Preface

Tensors are ubiquitous in the sciences. One reason for their ubiquity is that they provide a useful way to organize data. Geometry is a powerful tool for extracting information from data sets, and a beautiful subject in its own right. This book has three intended uses: as a classroom textbook, a reference work for researchers, and a research manuscript.

0.1. Usage

Classroom uses. Here are several possible courses one could give from this text:

(1) The first part of this text is suitable for an advanced course in multilinear algebra—it provides a solid foundation for the study of tensors and contains numerous applications, exercises, and examples. Such a course would cover Chapters 1–3 and parts of Chapters 4–6.

(2) For a graduate course on the geometry of tensors not assuming algebraic geometry, one can cover Chapters 1, 2, and 4–8 skipping §§2.9–12, 4.6, 5.7, 6.7 (except Pieri), 7.6 and 8.6–8.

(3) For a graduate course on the geometry of tensors assuming algebraic geometry and with more emphasis on theory, one can follow the above outline only skimming Chapters 2 and 4 (but perhaps add §2.12) and add selected later topics.

(4) I have also given a one-semester class on the complexity of matrix multiplication using selected material from earlier chapters and then focusing on Chapter 11.

(5) Similarly I have used Chapter 13 as a basis of a semester-long class on **P** v. **NP**, assuming some algebraic geometry. Here Chapter 8 is important.

(6) I have also given several intensive short classes on various topics. A short class for statisticians, focusing on cumulants and tensor decomposition, is scheduled for the near future.

Reference uses. I have compiled information on tensors in table format (e.g., regarding border rank, maximal rank, typical rank, etc.) for easy reference. In particular, Chapter 3 contains most what is known on rank and border rank, stated in elementary terms. Up until now there had been no reference for even the classical results regarding tensors. (Caveat: I do not include results relying on a metric or Hermitian metric.)

Research uses. I have tried to state all the results and definitions from geometry and representation theory needed to study tensors. When proofs are not included, references for them are given. The text includes the state of the art regarding ranks and border ranks of tensors, and explains for the first time many results and problems coming from outside mathematics in geometric language. For example, a very short proof of the well-known Kruskal theorem is presented, illustrating that it hinges upon a basic geometric fact about point sets in projective space. Many other natural subvarieties of spaces of tensors are discussed in detail. Numerous open problems are presented throughout the text.

Many of the topics covered in this book are currently very active areas of research. However, there is no reasonable reference for all the wonderful and useful mathematics that is *already* known. My goal has been to fill this gap in the literature.

0.2. Overview

The book is divided into four parts: I. First applications, multilinear algebra, and overview of results, II. Geometry and representation theory, III. More applications, and IV. Advanced topics.

Chapter 1: Motivating problems. I begin with a discussion of the *complexity of matrix multiplication*, which naturally leads to a discussion of basic notions regarding tensors (bilinear maps, rank, border rank) and the central question of determining equations that describe the set of tensors of border rank at most r. The ubiquitous problem of *tensor decomposition* is illustrated with two examples: *fluorescence spectroscopy* in chemistry and *cumulants* in statistics. A brief discussion of **P** v. **NP** and its variants is presented as a prelude to Chapter 13, where the study of symmetric tensors

plays an especially important role. *Tensor product states* arising in quantum information theory and *algebraic statistics* are then introduced as they are typical of applications where one studies subvarieties of spaces of tensors. I conclude by briefly mentioning how the geometry and representation theory that occupies much of the first part of the book will be useful for future research on the motivating problems.

This chapter should be accessible to anyone who is scientifically literate.

Chapter 2: Multilinear algebra. The purpose of this chapter is to introduce the language of tensors. While many researchers using tensors often think of tensors as n-dimensional $\mathbf{a}_1 \times \cdots \times \mathbf{a}_n$-tables, I emphasize coordinate-free definitions. The coordinate-free descriptions make it easier for one to take advantage of symmetries and to apply theorems. Chapter 2 includes: numerous exercises where familiar notions from linear algebra are presented in an invariant context, a discussion of rank and border rank, and first steps towards explaining how to decompose spaces of tensors. Three appendices are included. The first contains basic definitions from algebra for reference, the second reviews Jordan and rational canonical forms. The third describes *wiring diagrams*, a pictorial tool for understanding the invariance properties of tensors and as a tool for aiding calculations.

This chapter should be accessible to anyone who has had a first course in linear algebra. It may be used as the basis of a course in multilinear algebra.

Chapter 3: Elementary results on rank and border rank. Rank and border rank are the most important properties of tensors for applications. In this chapter I report on the state of the art. When the proofs are elementary and instructional, they are included as well, otherwise they are proven later in the text. The purpose of this chapter is to provide a reference for researchers.

Chapter 4: Algebraic geometry for spaces of tensors. A central task to be accomplished in many of the motivating problems is to test if a tensor has membership in a given set (e.g., if a tensor has *rank* r). Some of these sets are defined as the zero sets of collections of polynomials, i.e., as *algebraic varieties*, while others can be expanded to be varieties by taking their *Zariski closure* (e.g., the set of tensors of border rank at most r is the Zariski closure of the set of tensors of rank at most r). I present only the essentials of projective geometry here, in order to quickly arrive at the study of groups and their modules essential to this book. Other topics in algebraic geometry are introduced as needed.

This chapter may be difficult for those unfamiliar with algebraic geometry—it is terse as numerous excellent references are available (e.g., [**157, 289**]). Its purpose is primarily to establish language. Its prerequisite is Chapter 2.

Chapter 5: Secant varieties. The notion of border rank for tensors has a vast and beautiful generalization in the context of algebraic geometry, to that of *secant varieties of projective varieties*. Many results on border rank are more easily proved in this larger geometric context, and it is easier to develop intuition regarding the border ranks of tensors when one examines properties of secant varieties in general.

The prerequisite for this chapter is Chapter 4.

Chapter 6: Exploiting symmetry: Representation theory for spaces of tensors. Representation theory provides a language for taking advantage of symmetries. Consider the space $Mat_{n\times m}$ of $n \times m$ matrices: one is usually interested in the properties of a matrix up to changes of bases (that is, the underlying properties of the linear map it encodes). This is an example of a vector space with a group acting on it. Consider polynomials on the space of matrices. The minors are the most important polynomials. Now consider the space of $n_1 \times n_2 \times \cdots \times n_k$-way arrays (i.e., a space of tensors) with $k > 2$. What are the spaces of important polynomials? Representation theory helps to give an answer.

Chapter 6 discusses representations of the group of permutations on d elements, denoted $\mathfrak{S}_d$, and of the group of invertible $n \times n$ matrices, denoted $GL_n\mathbb{C}$, and applies it to the study of homogeneous varieties. The material presented in this chapter is standard and excellent texts already exist (e.g., [**268, 135, 143**]). I focus on the aspects of representation theory useful for applications and its implementation.

The prerequisite for this chapter is Chapter 2.

Chapter 7: Tests for border rank: Equations for secant varieties. This chapter discusses the equations for secant varieties in general and gives a detailed study of the equations of secant varieties of the varieties of rank one tensors and symmetric tensors, i.e., the varieties of tensors, and symmetric tensors of border rank at most r. These are the most important objects for tensor decomposition, so an effort is made to present the state of the art and to give as many different perspectives as possible.

The prerequisite to Chapter 7 is Chapter 6.

Chapter 8: Additional varieties useful for spaces of tensors. In addition to secant varieties, there are general classes of varieties, such as

tangential varieties, *dual varieties*, and the *Fano varieties of lines* that generalize certain attributes of tensors to a more general geometric situation. In the special cases of tensors, these varieties play a role in classifying normal forms and the study of rank. For example, dual varieties play a role in distinguishing the different *typical ranks* that can occur for tensors over the real numbers. They should also be useful for future applications. Chapter 8 discusses these as well as the *Chow variety* of polynomials that decompose to a product of linear factors. I also present differential-geometric tools for studying these varieties.

Chapter 8 can mostly be read immediately after Chapter 4.

Chapter 9: Rank. It is more natural in algebraic geometry to discuss border rank than rank because it relates to projective varieties. Yet, for applications sometimes one needs to determine the ranks of tensors. I first regard rank in a more general geometric context, and then specialize to the cases of interest for applications. Very little is known about the possible ranks of tensors, and what little is known is mostly in cases where there are normal forms, which is presented in Chapter 10. The main discussion in this chapter regards the ranks of symmetric tensors, because more is known about them. Included are the Comas-Seguir theorem classifying ranks of symmetric tensors in two variables as well as results on maximum possible rank. Results presented in this chapter indicate there is beautiful geometry associated to rank that is only beginning to be discovered.

Chapter 9 can be read immediately after Chapter 5.

Chapter 10: Normal forms for small tensors. The chapter describes the spaces of tensors admitting normal forms, and the normal forms of tensors in those spaces, as well as normal forms for points in small secant varieties.

The chapter can be read on a basic level after reading Chapter 2, but the proofs and geometric descriptions of the various orbit closures require material from other chapters.

The next four chapters deal with applications.

Chapter 11: The complexity of matrix multiplication. This chapter brings the reader up to date on what is known regarding the complexity of matrix multiplication, including new proofs of many standard results.

Much of the chapter needs only Chapter 2, but parts require results from Chapters 5 and 6.

Chapter 12: Tensor decomposition. In many applications one would like to express a given tensor as a sum of rank one tensors, or some class of simple tensors. In this chapter I focus on examples coming from signal processing and discuss two such: blind source separation and deconvolution of DS-CMDA signals. The blind source separation problem is similar to many questions arising in statistics, so I explain the larger context of the study of cumulants.

Often in applications one would like unique expressions for tensors as a sum of rank one tensors. I bring the reader up to date on what is known regarding when a unique expression is possible. A geometric proof of the often cited Kruskal uniqueness condition for tensors is given. The proof is short and isolates the basic geometric statement that underlies the result.

The chapter can be read after reading Chapters 2 and 3.

Chapter 13: P versus NP. This chapter includes an introduction to several algebraic versions of **P** and **NP**, as well as a discussion of Valiant's *holographic algorithms*. It includes a discussion of the GCT program of Mulmuley and Sohoni, which requires a knowledge of algebraic geometry and representation theory, although the rest of the chapter is elementary and only requires Chapter 2.

Chapter 14: Varieties of tensors in phylogenetics and quantum mechanics. This chapter discusses two different applications with very similar underlying mathematics. In both cases one is interested in isolating subsets (subvarieties) of spaces of tensors with certain attributes coming from physics or statistics. It turns out the resulting varieties for *phylogenetics* and *tensor network states* are strikingly similar, both in their geometry, and in the methods they were derived (via auxiliary graphs).

Much of this chapter only requires Chapter 2 as a prerequisite.

The final three chapters deal with more advanced topics.

Chapter 15: Outline of the proof of the Alexander-Hirschowitz theorem. The dimensions of the varieties of symmetric tensors of border rank at most r were determined by Alexander and Hirschowitz. A brief outline of a streamlined proof appearing in [**257**] is given here.

This chapter is intended for someone who has already had a basic course in algebraic geometry.

Chapter 16: Representation theory. This chapter includes a brief description of the rudiments of the representation theory of complex simple Lie groups and algebras. There are many excellent references for this subject, so I present just enough of the theory for our purposes: the proof of

Kostant's theorem that the ideals of homogeneous varieties are generated in degree two, the statement of the Bott-Borel-Weil theorem, and a discussion of the *inheritance* principle of Chapter 6 in a more general context.

This chapter is intended for someone who has already had a first course in representation theory.

Chapter 17: Weyman's method. The study of secant varieties of triple Segre products naturally leads to the Kempf-Weyman method for determining ideals and singularities of G-varieties. This chapter contains an exposition of the rudiments of the method, intended primarily to serve as an introduction to the book [**333**].

The prerequisites for this chapter include Chapter 16 and a first course in algebraic geometry.

0.3. Clash of cultures

In the course of preparing this book I have been fortunate to have had many discussions with computer scientists, applied mathematicians, engineers, physicists, and chemists. Often the beginnings of these conversations were very stressful to all involved. I have kept these difficulties in mind, attempting to write both to geometers and researchers in these various areas.

Tensor practitioners want practical results. To quote Rasmus Bro (personal communication): "*Practical* means that a user of a given chemical instrument in a hospital lab can push a button and right after get a result."

My goal is to initiate enough communication between geometers and scientists so that such practical results will be realized. While both groups are interested in communicating, there are language and even philosophical barriers to be overcome. The purpose of this paragraph is to alert geometers and scientists to some of the potential difficulties in communication.

To quote G. Folland [**126**] "For them [scientists], mathematics is the discipline of manipulating symbols according to certain sophisticated rules, and the external reality to which those symbols refer lies not in an abstract universe of sets but in the real-world phenomena that they are studying."

But mathematicians, as Folland observed, are Platonists, we think the things we are manipulating on paper have a higher existence. To quote Plato [**266**]: "*Let us take any common instance; there are beds and tables in the world—plenty of them, are there not?*

Yes. But there are only two ideas or forms of them—one the idea of a bed, the other of a table.

True. And the maker of either of them makes a bed or he makes a table for our use, in accordance with the idea—that is our way of speaking in

this and similar instances—but no artificer makes the ideas themselves: how could he?

And what of the maker of the bed? Were you not saying that he too makes, not the idea which, according to our view, is the essence of the bed, but only a particular bed?

Yes, I did. Then if he does not make that which exists he cannot make true existence, but only some semblance of existence; and if any one were to say that the work of the maker of the bed, or of any other workman, has real existence, he could hardly be supposed to be speaking the truth."

This difference of cultures is particularly pronounced when discussing tensors: for some practitioners these are just multi-way arrays that one is allowed to perform certain manipulations on. For geometers these are spaces equipped with certain group actions. To emphasize the geometric aspects of tensors, geometers prefer to work invariantly: to paraphrase W. Fulton: "Don't use coordinates unless someone holds a pickle to your head."[1]

0.4. Further reading

For gaining a basic grasp of representation theory as used in this book, one could consult [**143, 135, 268, 170**]. The styles of these books vary significantly and the reader's taste will determine which she or he prefers. To go further with representation theory [**187**] is useful, especially for the presentation of the Weyl character formula. An excellent (and pictorial!) presentation of the implementation of the Bott-Borel-Weil theorem is in [**19**].

[1]This modification of the actual quote in tribute to my first geometry teacher, Vincent Gracchi. A problem in our 9-th grade geometry textbook asked us to determine if a 3-foot long rifle could be packed in a box of certain dimensions, and Mr. Gracchi asked us all to cross out the word 'rifle' and substitute the word 'pickle' because he "did not like guns". A big $5q + 5q$ to Mr. Gracchi for introducing his students to geometry!

For basic algebraic geometry as in Chapter 4, [**157, 289**] are useful. For the more advanced commutative algebra needed in the later chapters [**119**] is written with geometry in mind. The standard and only reference for the Kempf-Weyman method is [**333**].

The standard reference for what was known in algebraic complexity theory up to 1997 is [**54**].

0.5. Conventions, acknowledgments

0.5.1. Notations. This subsection is included for quick reference. All notations are defined properly the first time they are used in the text.

Vector spaces are usually denoted A, B, C, V, W, and A_j, and the dimensions are usually the corresponding bold letters $\mathbf{a}, \mathbf{b}, \mathbf{c}$, etc. If $v_1, \dots, v_p \in V$, $\langle v_1, \dots, v_p \rangle$ denotes the span of $v_1, \dots, v_p$. If $e_1, \dots, e_{\mathbf{v}}$ is a basis of V, $e^1, \dots, e^{\mathbf{v}}$ denotes the dual basis of V^*. $GL(V)$ denotes the general linear group of invertible linear maps $V \to V$ and $\mathfrak{gl}(V)$ its Lie algebra. If G denotes a Lie or algebraic group, $\mathfrak{g}$ denotes its associated Lie algebra.

If $X \subset \mathbb{P}V$ is an algebraic set, then $\hat{X} \subset V$ is the cone over it, its inverse image plus 0 under $\pi : V \backslash 0 \to \mathbb{P}V$. If $v \in V$, $[v] \in \mathbb{P}V$ denotes $\pi(v)$. The linear span of a set $X \subset \mathbb{P}V$ is denoted $\langle X \rangle \subseteq V$.

For a variety X, X_{smooth} denotes its smooth points and X_{sing} denotes its singular points. $X_{general}$ denotes the set of general points of X. Sometimes I abuse language and refer to a point as a general point with respect to other data. For example, if $L \in G(k, V)$ and one is studying the pair (X, L) where $X \subset \mathbb{P}V$ is a subvariety, I will call L a general point if L is in general position with respect to X.

$\Lambda^k V$ denotes the k-th exterior power of the vector space V. The symbols $\wedge$ and $\bigwedge$ denote exterior product. $S^k V$ is the k-th symmetric power. The tensor product of $v, w \in V$ is denoted $v \otimes w \in V^{\otimes 2}$, and symmetric product has no marking, e.g., $vw = \frac{1}{2}(v \otimes w + w \otimes v)$. If $p \in S^d V$ is a homogeneous polynomial of degree d, write $p_{k,d-k} \in S^k V \otimes S^{d-k} V$ for its partial polarization and $\overline{p}$ for p considered as a d-multilinear form $V^* \times \cdots \times V^* \to \mathbb{C}$. When needed, $\circ$ is used for the symmetric product of spaces, e.g., given a subspace $W \subset S^q V$, $W \circ S^p V \subset S^{q+p} V$.

$\mathfrak{S}_d$ denotes the group of permutations on d elements. To a partition $\pi = (p_1, \dots, p_r)$ of d, i.e., a set of integers $p_1 \geq p_2 \geq \cdots \geq p_r$, $p_i \in \mathbb{Z}_+$, such that $p_1 + \cdots + p_r = d$, $[\pi]$ denotes the associated irreducible $\mathfrak{S}_d$-module, and $S_\pi V$ denotes the associated irreducible $GL(V)$-module. I write $|\pi| = d$ and $\ell(\pi) = r$.

0.5.2. Layout. All theorems, propositions, remarks, examples, etc., are numbered together within each section; for example, Theorem 1.3.2 is the second numbered item in Section 1.3. Equations as well as figures are numbered sequentially within each section. I have included hints for selected exercises, those marked with the symbol ◎ at the end, which is meant to be suggestive of a life preserver.

0.5.3. Acknowledgments. This project started as a collaboration with Jason Morton, who contributed significantly to the writing and editing of Chapters 2, 4, 5, and 6. The book has greatly benefited from his input. The first draft of this book arose out of a series of lectures B. Sturmfels invited me to give for his working group at UC Berkeley in Spring 2006. I then gave a graduate class at Texas A&M University in Fall 2007 on the complexity of matrix multiplication and a class on complexity theory in spring 2009. J. Morton and I gave lectures from the notes at a summer graduate workshop at MSRI in July 2008, as well as several lectures at a follow-up research workshop at AIM in July 2008. I also gave a GNSAGA lecture series in Florence, Italy in June 2009 on secant varieties, a short course on the geometry of tensors in June 2010 in Nordfjordeid, Norway, and a one-semester graduate class on the same subject at Texas A&M, Fall 2010. It is a pleasure to thank the students in these classes as well as my hosts B. Sturmfels, MSRI, AIM, G. Ottaviani and K. Ranestad. Much of the material in this book comes from joint work with L. Manivel, G. Ottaviani, and J. Weyman. It is a pleasure to thank these three collaborators for significant help at each step along the way. Other material comes from joint work with the (at the time) post-docs J. Buczyński, J. Morton, S. Norine, and Z. Teitler, who have also significantly helped with the book. I also thank my students and post-docs A. Boralevi, L. Nguyen, L. Oeding, Y. Qi, D. The, M. Yang, and K. Ye for their useful comments and questions, as well as R. Bardeli, M. Bläser, P. Bürgisser, E. Briand, J-Y Cai, P. Comon, L. De Lathauwer, M. Dillon, M. Eastwood, J. von zur Gathen, J. Grochow, C. Hammond, L-H Lim, G. Paouris, and A. Stegeman for useful comments and patiently answering my questions. It would not have been possible to write this book without such wonderful help. I was very fortunate to have careful and thoughtful anonymous referees who made numerous outstanding suggestions for improving the manuscript for which I am truly grateful. Finally I thank my family for their help, support and understanding.