_{\tilde\Gamma\cap\{0\le t\le s\}} u_t\ \Box u\ dt\, dx\ \ge\ \phi(s) - \phi(0)\,,$$

completing the proof. $\Box$

1.6.1. The method of images. Introduce the notations

$$x = (x_1, x'),\quad x' := (x_2, \ldots, x_d),\qquad \xi = (\xi_1, \xi'),\quad \xi' := (\xi_2, \ldots, \xi_d).$$

Definitions 1.6.2. *A function f on $\mathbb{R}^{1+d}$ is* **even** *(resp.,* **odd***) in x_1 when*

$$f(t, x_1, x') = f(t, -x_1, x'),\qquad \textit{respectively,}\quad f(t, -x_1, x') = -f(t, x_1, x').$$

Define the **reflection operator** *R by*

$$(Rf)(t, x_1, x')\ :=\ f(t, -x_1, x')\,.$$

The even (resp., odd) parts of a function f are defined by

$$\frac{f + Rf}{2},\qquad \textit{respectively,}\qquad \frac{f - Rf}{2}.$$

Proposition 1.6.3. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ satisfies $\Box u = 0$ and is odd in x_1, then the restriction of u to $\{x_1 \le 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Dirichlet boundary condition (1.6.1).*

ii. *Conversely, if $u \in C^\infty(\{x_1 \le 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.1), define $\tilde u$ to be the odd extension of u to $\mathbb{R}^{1+d}$. Then $\tilde u$ is a smooth odd solution of $\Box \tilde u = 0$.*

Proof. i. Setting $x_1 = 0$ in the identity $u(t, x_1, x') = -u(t, -x_1, x')$ shows that (1.6.1) is satisfied.

ii. First prove by induction on n that

$$\forall n \ge 0, \qquad \frac{\partial^{2n} u}{\partial^{2n} x_1}\bigg|_{x_1=0} \;=\; 0\,. \tag{1.6.3}$$

The case $n = 0$ is (1.6.1).

Since the derivatives ∂_t and ∂_j for $j > 1$ are parallel to the boundary along which $u = 0$, it follows that u_{tt} and $\partial_j^2 u$ with $j > 1$ vanish at $x_1 = 0$. The equation $\Box u = 0$ implies

$$\frac{\partial^2 u}{\partial x_1^2} \;=\; \frac{\partial^2 u}{\partial t^2} \;-\; \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

The right-hand side vanishes on $\{x_1 = 0\}$ proving the case $n = 1$.

If the case $n \ge 1$ is known, consider $v := \partial_1^{2n} u$. It satisfies the wave equation in $x_1 \le 0$ and, by the inductive hypothesis, satisfies the Dirichlet boundary condition at $x_1 = 0$. The case $n = 1$ applied to v proves the case $n + 1$. This completes the proof of (1.6.3).

It is not hard to prove using Taylor's theorem that (1.6.3) is a necessary and sufficient for the odd extension $\tilde u$ to belong to $C^\infty(\mathbb{R}^{1+d})$. The equation $\Box \tilde u = 0$ for $x_1 \ge 0$ follows from the equation in $x_1 \le 0$ since $\Box \tilde u$ is odd. $\Box$

Example 1.6.1. Suppose that $d = 1$ and that $f \in C_0^\infty(]-\infty, 0[)$ so that for $0 \le t$ small $u = f(x - t)$ is a solution of the wave equation supported to the left of and approaching the boundary $x_1 = 0$. To describe the continuation as a solution satisfying the Dirichlet condition, use the method of images as follows. The solution in $\{x < 0\}$ is the restriction to $x < 0$ of an odd solution of the wave equation. For $t < 0$ the odd extension is equal to the given function in $x < 0$ and to minus its reflection in $\{x > 0\}$,

$$\tilde u \;=\; f(x - t) \;-\; f(-x - t)\,.$$

The formula on the right is the unique odd solution of the wave equation that is equal to u in $\{t < 0\} \cap \{x < 0\}$. The solution u is the restriction of $\tilde u$ to $x < 0$.

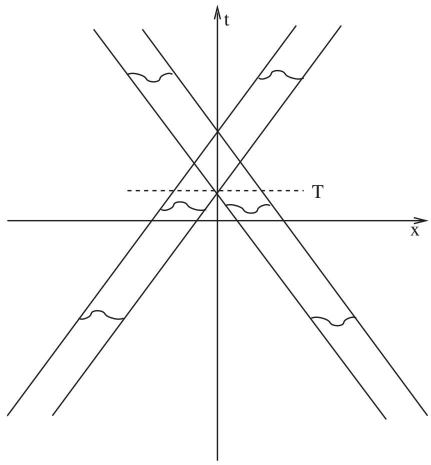

FIGURE 1.6.1. Reflection in dimension $d = 1$

An example is sketched in the Figure 1.6.1. In $\mathbb{R}^{1+1}$ one has an odd solution of the wave equation. There is a rightward moving wave with positive profile and a leftward moving wave with negative profile equal to -1 times the reflection of the first.

Viewed from $x < 0$, there is a wave with positive profile which arrives at the boundary at time T. At that time a leftward moving wave seems to emerge from the boundary. It is the reflection of the wave arriving at the boundary. If the wave arrives at the boundary with amplitude a on an incoming ray, the reflected wave on the reflected ray has amplitude $-a$. The *coefficient of reflection* is equal to -1 (see Figure 1.6.1).

Example 1.6.2. Suppose that $d = 3$ and in $t < 0$ one has an outgoing spherically symmetric wave centered at a point $\underline{x}$ with $\underline{x}_1 < 0$.[2] Until it reaches the boundary, the boundary condition does not play a role. The reflection is computed by extending the incoming wave to an odd solution consisting of the given solution and its negative in mirror image. The moment when the original wave reaches the boundary from the left, its image arrives from the right.

[2]The smooth rotationally symmetric solutions u of $\square_{1+3} u = 0$ centered at the origin are given by (see [Rauch, 1991])

$$u(t,x) \;=\; \frac{f(t+|x|)\;-\;f(t-|x|)}{|x|}\,, \quad \text{when} \quad x \neq 0, \qquad u(t,0) = 2f'(t)\,,$$

where $f \in C^\infty(\mathbb{R})$ is arbitrary. Equivalently, $ru(t,r)$ is an odd solution of $\square_{1+1}\big(ru(t,r)\big) = 0$.

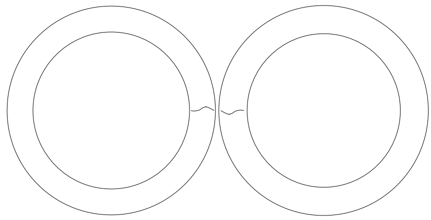

FIGURE 1.6.2. Spherical wave arrives at the boundary

In Figure 1.6.2 the wave on the left has positive profile; and that on the right, a negative profile.

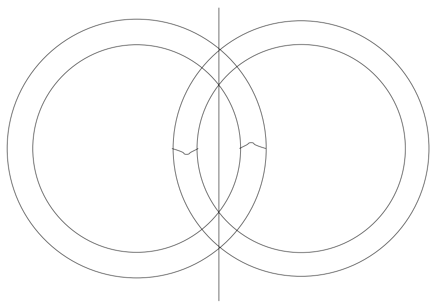

FIGURE 1.6.3. Spherical wave with reflection

In Figure 1.6.3. the middle line represents the boundary. Viewed from $x_1 < 0$, the wave on the left disappears into the boundary and a reflected spherical wave emerges with profile flipped. The profiles of outgoing spherical waves in three-space preserve their thickness and shape. They decrease in amplitude as time increases.

1.6.2. The plane wave derivation. In many texts you will find a derivation that goes as follows. Begin with the plane wave solution

$$e^{i(x\xi - t|\xi|)}, \qquad \xi \in \mathbb{R}^d .$$

The solution is everywhere of modulus one, so it cannot satisfy the Dirichlet boundary condition.

Seek a solution of the initial boundary value problem which is a sum of two plane waves,

$$e^{i(x\xi - t|\xi|)} \ + \ A\, e^{i(x\eta + t\sigma)}, \qquad A \in \mathbb{C} .$$

To satisfy the wave equation, one must have $\sigma^2 = |\eta|^2$. In order that the plane waves sum to zero at $x_1 = 0$, it is necessary and sufficient that $\eta' = \xi'$, $\sigma = -|\xi|$, and $A = -1$. Since $\sigma^2 = |\eta|^2$, it follows that $|\eta| = |\xi|$, so

$$\eta = (\pm\xi_1, \xi_2, \ldots, \xi_d) .$$

The sign $+$ yields the solution $u = 0$. Denote

$$\tilde{x} \ := \ (-x_1, x_2, \ldots, x_d), \qquad \tilde{\xi} \ := \ (-\xi_1, \xi_2, \ldots, \xi_d).$$

The sign $-$ yields the interesting solution

$$e^{i(x\xi - t|\xi|)} \; - \; e^{i(x\tilde{\xi} - t|\tilde{\xi}|)}\,,$$

which is twice the odd part of $e^{i(x\xi - t|\xi|)}$.

The textbook interpretation of the solution is that $e^{i(x\xi - t|\xi|)}$ is a plane wave approaching the boundary $x_1 = 0$, and $e^{i(x\tilde{\xi} - t|\tilde{\xi}|)}$ moves away from the boundary. The first is an incident wave and the second is a reflected wave. The factor $A = -1$ is the reflection coefficient. The direction of motions are computed using the group velocity computed from the dispersion relation.

Both waves are of infinite extent and of modulus one everywhere in space-time. They have finite energy density but infinite energy. They both meet the boundary at all times. It is questionable to think of either one as incoming or reflected. The next subsection shows that there are localized waves which are clearly incoming and reflected waves with the property that when they interact with the boundary, the local behavior resembles the sum of plane waves just constructed.

To analyze reflections for more general mixed initial boundary value problems, wave forms more general than plane waves need to be included. All solutions of the form $e^{i(x\xi + t\tau)}$ with ξ', τ real and $\operatorname{Im}\xi_1 \le 0$ must be considered. When $\operatorname{Im}\xi_1 < 0$, the associated waves are localized near the boundary. The Rayleigh waves in elasticity are a classic example. They carry the devastating energy of earthquakes. Waves of this sort are needed to analyze total reflection described at the end of §1.7. The reader is referred to [Benzoni-Gavage and Serre, 2007], [Chazarain and Piriou, 1982], [Taylor, 1981], [Hörmander, 1983, v.II], and [Sakamoto, 1982] for more information.

1.6.3. Reflected high frequency wave packets. Consider functions that for small time are equal to high frequency solutions from §1.3,

$$u^\epsilon = e^{i(x\xi - t|\xi|)/\epsilon}\, a(\epsilon, t, x)\,, \quad a(\epsilon, t, x) \sim a_0(t,x) + \epsilon\, a_1(t,x) + \cdots\,, \tag{1.6.4}$$

with

$$\xi = (\xi_1, \xi_2, \dots, \xi_d)\,, \qquad \xi_1 > 0\,.$$

Then $a_0(t,x) \;=\; h(x - t\xi/|\xi|)$ is constant on the rays $\underline{x} + t\xi/|\xi|$. If the Cauchy data are supported in a set $\mathcal{O} \Subset \{x_1 < 0\}$, then the amplitudes a_j are supported in the tube of rays

$$\mathcal{T} \;:=\; \Big\{(t,x) \;:\; x = \underline{x} + t\xi/|\xi|, \quad \underline{x} \in \mathcal{O}\Big\}. \tag{1.6.5}$$

Finite speed shows that the wave as well as the geometric optics approximation stays strictly to the left of the boundary for small $t > 0$.

The method of images computes the reflection. Define v^ϵ to be the reversed mirror image solution,

$$v^\epsilon(t, x_1, x_2, \ldots, x_d) \ := \ -u^\epsilon(t, -x_1, x_2, \ldots, x_d)\,.$$

The solution of the Dirichlet problem is then equal to the restriction of $u^\epsilon + v^\epsilon$ to $\{x_1 \le 0\}$.

Then

$$v^\epsilon = -\, e^{i(\tilde{x}\xi - |\xi| t)/\epsilon}\, h(\tilde{x} - t\xi/|\xi|) + \text{h.o.t.} = e^{i(x\tilde{\xi} - |\tilde{\xi}|t)/\epsilon}\, (-Rh)(x - t\tilde{\xi}/|\tilde{\xi}|) + \text{h.o.t}\,.$$

To leading order, $u^\epsilon + v^\epsilon$ is equal to

$$e^{i(x\xi - t|\xi|)/\epsilon}\, h(x - t\xi/|\xi|) \ - \ e^{i(x\tilde{\xi} - |\xi| t)/\epsilon}\, (Rh)(x - t\tilde{\xi}/|\tilde{\xi}|)\,. \tag{1.6.6}$$

The wave represented by u^ϵ has a leading term which moves with velocity $\xi/|\xi|$. The wave corresponding to v^ϵ has a leading term with velocity $\tilde{\xi}/|\tilde{\xi}|$ that comes from $\xi/|\xi|$ by reversing the first component. At the boundary $x_1 = 0$, the tangential components of $\xi/|\xi|$ and $\tilde{\xi}/|\tilde{\xi}|$ are equal and their normal components are opposite. The directions are related by the standard law that the angle of incidence equals the angle of reflection. The amplitude of the reflected wave v^ϵ on the reflected ray is equal to -1 times the amplitude of the incoming wave u^ϵ on the incoming wave. This is summarized by the statement that the reflection coefficient is equal to -1.

Suppose that $\underline{t}, \underline{x}$ is a point on the boundary and $\mathcal{O}$ is a neighborhood of size large compared to the wavelength ϵ and small compared to the scale on which h varies. Then, on $\mathcal{O}$, the solution is approximately equal to

$$e^{i(x\xi - t|\xi|)/\epsilon}\, h(\underline{x} - \underline{t}\xi/|\xi|) \ - \ e^{i(x\tilde{\xi} - t|\tilde{\xi}|)/\epsilon}\, \tilde{h}(\underline{x} - \underline{t}\tilde{\xi}/|\tilde{\xi}|)\,.$$

This recovers the reflected plane waves of §1.6.2. An observer on such an intermediate scale sees the structure of the plane waves. Thus, even though the plane waves are completely nonlocal, the asymptotic solutions of geometric optics shows that they predict the local behavior at points of reflection.

The method of images also solves the Neumann boundary value problem in a half-space using an *even* mirror reflection in $x_1 = 0$. It shows that for the Neumann condition, the reflection coefficient is equal to 1.

Proposition 1.6.4. i. *If $u \in C^\infty(\mathbb{R}^{1+d})$ is an even solution of $\Box u = 0$, then its restriction to $\{x_1 \le 0\}$ is a smooth solution of $\Box u = 0$ satisfying the Neumann boundary condition*

$$\partial_1 u|_{x_1 = 0} = 0. \tag{1.6.7}$$

ii. *Conversely, if $u \in C^\infty(\{x_1 \le 0\})$ is a smooth solution of $\Box u = 0$ satisfying (1.6.8), then the even extension of u to $\mathbb{R}^{1+d}$ is a smooth even solution of $\Box u = 0$.*

The analogue of (1.6.3) in this case is

$$\forall n \geq 0, \qquad \left.\frac{\partial^{2n+1} u}{\partial x_1^{2n+1}}\right|_{x_1=0} = 0\,. \tag{1.6.8}$$

Exercise 1.6.1. *Prove Proposition 1.6.4.*

Exercise 1.6.2. *Prove uniqueness of solutions by the energy method.* **Hint.** Use the local energy identity.

Exercise 1.6.3. *Verify the assertion concerning the reflection coefficient by following the examples above. That is, consider the case of dimension $d = 1$, the case of spherical waves with $d = 3$, and the behavior in the future of a solution which near $t = 0$ is a high frequency asymptotic solution approaching the boundary.*

1.7. Snell's law of refraction

Refraction is the bending of waves as they pass through media whose propagation speeds vary from point to point. The simplest situation is when media with different speeds occupy half-spaces, for example $x_1 < 0$ and $x_1 > 0$. The classical physical situations are when light passes from air to water or from air to glass. Snell observed that for fixed materials, the ratio of the sines of the angles of incidence and refraction $\sin\theta_i/\sin\theta_r$ is independent of the incidence angle. Fermat observed that this would hold if the speed of light were different in the two media and the light chose a path of least time. In that case, the quotient of sines is equal to the ratio of the speeds, c_i/c_r. In this section we derive this behavior for a model problem quite close to the natural Maxwell equations.

The simplified model with the same geometry is

$$u_{tt} - \Delta\, u = 0 \ \text{ in } \ x_1 < 0\,, \quad u_{tt} - c^2\,\Delta\, u = 0 \ \text{ in } \ x_1 > 0\,, \quad 0 < c < 1\,. \tag{1.7.1}$$

In $x_1 < 0$, the speed is equal to 1 which is greater than the speed c in $x > 0$.[3]

A transmission condition is required at $x_1 = 0$ to encode the interaction of waves with the interface. In the one-dimensional case, there are waves that approach the boundary from both sides. The waves that move from the boundary into the interior must be determined from the waves that arrive from the interior. There are two arriving waves and two departing waves. One needs two boundary conditions.

[3] To see that c is the speed of the latter equation, one can (in order of increasing sophistication) factor the one-dimensional operator $\partial_t^2 - c^2\partial_x^2 = (\partial_t + c\partial_x)(\partial_t - c\partial_x)$, or use the formula for group velocity with dispersion relation $\tau^2 = c^2|\xi|^2$, or prove finite speed using the differential law of conservation of energy or Fritz John's Global Hölmgren Theorem.

We analyze the transmission condition that imposes continuity of u and $\partial_1 u$ across $\{x_1 = 0\}$. Seek solutions of (1.7.1) satisfying the transmission condition

$$u(t,0^-,x') \ = \ u(t,0^+,x')\,, \qquad \partial_1 u(t,0^-,x') \ = \ \partial_1 u(t,0^+,x')\,. \tag{1.7.2}$$

Denote by square brackets the jump

$$\big[u\big](t,x') \ := \ u(t,0^+,x') \ - \ u(t,0^-,x')\,.$$

The transmission condition is then

$$\big[u\big] \ = \ 0\,, \qquad \big[\partial_1 u\big] \ = \ 0\,.$$

For solutions that are smooth on both sides of the boundary $\{x_1 = 0\}$, the transmission condition (1.7.2) can be differentiated in t or $x_2, \dots, x_d$ to find

$$\Big[\partial^\beta_{t,x'} u\Big] \ = \ 0\,, \qquad \Big[\partial^\beta_{t,x'} \partial_1 u\Big] \ = \ 0\,. \tag{1.7.3}$$

The partial differential equations then imply that in $x_1 < 0$ and $x_1 > 0$, respectively, one has

$$\frac{\partial^2 u}{\partial x_1^2} \ = \ \frac{\partial^2 u}{\partial t^2} \ - \ \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}\,, \qquad \frac{\partial^2 u}{\partial x_1^2} \ = \ \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} \ - \ \sum_{j=2}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

Therefore at the boundary

$$\left[\frac{\partial^2 u}{\partial x_1^2}\right] \ = \ \left(1 - \frac{1}{c^2}\right)\frac{\partial^2 u}{\partial t^2}\,,$$

the second derivative $\partial_1^2 u$ is expected to be discontinuous at $\{x_1 = 0\}$.

The physical conditions for Maxwell's equations at an air-water or air-glass interface can be analyzed in the same way. In that case, the dielectric constant is discontinuous at the interface.

Define

$$\gamma(x) \ := \ \begin{cases} 1 & \text{when } x_1 > 0, \\ c^{-2} & \text{when } x_1 < 0\,, \end{cases} \qquad e(t,x) \ := \ \frac{\gamma\, u_t^2 + |\nabla_x u|^2}{2}\,.$$

From (1.7.1) it follows that solutions suitably small at infinity satisfy

$$\partial_t \int_{x_1<0} e\ dx \ = \ \int u_t(t,0^-,x')\ \partial_1 u(t,0^+,x')\ dx'\,,$$

$$\partial_t \int_{x_1>0} e\ dx \ = \ -\int u_t(t,0^+,x')\ \partial_1 u(t,0^+,x')\ dx'\,.$$

The transmission condition guarantees that the terms on the right compensate exactly so

$$\partial_t \int_{\mathbb{R}^3} e\ dx \ = \ 0\,.$$

This suffices to prove uniqueness of solutions. A localized argument as in §1.6.1, shows that signals travel at most at speed one.

Exercise 1.7.1. *State and prove this finite speed result.*

A function $u(t,x)$ is called *piecewise smooth* if its restriction to $x_1 < 0$ (resp., $x_1 > 0$) has a C^∞ extension to $x_1 \le 0$ (resp., $x_1 \ge 0$). The Cauchy data of piecewise smooth solutions must be piecewise smooth (with the analogous definition for functions of x only). They must, in addition, satisfy conditions analogous to (1.6.3).

Proposition 1.7.1. *If u is a piecewise smooth solution u of the transmission problem, then the partial derivatives satisfy the sequence of compatibility conditions, for all $j \ge 0$,*

$$\begin{aligned}\Delta^j\{u, u_t\}(t, 0^-, x_2, x_3) &= (c^2\Delta)^j\{u, u_t\}(t, 0^+, x_2, x_3)\,,\\ \Delta^j\partial_1\{u, , u_t\}(t, 0^-, x_2, x_3) &= (c^2\Delta)^j\partial_1\{u, u_t\}(t, 0^+, x_2, x_3)\,.\end{aligned}$$

ii. *Conversely, if the piecewise smooth f, g satisfy for all $j \ge 0$,*

$$\begin{aligned}\Delta^j\{f, g\}(0^-, x_2, x_3) &= (c^2\Delta)^j\{f, g\}(0^+, x_2, x_3)\,,\\ \Delta^j\partial_1\{f, g\}(0^-, x_2, x_3) &= (c^2\Delta)^j\partial_1\{f, g\}(0^+, x_2, x_3)\,,\end{aligned}$$

then there is a piecewise smooth solution with these Cauchy data.

Proof. i. Differentiating (1.7.2) with respect to t yields

$$\big[\partial_t^j u\big] \;=\; 0\,, \qquad \big[\partial_t^j \partial_1 u\big] \;=\; 0\,. \tag{1.7.4}$$

Compute for $k \ge 1$,

$$\partial_t^{2k} u \;=\; \begin{cases}\Delta^k u & \text{when } x_1 < 0,\\ (c^2\Delta)^k u & \text{when } x_1 > 0,\end{cases} \qquad \partial_t^{2k} u_t \;=\; \begin{cases}\Delta^k u_t & \text{when } x_1 < 0,\\ (c^2\Delta)^k u_t & \text{when } x_1 > 0.\end{cases}$$

The transmission conditions (1.7.4) prove **i.**

The proof of **ii** is technical, interesting, and omitted. One can construct solutions using finite differences almost as in §2.2. The shortest existence proof to state uses the spectral theorem for selfadjoint operators.[4] The general regularity theory for such transmission problems can be obtained by

[4] For those with sufficient background, the Hilbert space is $\mathcal{H} := L^2(\mathbb{R}^d\,;\,\gamma\,dx)$.

$$D(\mathcal{A}) \;:=\; \Big\{w \in H^2(\mathbb{R}^d_+) \cap H^2(\mathbb{R}^d_-) \;:\; [w] = [\partial_1 w] = 0\Big\}\,,$$

$$\mathcal{A}w := \Delta w \quad \text{in} \quad x_1 < 0, \qquad \mathcal{A}w := c^2\Delta \quad \text{in} \quad x_1 > 0\,.$$

Then,

$$(\mathcal{A}u, v)_{\mathcal{H}} \;=\; (u, \mathcal{A}v)_{\mathcal{H}} \;=\; -\int \nabla u \cdot \nabla v \; dx\,,$$

so $-\mathcal{A} \ge 0$. The Elliptic Regularity Theorem implies that $\mathcal{A}$ is selfadjoint. The regularity theorem is proved, for example, by the methods in [Rauch, 1991, Chapter 10]. The solution of the initial folding them to a boundary value problem and using the results of [Rauch and Massey, 1974] and [Sakamoto, 1982]. □

Next consider the mathematical problem whose solution explains Snell's law. The idea is to send a wave in $x_1 < 0$ toward the boundary and ask how it behaves in the future. Suppose

$$\xi \in \mathbb{R}^d, \qquad |\xi| = 1\,, \qquad \xi_1 > 0\,,$$

and consider a short wavelength asymptotic solution in $\{x_1 < 0\}$ as in §1.6.3,

$$I^\epsilon \sim e^{i(x\xi - t)/\epsilon}\, a(\epsilon, t, x)\,, \qquad a(\epsilon, t, x) \sim a_0(t, x) + \epsilon\, a_1(t, x) + \cdots\,, \tag{1.7.5}$$

where for $t < 0$ the support of the a_j is contained in a tube of rays with compact cross section and moving with speed ξ. Take a to vanish outside the tube. Since the incoming waves are smooth and initially vanish identically on a neighborhood of the interface $\{x_1 = 0\}$, the compatibilities are satisfied and there is a family of piecewise smooth solutions u^ϵ defined on $\mathbb{R}^{1+d}$. We construct an infinitely accurate description of the family of solutions u^ϵ.

Seek an asymptotic solution that at $\{t = 0\}$ is equal to this incoming wave. A first idea is to find a transmitted wave which continues the incoming wave into $\{x_1\} > 0$.

Seek the transmitted wave in $x_1 > 0$ in the form

$$T^\epsilon \sim e^{i(x\eta + t\tau)/\epsilon}\, d(\epsilon, t, x)\,, \qquad d(\epsilon, t, x) \sim d_0(t, x) + \epsilon\, d_1(t, x) + \cdots\,.$$

On the interface, the incoming wave oscillates with phase $(x'\xi' - t)/\epsilon$ and the proposed transmitted wave oscillates with phase $(x'\eta' + t\tau)/\epsilon$. In order that there be any chance at all of satisfying the transmission conditions one must take

$$\eta' = \xi', \qquad \tau = -1,$$

so that the two expressions oscillate together. In order that the transmitted wave be an approximate solution on the right with positive velocity, one must have

$$\tau^2 = c^2|\eta|^2, \qquad \eta_1 > 0\,.$$

The equation $\tau^2 = c^2|\eta|^2$ implies

$$\eta_1^2 = \frac{\tau^2}{c^2} - |\eta'|^2 = \frac{1}{c^2} - |\xi'|^2\,, \qquad \text{so} \qquad \eta_1 = \left(\frac{1}{c^2} - |\xi'|^2\right)^{1/2} > \xi_1 > 0.$$

Thus,

$$T^\epsilon \sim e^{i(x\eta - t)/\epsilon}\, d(\epsilon, t, x)\,, \qquad \eta = \big(\,(c^{-2} - |\xi'|^2)^{1/2}\,,\ \xi'\,\big)\,. \tag{1.7.6}$$

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value problem is

$$u = \cos t\sqrt{-\mathcal{A}}\; f + \frac{\sin t\sqrt{-\mathcal{A}}}{\sqrt{-\mathcal{A}}}\; g\,.$$

For piecewise H^∞ data, the sequence of compatibilities is equivalent to the data belonging to $\bigcap_j D(\mathcal{A}^j)$.

From section 1.6.3, the leading amplitude d_0 must be constant on the rays $t \mapsto (t\,,\, \underline{x} + c\,t\,\eta/|\eta|)$. To determine d_0, it suffices to know the values $d_0(t, 0^+, x')$ at the interface. One could choose d_0 to guarantee the continuity of u or of $\partial_1 u$, but not both. One cannot construct a good approximate solution consisting of just an incident and transmitted wave.

Add to the recipe a reflected wave. Seek a reflected wave in $x_1 \geq 0$ in the form

$$R^\epsilon \;\sim\; e^{i(x\zeta + t\sigma)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots\,.$$

In order that the reflected wave oscillate with the same phase as the incident wave in the interface $x_1 = 0$, one must have $\zeta' = \xi'$ and $\sigma = -1$. To satisfy the wave equation in $x_1 < 0$ requires $\sigma^2 = |\zeta|^2$. Together these imply $\zeta_1^2 = \xi_1^2$. To have propagation away from the boundary requires $\zeta_1 = -\xi_1$ so $\zeta = \tilde{\xi}$. Therefore,

$$R^\epsilon \;\sim\; e^{i(x\tilde{\xi} - t)/\epsilon}\, b(\epsilon, t, x)\,, \qquad b(\epsilon, t, x) \;\sim\; b_0(t, x) + \epsilon\, b_1(t, x) + \cdots\,. \tag{1.7.7}$$

Summarizing, seek

$$v^\epsilon \;=\; \begin{cases} I^\epsilon + R^\epsilon & \text{in} \quad x_1 < 0, \\ \quad T^\epsilon & \text{in} \quad x_1 > 0. \end{cases}$$

The continuity required at $x_1 = 0$ forces

$$e^{i(x'\xi' - t)/\epsilon}\,\big(a(\epsilon, t, 0, x') + b(\epsilon, t, 0, x')\big) \;=\; e^{i(x'\xi' - t)/\epsilon}\, d(\epsilon, t, 0, x')\,. \tag{1.7.8}$$

The continuity of u and $\partial_1 u$ hold if and only if at $x_1 = 0$ one has

$$a \;+\; b \;=\; d\,, \qquad \text{and} \qquad \frac{i\xi_1}{\epsilon} a + \partial_1 a \;-\; \frac{i\xi_1}{\epsilon} b + \partial_1 b = \frac{i\eta_1}{\epsilon} d + \partial_1 d\,. \tag{1.7.9}$$

The first of these relations yields

$$\Big(a_j \;+\; b_j \;-\; d_j\Big)_{x_1 = 0} = 0, \qquad j = 0, 1, 2, \ldots. \tag{1.7.10}$$

The second relation in (1.7.9) is expanded in powers of ϵ. The coefficients of ϵ^j must match for all all $j \geq -1$. The leading order is ϵ^{-1} and yields

$$\big(a_0 - b_0 - (\eta_1/\xi_1) d_0\big)_{x_1 = 0} = 0\,. \tag{1.7.11}$$

Since a_0 is known, the $j = 0$ equation from (1.7.10) together with (1.7.11) is a system of two linear equations for the two unknown b_0, d_0,

$$\begin{pmatrix} -1 & 1 \\ 1 & \eta_1/\xi_1 \end{pmatrix} \begin{pmatrix} b_0 \\ d_0 \end{pmatrix} \;=\; \begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\,.$$

Since the matrix is invertible, this determines the values of b_0 and d_0 at $x_1 = 0$.

The amplitude b_0 (resp., d_0) is constant on rays with velocity $\tilde{\xi}$ (resp., $c\eta/|\eta|$). Thus the leading amplitudes are determined throughout the half-spaces on which they are defined.

Once these leading terms are known, the ϵ^0 term from the second equation in (1.7.9) shows that on $x_1 = 0$,

$$a_1 - b_1 - d_1 = \text{known}\,.$$

Since a_1 is known, this together with the case $j = 2$ from (1.7.10) suffices to determine b_1, d_1 on $x_1 = 0$. Each satisfies a transport equation along rays that are the analogue of (1.4.12). Thus from the initial values just computed on $x_1 = 0$, they are determined everywhere. The higher order correctors are determined analogously.

Once the b_j, d_j are determined, one can choose b, c as functions of ϵ with the known Taylor expansions at $x = 0$. They can be chosen to have supports in the appropriate tubes of rays and to satisfy the transmission conditions (1.7.9) exactly.

The function u^ϵ is then an infinitely accurate approximate solution in the sense that it satisfies the transmission and initial conditions exactly while the residuals

$$v^\epsilon_{tt} - \Delta\, v^\epsilon := r^\epsilon \quad \text{in} \quad x_1 < 0\,, \qquad v^\epsilon_{tt} - c^2\,\Delta\, v^\epsilon := \rho^\epsilon$$

satisfy for all N, s, T there is a C so that

$$\big\|r^\epsilon\big\|_{H^s([-T,T]\times\{x_1<0\})} \;+\; \big\|\rho^\epsilon\big\|_{H^s([-T,T]\times\{x_1>0\})} \;\le\; C\,\epsilon^N\,.$$

From the analysis of the transmission problem, it follows that with new constants,

$$\big\|u^\epsilon \;-\; v^\epsilon\big\|_{H^s([-T,T]\times\{x_1>0\})} \;\le\; C\,\epsilon^N\,.$$

The proposed problem of describing the family of solutions u^ϵ is solved.

The angles of incidence and refraction, θ_i and θ_r, are computed from the directions of propagation of the incident and transmitted waves. From Figure 1.7.1 one finds

$$\sin\theta_i \;=\; \frac{|\xi'|}{|\xi|}\,, \qquad \text{and} \qquad \sin\theta_r \;=\; \frac{|\eta'|}{|\eta|} \;=\; \frac{|\xi'|}{|\xi|/c}\,.$$

Therefore,

$$\frac{\sin\theta_i}{\sin\theta_r} \;=\; \frac{1}{c}\,,$$

is independent of θ_i. The high frequency asymptotic solutions explain Snell's law. This is the last of the three basic laws of geometric optics. The derivation of Snell's law only uses the phases of the incoming and transmitted waves. The phases are determined by the requirement that the restriction of the phases to $x_1 = 0$ are equal. They do not depend on the precise transmission condition that we chose. It is for this reason that the conclusion is the same for the correct transmission problem for Maxwell's equations.

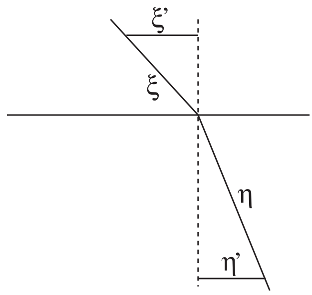

FIGURE 1.7.1.

On a neighborhood $(\underline{t}, \underline{x}) \in \{x_1 = 0\}$ that is small compared to the scale on which a, b, c vary and large compared to ϵ, the solution resembles three interacting plane waves. In science texts one usually computes for which such triples the transmission condition is satisfied in order to find Snell's law. The asymptotic solutions of geometric optics show how to overcome the criticism that the plane waves have modulus independent of (t, x) so cannot reasonably be viewed as either incoming or outgoing.

For a more complete discussion of reflection and refraction, see [Taylor, 1981] and [Benzoni-Gavage and Serre, 2007]. In particular these treat the phenomenon of *total reflection* that can be anticipated as follows. From Snell's law one sees that $\sin\theta_r < 1/c$ and approaches $1/c$ as θ_i approaches $\pi/2$. The refracted rays lie in the cone $\theta_r < \arcsin(1/c)$. Reversing time shows that light rays from below approaching the surface at angles smaller than this critical angle traverse the surface tracing backward the old incident rays. For angles larger than $\arcsin(1/c)$ there is no possible continuation as a ray above the surface. One can show by constructing infinitely accurate approximate solutions that there is total reflection. Below the surface there is a reflected ray with the usual law of reflection. The role of a third wave is played by a boundary layer of thickness $\sim \epsilon$ above which the solution is $O(\epsilon^\infty)$.