
Preface

Gröbner basis theory has become a fundamental field in algebra which provides a wide range of theoretical and computational methods in many areas of mathematics and other sciences. Bruno Buchberger defined the notion of Gröbner basis in 1965 [Bu65]. An intensive research in this theory, related algorithms and applications developed, and many books on this topic have appeared since then. Among them, the books of Adams and Loustaunau [AL94], Becker, Kredel, Weispfenning [BKW93], Cox, Little, O’Shea [CLO05], [CLO07], and Eisenbud [E95] give a fine introduction to Gröbner basis theory and its applications. Many computer algebra systems like CoCoA, Macaulay2, Magma, Maple, Mathematica, or Singular have implemented various versions of Buchberger’s algorithm.

This book aims to provide a concise but rather comprehensive introduction to the theory of Gröbner bases and to introduce the reader to different current trends in theoretical applications of this theory. The complexity level of the presentation increases gradually. The first three chapters and part of Chapter 4 are self-contained and lead to a quick insight into the basics of Gröbner basis theory. They require only a very basic knowledge of algebra. Therefore, this first part of the book would also be appropriate for those readers who are only familiar with elementary algebraic concepts. The second part of Chapter 4 and the last two chapters discuss more advanced topics related to the theory together with applications of it and require a reasonable knowledge in commutative and homological algebra.

Our purpose in writing this book was to provide young researchers and graduate students in commutative algebra and algebraic geometry with methods and techniques related to the Gröbner basis theory. Although it was not our goal to illustrate the algorithmic and computational attributes

of the Gröbner basis theory, the usage of the computer in testing examples is indispensable. Users of computer algebra systems may consult specialized monographs like [GP02], [Mac01] or [KR00], [KR05].

We give now a brief summary of the book's content. Chapter 1 presents polynomial rings in finitely many indeterminates over a field together with their basic properties and studies ideals in this class of rings. The last two sections are devoted to monomial ideals and standard operations on them. Chapter 2 provides a short but comprehensive exposition of the Gröbner basis notion and Buchberger's criterion and algorithm. In Chapter 3 we discuss first applications based on the Elimination Theorem. Chapter 4 is devoted to the extension of the Gröbner basis theory to submodules of free modules over polynomial rings. The chapter begins with a quick introduction to module theory. The more general concepts discussed here lead to a proof, due to Schreyer, for the celebrated Hilbert's Syzygy Theorem. Chapter 5 opens the series of applications of Gröbner basis theory in commutative algebra and combinatorics. In this chapter we discuss semigroup rings and toric ideals which are intensively studied nowadays from different points of view. Chapter 6 intends to introduce the reader to more advanced topics and to subjects of recent research. The topics treated in this section are not presented in the largest possible generality. Instead, one of the main goals of this last chapter of this monograph is to inspire and to enable the reader, who is interested in further developments and other aspects of the theory, to read more advanced monographs and articles on these subjects, for example, the monograph of Sturmfels [St95] which influenced the writing of this chapter substantially, the book [MS05] of Miller and Sturmfels, the influential monograph [S96] of Stanley and the book of Hibi [Hi92]. A compact and detailed presentation of determinantal ideals can be found in the article of Bruns and Conca [BC03]. A reader who is interested in further results on monomial ideals should consult the book [V01] of Villarreal and the book [HH10] of Herzog and Hibi. Further references to research articles are given in the text of Chapter 6. In the first section of the chapter Gröbner bases are used to study Koszul algebras. In particular, it is shown that algebras whose defining ideal has a quadratic Gröbner basis are Koszul. Large classes of algebras whose defining ideal has a quadratic Gröbner basis are provided by algebras generated by sortable sets. This is the topic of the next section. The theory of sortable sets is then applied in the following sections to study generalized Hibi rings and Rees algebras. Next we outline the approach to the theory of determinantal ideals via Gröbner basis with the conclusion that these ideals are all Cohen–Macaulay. Then we give a short introduction to Sagbi bases and apply the theory to show that the coordinate ring of the Grassmannian of m -dimensional vector K -subspaces of K^n is a Gorenstein ring and compute its dimension. The last two sections

of the chapter deal with binomial edge ideals and some aspects of algebraic statistics.

The book contains over one hundred problems with a moderate level of difficulty which illuminate the theory and help the reader to fully understand the results discussed in the text. Some of them complete the proofs. Other problems are more complex and encourage the reader to re-examine the simple but essential ideas, to establish connections, and to become interested in further reading.

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