

Introduction

The purpose of this monograph is to provide an introduction to Shimura curves from a theoretical and algorithmic perspective. Shimura curves lie at the crossroads of many areas, including complex analysis, p -adic analysis, arithmetic, Diophantine geometry, algebraic geometry, algebra, and noncommutative algebra. Our approach to them has two objectives: to construct fundamental domains in the Poincaré half-plane, and to determine their complex multiplication points. Our presentation is based on a previous study of quadratic forms attached to orders in quaternion algebras. The algorithms needed for the computations have been compiled in a package, named `Poincare`, which has been implemented in *Maple V*.

Let H denote a quaternion algebra defined over a totally real number field K of degree d , and let \mathcal{O} be an order in H . We shall deal exclusively with indefinite algebras and, moreover, we suppose that $\mathbb{R} \otimes_{\mathbb{Q}} H = M(2, \mathbb{R}) \times \mathbb{H}^{d-1}$, where \mathbb{H} denotes the skew field of the Hamilton quaternions. By considering an embedding of H into the matrix algebra $M(2, \mathbb{R})$, the group of the units of \mathcal{O} of positive norm provides an arithmetic Fuchsian group $\Gamma \subseteq \mathrm{SL}(2, \mathbb{R})$. The group Γ acts on the Poincaré half-plane \mathcal{H} and the quotient $\Gamma \backslash \mathcal{H}$ yields a Riemann surface which is compact unless $H = M(2, K)$. The projective nonsingular curve attached to this Riemann surface will be denoted by $X(\Gamma)$. From 1960 onward, Shimura outlined a theory for these curves, which came to be known as *Shimura curves*. If $K = \mathbb{Q}$ and Γ is commensurable with a subgroup of the modular group $\mathrm{SL}(2, \mathbb{Z})$, then the curves $X(\Gamma)$ are the classical modular curves, whose study goes back to Klein and Fricke, in the nineteenth century.

One of Shimura's main contributions in the topic that concerns us is the theory of the canonical models. To prove their existence, Shimura describes the curves $X(\Gamma)$ as moduli spaces of principally polarized abelian surfaces with quaternion multiplication and level structure. To prove their uniqueness, Shimura considers what today are called the complex multiplication points of the curves, which correspond to those abelian surfaces which are also endowed with complex multiplication. This modular point of view generalizes the well-known interpretation of the classical modular curves as moduli spaces of elliptic curves with level structure and sheds new light on the theory of complex multiplication.

Our study will focus mainly on the compact case and $K = \mathbb{Q}$. The algorithmic approach to Shimura curves which are nonmodular differs considerably from the modular case, mainly due to the absence of cusps and the lack of numerical information on abelian surfaces, in contrast to the abundant numerical data available for elliptic curves.

In order to develop an effective approach to Shimura curves, quadratic forms are important tools. An interplay mentioned by Poincaré between Fuchsian groups and

indefinite ternary quadratic forms has been especially valuable. Indeed, Poincaré used Fuchsian groups in his studies of indefinite ternary quadratic forms:

Envisageons une forme quadratique indéfinie F à coefficients entiers [...]. Considérons le groupe principal de F formé de toutes les substitutions à coefficients entiers qui n'altèrent pas cette forme. [...] au groupe principal de F correspondra un groupe fuchsien G , qui sera le groupe fuchsien principal de F .

H. Poincaré [Poi87]

In one of his early articles, Shimura also mentioned Poincaré's point of view:

[...] part of the paper is devoted to the theory of a certain type of automorphic functions of one variable known in the literature as functions belonging to indefinite ternary quadratic forms [Poi87, FK97]. They occur as moduli of abelian varieties of dimension 2 whose endomorphism rings are isomorphic to an order of an indefinite quaternion algebra.

G. Shimura [Shi59]

The study of complex multiplication points in Shimura curves leads to the study of families of binary quadratic forms with algebraic coefficients and to their classification by arithmetic Fuchsian groups. Once a fundamental domain is determined, each Shimura curve provides a reduction theory for a family of binary quadratic forms attached to it. In this regard, the theory parallels Gauss' theory on the classification of binary quadratic forms with integral coefficients by the action of the modular group $SL(2, \mathbb{Z})$.

Our presentation draws on a variety of papers and books on hyperbolic geometry, quadratic forms, and quaternion orders.

For the study of the action of Fuchsian groups on the Poincaré half-plane, we should mention the works of Ford [For51], Lehner [Leh64] and Siegel [Sie71], which contain results for isometric circles and their role in the construction of fundamental domains.

The main sources for the arithmetic of quaternion orders are still Eichler's articles [Eic37, Eic38, Eic55b, Eic55a]. Some of Eichler's results can be found in the text of Vignéras [Vig80]. For more general results on orders, we also quote Deuring [Deu68] and Reiner [Rei75]. We note the works of Hijikata [Hij74], in which local Eichler orders are taken into account, and also quote the works of Pizer [Piz73, Piz76a, Piz76b, Piz80], where connections between quadratic forms and the arithmetic of Eichler orders are considered, in the context of definite quaternion algebras.

Binary quadratic forms were studied by Gauss in *Disquisitiones Arithmeticae* [Gau01]. For higher numbers of variables, we have Ogg [Ogg69], Jones [Jon67], Serre [Ser73], and Lehman [Leh92], among many others. Mostly, the forms considered in these works have rational, or rational integer, coefficients. We also note that in the usual computational packages the commands dealing with quadratic forms are restricted to integral quadratic forms.

On relations between quadratic forms and quaternion algebras, we mention the classical work of Latimer [Lat37], where a partial correspondence was stated, and those of Brandt [Bra24, Bra28, Bra43], where a complete correspondence was presented. The works of Brzezinski ([Brz80, Brz82, Brz83, Brz90, Brz95])

focus on the algebraic study of orders; he uses a relationship between quadratic forms and orders and generalizes some results known for Eichler orders.

Next we describe the contents of the monograph, summarizing the contributions contained in each chapter.

Chapter 1 is devoted to general facts on quaternion algebras and their orders. We give the simplest examples of them. To establish notation, the concepts are reviewed. We define what is meant by *nonramified* or by *small ramified* quaternion \mathbb{Q} -algebras and prove a classification theorem for them. We introduce the concepts of small ramified algebras of *type A* and of *type B*, in accordance with the classification theorem. These algebras will play a special role throughout the whole monograph, since they will provide the context in which our results will be made explicit. Section 1.2 is devoted to the arithmetic theory of quaternion orders. We derive some basic properties of Eichler orders $\mathcal{O}(D, N)$, of level N in a quaternion \mathbb{Q} -algebra of discriminant D . We provide tables of quaternion algebras and tables of Eichler orders for small ramified algebras of type A or B.

In Chapter 2, Shimura curves $X(D, N)$ are formally introduced. Section 2.1 contains a brief description of concepts relative to hyperbolic geometry in the Poincaré half-plane which are needed in the sequel. In Section 2.3, the Fuchsian groups $\Gamma(D, N)$ are introduced. We make them explicit for the small ramified algebras of type A or B. The tables contain constants for Shimura curves $X(D, N)$. For sake of completeness, we also provide tables of known equations of nonmodular Shimura curves, and the list of all hyperelliptic nonmodular Shimura curves, which was obtained by Ogg [Ogg83].

Chapter 3 presents a basic approach to ternary and quaternary quadratic forms attached to quaternion algebras or to their orders. The quaternary and ternary normic forms $n_{H,4}$ and $n_{H,3}$ attached to H turned out to be K-forms, a concept which goes back to Brandt. Section 3.6 is devoted to the study of quadratic forms associated to quaternion orders \mathcal{O} , in particular the quaternary and ternary normic forms, denoted by $n_{\mathcal{O},4}$ and $n_{\mathcal{O},3}$. We specify relationships between invariants associated to the order and to the quadratic forms, and characterize those normic forms which are K-forms. Explicit criteria are provided in the case of Eichler orders.

In Chapter 4, we reformulate Eichler's theory of optimal embeddings of quadratic orders into quaternion orders in terms of quadratic forms. This point of view has been particularly important for performing effective computations. In Section 4.4, we define the set $\mathcal{H}(\mathcal{O})$ of binary forms associated to an order and give explicit results. In Section 4.5, we obtain bijective mappings between the set of embeddings of orders into orders and families of binary quadratic forms. These forms turn out to have semi-integer quadratic coefficients. Specifically, we make explicit the binary forms which correspond to the nonramified case, and to small ramified cases of type A or B.

In Chapter 5, we construct hyperbolic polygons that are fundamental domains for Shimura curves in the nonramified and small ramified cases. Section 5.1 describes certain properties of fundamental domains for some modular curves of prime level. In Section 5.2, we characterize quaternion transformations using results for embeddings and quadratic forms from Chapter 4. In particular, in Section 5.3 we study the hyperbolic transformations fixing infinity and some symmetries, analogous to those of the modular case. We show the existence of a principal homothety

which replaces the usual translation of the nonramified case. In Section 5.5, we show the graphic representations of explicit fundamental domains for the Shimura curves $X(6, 1)$, $X(10, 1)$, and $X(15, 1)$. We include tables which contain cycles and presentations of the Fuchsian groups involved.

In Chapter 6, we study complex multiplication points of the Shimura curves $X(D, N)$ by using results on the hyperbolic uniformization and the interplay between embeddings and quadratic forms from previous chapters. We compile the graphic results in Sections 6.4 and 6.5. The commands implemented allow us to classify quadratic orders for which a given curve $X(D, N)$ has special complex multiplication points, to obtain the quadratic orders involved, and to compute the complex multiplication points for a fixed quadratic order.

In Chapter 7, we present the instructions for the `Poincare` package. We describe the technical characteristics and we provide a full index of the implemented instructions.

The `Poincare` package has been constructed in conjunction with the results presented throughout this work. It has contributed to the evolution of the results, giving support to the computations, and it has been improved constantly by the theoretical results. It is a good tool for dealing with the arithmetic of the quaternion orders and with the computation of hyperbolic domains for Shimura curves. It allows us to handle quaternions, quaternion orders, quadratic forms, embeddings of orders, points of Shimura curves, among others.

Tables in this monograph are collected in Appendix A. In Appendix B we present a survey of further results on Shimura curves. In Appendix C we also include a list of applications of these curves to the study of outstanding problems of number theory.

In general, at the beginning of each chapter or section, we give references for the known results for which the proofs are omitted.

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