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Introduction

The aim of this monograph is to describe the models of the elementary theory of an o-minimal expansion of the reals in which the exponential function is definable. We focus on polynomially bounded + (exp) expansions (cf. Section 4 of Chapter 6). Important examples of such expansions are: the expansion by restricted analytic functions and the unrestricted exponential function (cf. [D–M–M1]), the expansion by convergent generalized power series and the unrestricted exponential function (in which the Riemann ζ-function restricted to $(1,\infty)$ is definable: cf. [D–S1]), and the expansion by multisummable real power series and the unrestricted exponential function (in which the Γ-function restricted to $(0,\infty)$ is definable: cf. [D–S2]).

The notion of o-minimality (cf. Section 2 of Chapter 6) was introduced by van den Dries in [D3], while studying the expansion $(\mathbb{R}, \exp)$ of the ordered field of the real numbers by the real exponential function. Van den Dries observed that the subsets of the cartesian products $\mathbb{R}^n$ which are parametrically definable in an o-minimal expansion of $\mathbb{R}$ share many of the geometric properties of semi-algebraic sets. For example, a semi-algebraic set has only finitely many connected components, each of them semi-algebraic (cf. [CO]). Van den Dries showed that this result remains true if one replaces "semi-algebraic" by "parametrically definable in an o-minimal expansion of $\mathbb{R}$" (cf. also *cell-decomposition* for o-minimal structures [K–P–S]). This is a *finiteness theorem*, and van den Dries has set out as a goal to explain the other finiteness phenomena in real algebraic and real analytic geometry as consequences of o-minimality (cf. [D4]). The breakthrough was achieved with Wilkie's results on the o-minimality of the reals with exponentiation (cf. [W1]). He showed that the expansion of $\mathbb{R}$ by Pfaffian functions *restricted to the closed unit box* (i.e., the functions are set to be identically zero outside the unit box) has a model complete theory. This result may be viewed as a strong refinement of Gabrielov's Theorem (cf. [GA]). The latter states that the class of sub-analytic sets is closed under taking complements. Wilkie's Theorem shows that if the restricted analytic functions used to describe a given sub-analytic set A are Pfaffian, then the complement of A may also be described by Pfaffian functions. Wilkie also establishes the model completeness of the elementary theory $T(\exp)$ of $\mathbb{R}$ with the real exponential function exp. This theorem has an important geometric interpretation (cf. [W1], p. 1054): call a subset of $\mathbb{R}^n$ *semi-exponential-algebraic* (semi-EA) if it is defined by exponential-polynomial equations and inequalities, and a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ semi-EA if its graph is so, and finally a set to be *sub*-EA if it is the image of a semi-EA set under a semi-EA map. Then the theorem is equivalent to the assertion that the complement of a sub-EA set is a sub-EA set. This, as for the semi-algebraic case, implies that the class of sub-EA sets is also closed under taking closures, interiors and boundaries. Recently (cf. [W2]) Wilkie proved a far-reaching generalization, from which it follows that the expansion of the reals by *total* Pfaffian functions is o-minimal as well.

When proving results such as model completeness, the model theorist has to investigate the class of *all* models of a given theory, instead of studying one particular model. At the center of the class of models that we consider lies the ordered field of the reals. But in this class there are also non-archimedean models, which we undertake to describe. Our basic tool for this is valuation theory. Inspired by the theory of real places and real closed fields, and seeking the analogy to the semi-algebraic case, we systematically develop an exponential analogue for all important notions and methods. We use this abstract machinery to describe explicitly the algebraic structure of the models, and to give concrete constructions. These constructions use power series fields (cf. Theorem 5.7 and Theorem 5.11).

The valuation theory that we need is basic. Indeed, we consider ordered fields with convex valuations, whose residue fields are ordered and hence of characteristic zero. There are no deep mysteries in the theory of valued fields in the characteristic zero case. We use, so to say, "shadows" of valuations. In fact, we reduce the valuation theory to the "bones" by going down to the skeletons of the value groups, and yet further down to the value sets of the value groups. This descent makes us deal with those value sets, which are often lexicographic orderings and have some kind of ultrametric structure that is reminiscent of the valuations way up on the fields.

At first sight, this approach may seem too simplistic. However, the power of this method comes from the fact that in most important cases, as in the case of the power series fields, we can lift the whole situation from the value set back up to the field, via the value group.

At the beginning of this work, our approach to the exponential function is equally naive. Although our ultimate aim is to construct extremely well-behaved exponentials (that is, satisfying all the elementary theorems that the real exponential function does), we start by working on very simple exponentials. We just demand that they are order preserving group isomorphisms from the additive group of the field onto the multiplicative group of its positive elements. But even those almost ridiculously weak exponentials impose tremendous restrictions on the structure of the ordered fields that carry them. At the point of this work where we discover that power series fields cannot carry exponentials, we are forced to make our demands on the exponentials, (or more precisely on their compositional inverses, the *logarithms*) even more modest than before. Then, we have to learn to work with prelogarithms, that is non-surjective logarithms. We just have these logarithms that are not even surjective and that enjoy no reasonable properties whatsoever.

Luckily however, we have three important keys to modify and improve those weak prelogarithms, all the way through.

The first is the discovery that power series fields, whilst not carrying surjective logarithms, always carry *prelogarithms*. Thus we develop a standard method to get a surjective logarithm on a countable union of power series fields.

The second key is a result of [D–M–M1] who show that power series fields can be naturally made into models of *restricted* real exponentiation, and even more, of all restricted analytic functions. This structure is preserved by taking countable unions of power series fields. Thus we now end up with ordered fields, endowed with an exponential whose *restriction to the unit box* of the field enjoys the same elementary properties as the real exponential restricted to the unit box in the reals.

But we are not done. In fact the exponential thus constructed does *not yet* satisfy the elementary properties of the *unrestricted* real exponential! Here, we use the last key that opens the last door: Ressayre's Theorem [RE]. It states that an exponential on an ordered field which satisfies the elementary properties of the restricted real exponential, *and* satisfies the growth axiom scheme (GA), satisfies the elementary properties of the *unrestricted* real exponential as well.

This leads us to a substantial part of the research presented in this monograph. We undertake a systematic study of the growth axiom scheme. Our main idea is to investigate what (GA) imposes on the value group of the field. Descending even further down, we are able to *encode* the growth axiom scheme in the *value set* of the value group of the field. Heuristically, (GA) is encoded in the order automorphisms which the value set carries. It is certainly much easier to understand a totally ordered *set* (the value set) endowed with an automorphism, than to understand a totally ordered *field* endowed with a logarithm. So to say, we get rid of the algebraic structure, but we lose nothing. Indeed, as mentioned already, we are in the privileged situation where we can lift this information up again, from the value set to the ordered field. We develop a canonical method to achieve this lifting. Finally, this allows us to describe the non-archimedean ordered fields endowed with exponentials and restricted analytic functions which satisfy all the axioms of real exponentiation, and which moreover enjoy further interesting properties.

Related to (GA) is the notion of exponential rank. It is a finer measurement of the growth rates of exponentials than just (GA). We encode the exponential rank as well in the automorphisms of the value set. We construct fields of arbitrary exponential ranks, the "exponential-logarithmic power series fields". These canonically defined exponential fields are the exponential analogue of power series fields in the real case. They can carry a multitude of exponentials of *distinct* growth rates, and enjoy further surprising properties of which we provide a detailed account.

Hardy fields provide the most beautiful example of non-archimedean exponential fields. They were introduced by Hardy (cf. [HD2]), as "the natural domain for the study of asymptotic analysis". We apply our general structure theorems for exponential fields to this particularly important case. Inspired by Rosenlicht's work on Hardy fields of finite rank (cf. [RO1]), we extend the study to the infinite rank case. We give a detailed description of the value groups and residue fields of Exponential Hardy fields (which necessarily have infinite rank). Using our results, we present at the end of the monograph a new proof to a conjecture raised by Hardy concerning the asymptotic behaviour of the "Logarithmico-Exponential" functions. This conjecture was first established in [D–M–M2] by different methods.

In Chapter 0, we gather some preliminaries about valuations on *ordered Abelian groups*. We introduce the value set and the skeleton of such a group. These invariants turn out to be very handy throughout this research.

In Chapter 1, we study the necessary conditions that an ordered exponential field K has to satisfy. We show at the end of Chapter 2 that the exponential induces canonically an amazing map (which we call a contraction) on the value group G of K with respect to the natural valuation. A contraction contracts every archimedean class $\neq \{0\}$ to a set $\{a, -a\}$ of two points, and yet maps G *surjectively* onto G in an order preserving way. The class of ordered Abelian groups that are able to carry contractions is much smaller than the class of *all* ordered Abelian groups. It is an elementarily axiomatizable class and has a very well-behaved model theory. This has been worked out in [KF1] and [KF2], and we present the results of these two papers in the Appendix of this monograph.

At the end of Chapter 1, we study "small" non-archimedean exponential fields. We investigate the following problem: Given an ordered field K, and assuming that its residue field $\overline{K}$ with respect to the natural valuation (which is a subfield of $\mathbb{R}$) is an exponentially closed subfield of the reals, is it possible to lift the real exponential exp to an exponential f of K? We answer this problem for non-archimedean countable fields that are root closed for positive elements. We get a structure theorem for those fields (cf. Theorem 1.44), and show that they admit exponentials f lifting exp from their residue fields if and only if their value group is isomorphic to the lexicographic sum of copies of the additive ordered group $(\overline{K}, +, 0, <)$, taken over the rationals. In Section 6 of Chapter 2, we show that this condition is indeed sufficient to get exponentials satisfying (GA) on the countable field. This theorem provides a method to construct non-archimedean countable exponential fields, given archimedean ones (cf. Example 1.45).

In Chapter 2, we translate the meaning of (GA) and the Taylor axiom scheme (T) into a valuation theoretic language. For example, we show that (T) is equivalent to assertions of the form $v(f(x)-E_n(x)) > v(x^n)$ (where E_n denotes the n-th partial sum in the Taylor expansion of exp). These results are used throughout the later chapters.

In Chapter 3, we introduce prelogarithms and define the exponential rank to be the chain of convex valuation rings which are compatible with the prelogarithm. We characterize the exponential rank through *exponential equivalence* (cf. Section 4 of Chapter 3), as the rank is characterized by the "multiplicative equivalence relation" in the real case. It is worthwhile mentioning here that if K is a model of the elementary theory T of an exponentially bounded expansion of the reals, such that the exponential f is definable, then the valuation rings R_w of valuations w compatible with f are precisely the T-convex valuation rings of K, in the sense of [D–L].

So far, we have only described results that are in nice analogy to the theory of real places. But when it comes to existence results, the analogy breaks down. If a field has a place onto an ordered residue field, then the order can be lifted up to the field through the place. It is not surprising that exponentials cannot be lifted through arbitrary places. Indeed, we show in Chapter 4 that power series fields *never* admit exponentials compatible with their canonical valuation. (It is interesting to note that there is an exponential on the surreal numbers, cf. [G], but this "power series field" is a proper class.) However, we show that every power series field $\mathbb{R}((G))$ carries a prelogarithm. Indeed, in Section 3 of Chapter 4, we give an explicit formula for the basic prelogarithm $\log_0$ on power series fields with any given value group of the form $\mathbb{R}^{\Gamma_0}$, where Γ_0 is a totally ordered set. Going to the union over an increasing chain of power series fields, we make this logarithm $\log_0$ surjective (cf. Section 4 of Chapter 4). We call the so-obtained field the *exponential-logarithmic power series field* and denote it by $R((\Gamma_0))^{EL}$. The logarithm $\log_0$ does not satisfy (GA), and we develop a method in Chapter 5 to modify $\log_0$. We show that one can use any order preserving map σ on Γ_0 satisfying that $\sigma\gamma > \gamma$ for all γ, to derive from $\log_0$ a logarithm $\log_\sigma$ having the right growth rate. Its inverse $\exp_\sigma$ will then yield a model of real exponentiation (and restricted analytic functions). This method enables us also to construct exponential fields with arbitrary principal exponential ranks. In this way, our construction exhibits the relation between order endomorphisms of the value groups and the growth rates of the constructed exponentials. We also show that $\mathbb{R}((\Gamma_0))^{EL}$ admits countably infinitely many exponentials of distinct exponential rank, for *any* Γ_0. This contrasts the impression of rigidity which is given by the notation $\mathbb{R}((\Gamma_0))^{EL}$ (cf. Example 5.10 and Remark 5.14).

Analogous constructions, using power series fields, are given in [D–M–M2]. There, a first limit process is employed to obtain a field with non-surjective exponential, and then a second (inverse) limit process renders the exponential surjective. The outcome is a model called the "logarithmic-exponential power series field". In contrast to the construction given in [D–M–M2], our construction uses only one limit process. It is an interesting task for future research to compare the models obtained by these two different approaches.

Chapter 6 answers a question raised by Macintyre in a course given at The Fields Institute, during the Algebraic Model Theory Program, November 1996. In [D–M–M2], the authors use results of Ressayre and Mourgues to show that Hardy's field LE of Logarithmico-exponential functions admits a truncation-closed embedding in the logarithmic-exponential power series field (see above). Then, they use this particular embedding to prove Hardy's conjecture (cf. Section 9 of Chapter 6 for details) and to show that certain functions, including the Gamma-function and the Riemann ζ-function, cannot be defined using exponential function, logarithm and restricted analytic functions. While lecturing on the results of [D–M–M2], Macintyre asked whether their results could be deduced by a "more invariant" version of truncation. Indeed, we derive the results of [D–M–M2] without using embeddings in the logarithmic exponential power series field. We replace truncation results by an intrinsic property, which is an assertion about the residue fields of the Hardy fields with respect to arbitrary convex valuations. It is invariant because it does not depend on an embedding in logarithmic-exponential power series fields. As a by-product, we get a structure theorem for the Hardy fields associated to a polynomially bounded + (exp) expansion of the reals (cf. Theorem 6.30) and show, amongst other results, that these Hardy fields have levels in the sense of Rosenlicht (cf. [RO3]).

Several results presented in this monograph were obtained in the joint papers [K–K1], [K–K2], [K–K3], [K–K4] and [K–K–S1]. I would like to thank my co-authors Franz-Viktor Kuhlmann and Saharon Shelah for their essential contributions in our joint work. Special thanks are due to Franz-Viktor Kuhlmann for allowing the inclusion of results of [KF1] and [KF2] as an Appendix, and for proofreading my manuscript.

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