CHAPTER 1

Introduction

1.1. An overview

Stein manifolds. A *Stein manifold* is a properly embedded complex submanifold of some $\mathbb{C}^N$. As we show in this book, Stein manifolds have symplectic geometry built into them, which is responsible for many phenomena in complex geometry and analysis. The goal of this book is a systematic exploration of this symplectic geometry (the "road from Stein to Weinstein") and its applications in the complex geometric world of Stein manifolds (the "road from Weinstein to Stein").

Stein manifolds are necessarily noncompact, and properly embedded complex submanifolds of Stein manifolds are again Stein. Stein manifolds arise, e.g., from closed complex projective manifolds $X \subset \mathbb{C}P^N$: if $H \subset \mathbb{C}P^N$ is any hyperplane, then the affine algebraic manifold $X \setminus H$ is Stein. Using this construction, it is not hard to see that every closed Riemann surface with at least one point removed is Stein. In fact, as we will see below, any open Riemann surface is Stein. Already this example shows that the class of Stein manifolds is much larger than the class of affine algebraic manifolds.

Stein manifolds can also be described intrinsically. The characterization most relevant for us is due to Grauert [**77**]. Let (V, J) be a complex manifold, where J denotes the complex multiplication on tangent spaces. To a smooth real-valued function $\phi : V \to \mathbb{R}$ we can associate the 1-form $d^{\mathbb{C}}\phi := d\phi \circ J$ and the 2-form $\omega_\phi := -dd^{\mathbb{C}}\phi$. The function is called *(strictly) plurisubharmonic* or, as we prefer to say, *J-convex* if $g_\phi(v, w) := \omega_\phi(v, Jw)$ defines a Riemannian metric. Since g_ϕ is symmetric, this is equivalent to saying that ω_ϕ is a symplectic (i.e., closed and nondegenerate) form compatible with J, i.e., $H_\phi = g_\phi - i\omega_\phi$ is a Hermitian metric. A function $\phi : V \to \mathbb{R}$ is called *exhausting* if it is proper (i.e., preimages of compact sets are compact) and bounded from below.

Since the function $\phi_{\mathrm{st}}(z) := |z|^2$ on $\mathbb{C}^N$ is exhausting and i-convex with respect to the *standard complex structure i* on $\mathbb{C}^N$, every Stein manifold admits an exhausting J-convex function (namely the restriction of ϕ_{st} to V). A combination of theorems of Grauert [**77**] and Bishop and Narasimhan [**18, 144**] asserts that the converse is also true: *a complex manifold which admits an exhausting J-convex function is Stein.*

Note that the space of exhausting J-convex functions on a given Stein manifold (V, J) is convex, and hence contractible. It is also open in $C^2(V)$, so a generic J-convex function is a *Morse function* (i.e., it has only nondegenerate critical points) and a generic path of J-convex functions consists of Morse and *generalized Morse* functions, i.e., functions with only nondegenerate and birth-death type critical points.

Weinstein manifolds. A *Weinstein structure* on a $2n$-dimensional manifold V is a triple (ω, X, ϕ), where ω is a symplectic form, $\phi : V \to \mathbb{R}$ is an exhausting generalized Morse function, and X is a complete Liouville vector field which is gradient-like for ϕ. Here the Liouville condition means that the Lie derivative $L_X\omega$ coincides with ω. The quadruple (V, ω, X, ϕ) is called a *Weinstein manifold*. We will see that homotopic (for an appropriate definition of homotopy, see Section 11.6) Weinstein manifolds are symplectomorphic. This structure was introduced in a slightly different form by A. Weinstein in [**187**] and then formalized in [**49**]. Since then it has become a central object of study in symplectic topology, see e.g. [**32, 169, 24**].

As it was explained above, after fixing an exhausting J-convex generalized Morse function $\phi : V \to \mathbb{R}$ on a Stein manifold (V, J) one can associate with the triple (V, J, ϕ) the symplectic form ω_ϕ. It turns out that the gradient vector field $X_\phi := \nabla_{g_\phi}\phi$ of ϕ, computed with respect to the metric g_ϕ which it generates, is Liouville with respect to the form ω_ϕ. After composing ϕ with a suitable function $\mathbb{R} \to \mathbb{R}$ we may further assume that the vector field X_ϕ is complete. Then the assignment

$$(J, \phi) \mapsto \mathfrak{W}(J, \phi) := (\omega_\phi, X_\phi, \phi)$$

yields a canonical map from Stein to Weinstein structures. A different choice of exhausting J-convex generalized Morse function leads to a homotopic, and hence symplectomorphic, Weinstein manifold. Note that this map forgets the most rigid datum, the integrable complex structure J. A major theme of this book is the reconstruction of Stein structures from Weinstein structures (the "road from Weinstein to Stein").

From Weinstein to Stein. We say that two functions $\phi, \phi' : V \to \mathbb{R}$ are *target equivalent* if there exists an increasing diffeomorphism $g : \mathbb{R} \to \mathbb{R}$ such that $\phi' = g \circ \phi$. In the following theorem we always have to allow for *target reparametrizations*, i.e., replacing functions by target equivalent ones, but we suppress this trivial operation from the notation.

THEOREM 1.1. *(a) Given a Weinstein structure $\mathfrak{W} = (\omega, X, \phi)$ on V, there exists a Stein structure (J, ϕ) on V such that $\mathfrak{W}(J, \phi)$ is Weinstein homotopic to $\mathfrak{W}$ with fixed function ϕ.*

(b) Given a Weinstein homotopy $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0, 1]$, on V beginning with $\mathfrak{W}_0 = \mathfrak{W}(J, \phi)$, there exists a Stein homotopy (J_t, ϕ_t) starting at $(J_0, \phi_0) = (J, \phi)$ such that the paths $\mathfrak{W}(J_t, \phi_t)$ and $\mathfrak{W}_t$ are homotopic with fixed functions ϕ_t and fixed at $t = 0$. Moreover, there exists a diffeotopy $h_t : V \to V$ with $h_0 = \mathrm{Id}$ such that $h_t^* J_t = J$ for all $t \in [0, 1]$.

(c) Given a Weinstein homotopy $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0, 1]$, on V connecting $\mathfrak{W}_0 = \mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}_1 = \mathfrak{W}(J_1, \phi_1)$ with $\phi_t = \phi_1$ for $t \in [\frac{1}{2}, 1]$, there exists a Stein homotopy (J_t, ϕ_t) connecting (J_0, ϕ_0) and (J_1, ϕ_1) such that the paths $\mathfrak{W}(J_t, \phi_t)$ and $\mathfrak{W}_t$ are homotopic with fixed functions ϕ_t and fixed at $t = 0, 1$.

Theorem 1.1 fits in the following more global, partially conjectural picture. To avoid discussing subtleties concerning the appropriate topologies on the spaces of Stein and Weinstein structures, we restrict our attention here to the compact case. Let W be a compact smooth manifold with boundary. In the following discussion we always assume that all considered functions on W have ∂W as their regular level set. A *Stein domain* structure on W is a pair (J, ϕ), where J is a complex

structure and $\phi : W \to \mathbb{R}$ is a J-convex generalized Morse function. A *Weinstein domain* structure on W is a triple (ω, X, ϕ) consisting of a symplectic form on W, a generalized Morse function $\phi : W \to \mathbb{R}$, and a Liouville vector field X which is gradient-like for ϕ. Let us denote by $\mathfrak{Stein}$ and $\mathfrak{Weinstein}$ the spaces of Stein and Weinstein domain structures on W, respectively. Let $\mathfrak{Morse}$ be the space of generalized Morse functions on W.

We have the following commutative diagram

$$
\begin{array}{ccc}
\mathfrak{Stein} & \xrightarrow{\;\;\mathfrak{W}\;\;} & \mathfrak{Weinstein} \\
& \searrow{\scriptstyle \pi_{\mathfrak{S}}} \quad \swarrow{\scriptstyle \pi_{\mathfrak{W}}} & \\
& \mathfrak{Morse} &
\end{array}
$$

where $\mathfrak{W}(J,\phi) = (\omega_\phi, X_\phi, \phi)$ as above, $\pi_{\mathfrak{W}}(\omega, X, \phi) := \phi$, and $\pi_{\mathfrak{S}}(J,\phi) := \phi$. Consider the fibers $\mathfrak{Stein}(\phi) := \pi_{\mathfrak{S}}^{-1}(\phi)$ and $\mathfrak{Weinstein}(\phi) := \pi_{\mathfrak{W}}^{-1}(\phi)$ of the projections $\pi_{\mathfrak{S}}$ and $\pi_{\mathfrak{W}}$ over $\phi \in \mathfrak{Morse}$.

THEOREM 1.2. *The map* $\mathfrak{W}_\phi := \mathfrak{W}|_{\mathfrak{Stein}(\phi)} : \mathfrak{Stein}(\phi) \to \mathfrak{Weinstein}(\phi)$ *is a weak homotopy equivalence.*

Note that (a compact version of) Theorem 1.1(a) is equivalent to the fact that the map $\mathfrak{W}_\phi$ induces an epimorphism on π_0, while Theorem 1.1(c) implies that the induced homomorphism is injective on π_0 and surjective on π_1. Conversely, it is easy to see that Theorem 1.1(b) and injectivity of $\mathfrak{W}_\phi$ on π_0 imply Theorem 1.1(c).

To put Theorem 1.1(b) into a more global framework, let us denote by $\mathcal{D}$ the identity component of the diffeomorphism group of W. Fix a Stein domain structure (J, ϕ_0) on W (the function ϕ_0 will play no role in what follows; the only important fact is that it exists). For a function $\phi \in \mathfrak{Morse}$, we introduce the spaces

$$\mathcal{D}_J(\phi) := \{h \in \mathcal{D} \mid \phi \text{ is } h^*J\text{-convex}\},$$

$$\mathcal{P}_J(\phi) := \{(h, \gamma) \mid h \in \mathcal{D}_J(\phi),\ \gamma : [0,1] \to \mathfrak{Weinstein}(\phi),\ \gamma(0) = \mathfrak{W}(h^*J, \phi)\},$$

$$\mathcal{P}_J := \bigcup_{\phi \in \mathfrak{Morse}} \mathcal{P}_J(\phi).$$

We denote by $\mathfrak{Weinstein}_J$ the connected component of $\mathfrak{W}(J, \phi_0)$ in $\mathfrak{Weinstein}$ (for any choice of ϕ_0, the component is independent of this choice).

CONJECTURE 1.3. *The projection* $\pi_{\mathcal{P}} : \mathcal{P}_J \to \mathfrak{Weinstein}_J$, $(h, \gamma) \mapsto \gamma(1)$ *is a Serre fibration.*

Note that (a compact version of) Theorem 1.1(b) is just the homotopy lifting property of $\pi_{\mathcal{P}}$ for homotopies of *points*, so it is a special case of Conjecture 1.3. We believe that this conjecture can be proven by further developing techniques discussed in this book. By an easy topological argument (see Appendix A.1), Conjecture 1.3 combined with Theorem 1.2 would imply

CONJECTURE 1.4. *The map* $\mathfrak{W} : \mathfrak{Stein} \to \mathfrak{Weinstein}$ *is a weak homotopy equivalence.*

Let us emphasize that in this book we are interested in the classification of Stein structures up deformation, and not up to biholomorphism. The classification of Stein complex structures up to biholomorphism is very subtle. For example, C^∞-small deformations of the round ball in $\mathbb{C}^n$, $n \geq 2$, give rise to uncountably

many pairwise non-biholomorphic Stein manifolds; see e.g. [**116**] for an exposition of this beautiful subject.

Existence of Stein structures.   Theorem 1.1 reduces complex-geometric questions about Stein manifolds to symplecto-geometric questions about Weinstein manifolds. Our next task is to develop techniques for answering those questions.

Let us first analyze necessary conditions for the existence of a Weinstein (or Stein) structure on a given smooth manifold V of real dimension $2n$. Clearly, one necessary condition is the existence of an *almost complex structure J*, i.e., an endomorphism of the tangent bundle with $J^2 = -\mathrm{Id}$. A second necessary condition arises from the following property of Morse functions with gradient-like Liouville fields (see Chapter 2): their Morse indices are $\leq n$. By Morse theory, this implies that V has a handlebody decomposition with handles of index at most n. This observation of Milnor [**139**] was the result of a long development, beginning with Lefschetz [**121**] and followed by Serre [**170**] and Andreotti and Frankel [**7**].

It turns out that for $\dim_{\mathbb{R}} V \neq 4$, these two conditions are sufficient for the existence of a Weinstein structure on V, so in combination with Theorem 1.1(a) we get the following existence theorem which was proved in [**42**]:

THEOREM 1.5 (Existence of Stein structures). *Let (V, J) be an almost complex manifold of dimension $2n \neq 4$ and $\phi : V \to \mathbb{R}$ an exhausting Morse function without critical points of index $> n$. Then there exists an integrable complex structure $\widetilde{J}$ on V homotopic to J for which the function ϕ is target equivalent to a $\widetilde{J}$-convex function. In particular, $(V, \widetilde{J})$ is Stein.*

We prove in this book several refinements and extensions of this result, some of which are due to Gompf [**71, 72, 73**] and Forstnerič and Slapar [**63**].

Theorem 1.5 settles the existence question for Stein structures in dimensions $\neq 4$. In dimension 4 the situation is drastically different. For instance, Lisca and Matič proved in [**125**] that $S^2 \times \mathbb{R}^2$ does not admit any Stein complex structure. On the other hand, Gompf proved the following topological analogue of Theorem 1.5:

THEOREM 1.6 (Gompf [**70**]). *Let V be an oriented open topological 4-manifold which admits a (possibly infinite) handlebody decomposition without handles of index > 2. Then V is homeomorphic to a Stein surface (i.e., a Stein manifold of complex dimension 2). Moreover, any homotopy class of almost complex structures on V is induced by an orientation preserving homeomorphism from a Stein surface.*

Let us point out that the Stein surfaces in Theorem 1.6 are usually not of finite type, where a Stein manifold is said to be of *finite type* if it admits an exhausting J-convex function with only finitely many critical points. Gompf's result which uses the technique of Casson handles, as well as Lisca and Matič's theorem which uses Seiberg–Witten theory, are beyond the scope of this book. See however Chapter 16 for some related discussion. For example, we prove that $S^2 \times \mathbb{R}^2$ is not homeomorphic to any Stein surface of finite type.

Deformations of Stein structures.   It turns out that the Weinstein problems in parts (b) and (c) of Theorem 1.1 cannot be reduced, in general, to differential topology even when $\dim V > 4$. On the contrary, they are tightly related to the core problems of symplectic topology.

It is easy to see that a Weinstein structure (ω, X, ϕ) on $\mathbb{R}^{2n}$ for which ϕ has no other critical points besides the minimum is symplectomorphic to the standard

structure on $\mathbb{R}^{2n}$. On the other hand, as we already pointed out, homotopic Weinstein structures are symplectomorphic. Seidel and Smith [167], McLean [137], and Abouzaid and Seidel [3] have recently constructed for each $n \geq 3$ infinitely many "exotic" Weinstein structures on $\mathbb{R}^{2n}$ which are not symplectomorphic to the standard one and which, moreover, are pairwise non-symplectomorphic. Then Theorem 1.1(a) allows us to transform these Weinstein structures to Stein structures which are not Stein homotopic among each other and to $\mathbb{C}^n$; in particular they do not admit exhausting J-convex functions without critical points of positive index.

One can also reformulate this result as a failure of the following "J-convex h-cobordism problem": Let W be a smooth cobordism between manifolds $\partial_- W$ and $\partial_+ W$. A *Stein structure* on W is a complex structure J on W which admits a J-convex function $\phi : W \to \mathbb{R}$ which has $\partial_\pm W$ as its regular level sets. Then the above results by Abouzaid, McLean, Seidel and Smith imply that *for each $n \geq 3$ there exists a Stein cobordism (W, J) diffeomorphic to $S^{2n-1} \times [0, 1]$ for which the corresponding J-convex function ϕ cannot be chosen without critical points.*

By contrast, we prove the following uniqueness theorem in complex dimension two (first sketched in [47]; for the diffeomorphism part see [83, 43, 133]).

THEOREM 1.7. *Let (W, J) be a minimal compact complex surface with J-convex boundary diffeomorphic to S^3. Suppose that there exists a symplectic form ω taming J, i.e., such that ω is positive on complex directions. Then W is diffeomorphic to the 4-ball and admits a J-convex Morse function $\phi : W \to \mathbb{R}$ which is constant on ∂W and has no other critical points besides the minimum.*

Here a *complex surface* (i.e., a complex manifold of complex dimension 2) is called *minimal* if it contains no embedded holomorphic spheres of self-intersection -1. See Chapter 16 for a discussion of related uniqueness results due to McDuff, Hind, Wendl, and others.

It turns out that the Weinstein problems can be reduced to differential topology in the *subcritical* case when the Morse functions have no middle-dimensional critical points. Moreover, based on work of Murphy [143], who discovered that in contact manifolds of dimension > 3 there is a class of Legendrian knots which obey an h-principle, we define a larger class of *flexible* Weinstein manifolds for which problems of symplectic topology can be reduced to differential topology. For example, we have

THEOREM 1.8. *Let V be a manifold of dimension $2n \neq 4$, Ω a homotopy class of nondegenerate (not necessarily closed) 2-forms on V, and $\phi : V \to \mathbb{R}$ an exhausting Morse function without critical points of index $> n$. Then:*
(a) There exists a flexible Weinstein structure (ω, X, ϕ) on V with $\omega \in \Omega$, and this structure is unique up to Weinstein homotopy.
(b) Every diffeomorphism of V preserving the homotopy class Ω is diffeotopic to a symplectomorphism of (V, ω).

Another application of the flexible technique is

THEOREM 1.9. *Let (V, J) be a contractible Stein manifold. Then $V \times \mathbb{C}$ admits an exhausting J-convex Morse function with exactly one critical point, the minimum.*

For our final application, recall that the *pseudo-isotopy problem* in differentiable topology concerns the topology of the space $\mathcal{E}(M)$ of functions on $M \times [0, 1]$ without

critical points that are constant on $M \times 0$ and $M \times 1$. Work of Cerf, Hatcher and Wagoner, Igusa, and Waldhausen has led to a description of $\pi_0 \mathcal{E}(M)$ for $\dim M \geq 7$ in terms of algebraic K-theory.

Given a topologically trivial Stein cobordism $(M \times [0,1], J)$, one can ask about the topology of the space $\mathcal{E}(M \times [0,1], J)$ of J-convex functions without critical points that are constant on $M \times 0$ and $M \times 1$ (provided that this space is nonempty). Understanding of the topology of the inclusion map $\mathcal{I} : \mathcal{E}(M \times [0,1], J) \to \mathcal{E}(M)$ is the content of the J-convex pseudo-isotopy problem. We prove the following result in this direction.

THEOREM 1.10. *If* $\dim M > 3$ *and the Stein structure* J *is flexible, the homomorphism* $\mathcal{I}_* : \pi_0 \mathcal{E}(M \times [0,1], J) \to \pi_0 \mathcal{E}(M)$ *is surjective.*

We conjecture that $\mathcal{I}_*$ is an isomorphism.

1.2. Plan of the book

This book is organized as follows.

In Chapters 2 and 3 we explore basic properties and examples of J-convex functions and hypersurfaces. In particular, we prove Richberg's theorem on smoothing of J-convex functions and derive several important corollaries.

In Chapter 4 we construct special hypersurfaces that play a crucial role in extending J-convex functions over handles.

The next two chapters contain background material which is standard but sometimes not easy to find in the literature. The necessary complex analytic background is discussed in Chapter 5, and the symplecto-geometric one in Chapter 6.

In Chapter 7 we review several h-principles that we use in this book. We begin with a review of the Smale–Hirsch immersion theory and Whitney's theory of embeddings. We then discuss Gromov's results about symplectic and contact isotropic immersions and embeddings, Murphy's h-principle for loose Legendrian knots, and Gromov's theory of directed embeddings and immersions with applications to totally real embeddings. We finish this chapter with an h-principle for totally real discs with Legendrian boundaries, which we deduce from previously discussed h-principles and which plays an important role in the proofs of the main results of this book.

Theorem 1.5 is proved in Chapter 8. This chapter also contains several new results concerning surrounding of subsets by J-convex hypersurfaces, with applications to holomorphic and polynomial convexity. We also prove here several refinements of Theorem 1.5, some of which are due to Gompf, and Forstnerič and Slapar.

In Chapter 9 we review Morse–Smale theory and the h-cobordism theorem. In particular, we discuss basic facts concerning gradient-like vector fields. We also review the "two-index theorem" of Hatcher and Wagoner and basic notions of pseudo-isotopy theory.

In Chapter 10 we develop a Morse–Smale type theory for J-convex functions. In particular, we show how the Morse-theoretic operations which are used in the proof of the h-cobordism theorem—reordering of critical points, handle-slides, and cancellation of critical points—can be performed in the class of J-convex functions.

In Chapter 11 we introduce Weinstein structures and study their basic properties. We discuss Stein and Weinstein homotopies, and we introduce the classes of

subcritical and flexible manifolds which play an important role for the "road from Morse to Weinstein".

In Chapter 12 we discuss modifications of Weinstein structures near critical points and stable manifolds and prove Weinstein analogues of the results proven in Chapter 10 for J-convex functions.

In Chapter 13 we prove a more precise version of Theorem 1.5 by first constructing a Weinstein structure and then proving Theorem 1.1(a).

Chapters 14 and 15 contain our main results about deformations of Weinstein and Stein structures. In Chapter 14 we classify flexible Weinstein structures up to homotopy and show that the problem of simplification of the Morse function corresponding to a flexible Weinstein structure can be reduced to Morse–Smale theory. In particular, we prove Theorem 1.8.

In Chapter 15 we show that every Weinstein homotopy can be transformed to a Stein homotopy. In particular, we prove Theorem 1.1(b) and (c) and deduce various corollaries, including Theorems 1.9 and 1.10.

Chapter 16 concerns the situation in complex dimension 2. In particular, we discuss the method of filling by holomorphic discs and prove Theorem 1.7. We also discuss the classification of Stein fillings of 3-dimensional contact manifolds and review known results about Stein surfaces.

Finally, in Chapter 17 we sketch McLean's construction of exotic Stein structures in higher dimension and explain how they are distinguished by symplectic homology.

Notation. Throughout this book we use the following notation. For a subset $A \subset X$ of a topological space we denote by Int A and $\bar{A}$ its interior resp. closure, and $A \Subset B$ means that $\bar{A}$ is a compact subset of Int B. For A closed we denote by $\mathcal{O}p\, A$ a sufficiently small (*but not specified*) open neighborhood of A.

Manifolds are always assumed to be smooth and second countable.

APPENDIX C

Biographical Notes on the Main Characters

In this appendix we sketch biographies of the mathematicians whose work is most relevant to the content of this book. We have grouped them according to their fields (complex analysis, respectively, differential and symplectic topology), and put them in chronological order within each field. The following sources were used in preparation: the internet site Wikipedia; several articles by J. O'Connor and E. Robertson under http://www-history.mcs.st-andrews.ac.uk/Biographies; the article by L. Dell'Aglio on E. Levi under http://www.treccani.it/enciclopedia/eugenio-elia-levi_(Dizionario-Biografico)/ (translated from Italian by A. Gnoatto); the articles [**106**] and [**105**] by A. Huckleberry on K. Stein and H. Grauert; the article [**22**] by R. Bott on M. Morse; the article by J. Zund under http://www.anb.org/articles/13/13-02523.html, and the interview [**193**] with H. Whitney; the book [**14**] by S. Batterson on S. Smale; and the preface to the book [**127**] by J. Marsden and T. Ratiu on A. Weinstein.

C.1. Complex analysis

Friedrich Hartogs (May 20, 1874 – August 18, 1943). Friedrich Hartogs was born in Brussels, Belgium, into the family of a German businessman. Hartogs' family were German Jews and he was brought up in Frankfurt am Main, Germany. He attended the Realgymnasium Wöhlerschule in Frankfurt, graduating from high school in the spring of 1892.

At that time the standard university career for German students involved moving between different institutions and Hartogs followed this route. First he spent a semester at the Technical College at Hannover, followed by a semester at the Technical College in Berlin. He then matriculated at the University of Berlin where he was taught mathematics by, among others, Georg Frobenius, Lazarus Fuchs, and Hermann Schwarz, and he attended physics lectures by Max Planck. Following his studies at the University of Berlin, he went to the University of Munich where he attended courses by Ferdinand von Lindemann and Alfred Pringsheim. In 1901 Pringsheim became a full professor at Munich and he became Hartogs' thesis advisor. In 1903 Hartogs was awarded his doctorate from Ludwig-Maximilians-Universität in Munich, and two years later he received his habilitation.

After that Hartogs became a privatdozent at the University of Munich. In 1909–10 he taught Abraham Fraenkel who, years later, wrote in his memoirs that Hartogs was by nature a consistently shy and rather anxious person. Perhaps for this reason he was promoted only slowly when the outstanding quality of his research would suggest that he might have risen more rapidly through his profession. He became an extraordinary professor in 1912, then ten years later was offered a full professorship at the University of Frankfurt. Hartogs was indeed a very cautious person and he turned down the offer of this chair because, in the difficult financial

climate of the time with hyperinflation gripping Germany, he did not feel confident that a privately owned institution, which the University of Frankfurt was, offered the security that he required to support his wife and four children.

In Munich, Hartogs had several outstanding colleagues such as Oskar Perron, Constantin Carathéodory, and Heinrich Tietze. These three professors all made representations to the university arguing that Hartogs should be appointed to a full professorship, and in 1927, five years after turning down the full professorship at Frankfurt, he had at last reached the top of his profession in Munich. Like all Jewish academics, after the Nazi Party came to power in 1933, Hartogs' life became increasingly difficult. In October 1935 he was forced to retire from his professorship, and on 10 November 1938, during the infamous "Kristallnacht", Hartogs was one of those arrested and taken to the Dachau concentration camp. After being held for several weeks during which he was appallingly treated he was, nevertheless, released.

Hartogs' wife was not Jewish, and in 1941 Hartogs and his wife were given advice by a lawyer that in order to protect Hartogs' wife she should divorce him. This was a painful process for Hartogs and the process was deliberately drawn out to be as lengthy as possible. In early 1943 the divorce was finalized but Hartogs continued to live in his wife's house and the authorities turned a blind eye. The indignity and humiliation that Hartogs had suffered for ten years finally became too much for him, and in August 1943 he took his own life.

Hartogs is best known for his discovery of the *Hartogs phenomenon*, contained in his habilitation thesis, that compact singularities of holomorphic functions in $n > 1$ complex variables are always removable (see Section 5.4). This result is in striking contrast to the case of one variable, and marks the beginning of the theory of functions of several complex variables.

Eugenio Elia Levi (October 18, 1883 – October 28, 1917). Eugenio Elia Levi was born in Torino, Italy. His older brother Beppo Levi was also a well-known mathematician. Eugenio Levi graduated in mathematics from the Scuola Normale di Pisa in 1905. From 1906 to 1909 he was assistant of Ulisse Dini in Pisa, then he moved to the University of Genova where he became full professor in 1912. Eugenio Levi was killed in World War I on October 28, 1917, in Cormons, Italy, on the border with today's Slovenia.

In his short life Eugenio Levi wrote 33 papers making fundamental contributions to group theory, the theory of partial differential operators, and the theory of functions of several complex variables. In his work in group theory he discovered what is now called the Levi decomposition, which was conjectured by Wilhelm Killing and proved by Élie Cartan in a special case. In the theory of partial differential operators he discovered the method of the parametrix, which is a way to construct fundamental solutions for elliptic partial differential operators with variable coefficients. The parametrix method is widely used in the theory of pseudo-differential operators.

In the theory of functions of several complex variables Eugenio Levi introduced the *Levi form* and the concept of *(Levi) pseudoconvexity* (called *J-convexity* in this book), which turned out to be one of the key concepts in the theory of functions of several complex variables. The question of whether a bounded domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary is a domain of holomorphy became known as the *Levi problem* and was one of the main driving forces for the development of complex

analysis in the first half of the twentieth century. It was only solved in the 1950s by Oka, Bremermann, and Norguet.

Kiyoshi Oka (April 19, 1901 – March 1, 1978). Kiyoshi Oka entered the Imperial University of Kyoto in 1922 to study physics. However, in 1923 he changed subjects to study mathematics, graduating with a degree in mathematics in 1925. In the same year he was appointed as a lecturer in the Faculty of Science at the Imperial University of Kyoto, and in 1929 he was promoted to assistant professor. 1929 was a very significant year for Oka for in that year he took a sabbatical leave and went to the University of Paris, where he met Gaston Julia and became interested in unsolved problems in the theory of functions of several complex variables.

Oka remained on the staff at the Imperial University of Kyoto while he was on leave in Paris, but on his return to Japan in 1932 he accepted a position as assistant professor in the Faculty of Science of Hiroshima University. In 1938 Oka went to Kimitoge in Wakayama to study by himself, and in 1940 he presented his doctoral thesis to the University of Kyoto. After obtaining his doctorate and a short period 1941–42 as research assistant at Hokkaido University, Oka spent the next seven years again at Kimitoge, supported by a scholarship of the Huju-kai Foundation. In 1949, Oka was appointed professor at the Nara University for Women, a post he held until 1964. From 1969 until his death in 1978 he was a professor of mathematics at the Industrial University of Kyoto.

Oka's most famous work was published over the 25-year period 1936–1961, during which he solved a number of important problems in the theory of functions of several complex variables, such as the Cousin problems and the Levi problem. He proved important foundational results such as Oka's coherence theorem (Section 5.6) and the Oka–Weil theorem (Theorem 5.4). *Oka's principle* on holomorphic approximation of continuous sections, introduced by Oka in his work on the Cousin problems and later generalized by Grauert, provided an early example of an *h*-principle, and it marked one of the points of departure for Gromov's later work on this subject. In the introduction to Oka's collected works [**155**], Henri Cartan describes the way that Oka came into the subject:

> The publication in 1934 of a monograph by Behnke-Thullen marked a crucial stage in the development of the theory of analytic functions of several complex variables. By giving a list of the open problems in the area, this work played an important role in deciding the direction of Oka's research. He set himself the almost super-human task of solving these difficult problems. One could say that he was successful, overcoming one after the other the obstacles he encountered on the way.

Henri Cartan (July 8, 1904 – August 13, 2008). Henri Cartan was born in Nancy, France, and grew up in Paris. His father, Élie Cartan, was a mathematician well known for his work on Lie groups. Henri had a sister and two younger brothers, Jean and Louis, who both died tragically. Jean, a composer, died of tuberculosis at the age of 25 while Louis, a physicist, was a member of the Resistance arrested by the Germans in 1942, deported to Germany in February 1943, and executed after 15 months in captivity.

Cartan studied at the École Normale Supérieure in Paris, where he met and became friends with André Weil who was one year ahead. It was on André Weil's suggestion that Cartan later began working on analytic functions of several complex variables. Among Cartan's teachers at the École Normale were Gaston Julia and his father Élie Cartan. He received his doctorate in 1928 under the supervision of Paul Montel. After positions at the Lycée Caen and the University of Lille, he took up a post at the University of Strasbourg in 1931. When World War II broke out in September 1939, the inhabitants of Strasbourg had to be evacuated and the university was displaced to Clermont-Ferrand. In November 1940 Cartan was appointed professor at the Sorbonne in Paris. He taught in Paris from that time until 1969 (with the exception of two years 1945–46 when he returned to the University of Strasbourg), and then at the Université de Paris-Sud in Orsay from 1970 to until his retirement in 1975.

At the École Normale Supérieure, Cartan started the Séminaire Cartan. Jean-Pierre Serre, who was one of Cartan's doctoral students, suggested that the seminars should be written up for publication and 15 ENS-Seminars written by Cartan were published between 1948 and 1964. These publications played a major role in shaping the modern theory of functions of several complex variables.

Cartan's most important contribution to mathematics is without doubt the introduction of sheaf-theoretical methods into complex analysis and his Theorems A and B for coherent analytic sheaves on Stein manifolds (see Section 5.6). These new techniques allowed him to treat many of the classical problems on several complex variables in a unified manner, thus moving the whole field into a new era. After Cartan had presented his Theorems A and B at the *Colloque sur les fonctions de plusieurs variables* in Brussels in 1953, the German participant Karl Stein commented, "Wir haben Pfeil und Bogen, die Franzosen haben Panzer."[1]

Cartan also made significant contributions to other areas of mathematics such as algebra and topology. His 1956 book *Homological Algebra* with Eilenberg is a classic text which has had a profound influence on the subject over half a century.

An important part of Cartan's mathematical life was taken up with Bourbaki. He was one of the founding members of this group in 1935 together with André Weil, Jean Dieudonné, Szolem Mandelbrojt, Claude Chevalley, René de Possel, and Jean Delsarte.

Cartan was also involved with politics and in particular supporting human rights. In 1974 the Russian authorities placed the mathematician Leonid Plyushch in a special psychiatric hospital. Andrei Sakharov pointed out that this was a political act and Cartan began a strenuous campaign for Plyushch's release. The International Congress of Mathematicians was held in Vancouver in 1974 and this presented an opportunity to gain wide international support for Plyushch with a thousand signatures to a petition for his release. After the Congress Cartan played a major role in setting up the Comité des Mathématiciens to support Plyushch and other dissident mathematicians. In January 1976 the Soviet authorities released Plyushch, which was a major success for Cartan and the Comité des Mathématiciens. But the Comité did not stop after this success. It has supported other mathematicians who have suffered for their political views, such as the Uruguayan mathematician José Luis Massera. For his outstanding work in assisting dissidents Cartan received the Pagels Award from the New York Academy of Sciences.

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[1]We have bows and arrows, the French have tanks.

Karl Stein (January 1, 1913 – October 19, 2000). Karl Stein was born in Hamm in Westfalen, Germany. He studied in Münster, where he received his doctorate under the supervision of H. Behnke in 1937. By that time, he had already been exposed to the fascinating developments in the area of complex analysis. The brilliant young Peter Thullen was proving fundamental theorems, Henri Cartan had visited Münster, and Behnke and Thullen had just written their classical book on the subject. The amazing phenomenon of analytic continuation in higher dimensions had already been exemplified more than 20 years before in the works of Hartogs and Levi, while the recent work of Thullen, Cartan, and Behnke had gone much further. It must have been clear to Stein that this was the way to go.

Even though the Third Reich was already invading academia, Behnke kept things going for as long as possible, but this phase of the Münster school of complex analysis could not go on forever. Although Stein was taken into the army, during a brief stay at home, he was able to prepare and submit the paper which contained the results from his Habilitationsarbeit which was accepted in 1940. At a certain point he was sent to the eastern front. Luckily, however, the authorities were informed of his mathematical abilities, and he was called back to Berlin to work until the end of the war in some form of cryptology.

Almost immediately after the war, in a setting of total destruction, Behnke began to rebuild his group, and very soon Stein became the mathematics guru in Münster. At the time there were only two professor positions in pure mathematics, those of Behnke and F. K. Schmidt. Although it must have been very difficult, Behnke somehow found a position for Stein which he held from 1946 to 1955.

In 1955 Stein took a chair of mathematics at the Ludwigs-Maximilians-Universität in Munich, a position he held until his retirement in 1981. There he continued his mathematics and built his own group in complex analysis, one of his best known students being Otto Forster.

Stein made important contributions to many areas of several complex variables. Until the early 1950s his main efforts were directed towards the Cousin problems. In his 1951 paper [**178**] on this subject, he pointed out that most of the results he considered were true under assumptions which now form the definition of a *Stein manifold*; see Section 5.3. The term "varieté de Stein" for these new spaces was introduced by H. Cartan at the *Colloque sur les fonctions de plusieurs variables in Brussels* in 1953. Stein manifolds and their generalizations, Stein spaces, continue to play a central role in complex analysis to this day.

Hans Grauert (February 8, 1930 – September 4, 2011). Hans Grauert was born in Haren-Ems in Niedersachsen (Lower Saxony) in the north of Germany close to the border with the Netherlands. He attended primary and middle school there from 1936 until the end of the war in 1945. He later recalled how he struggled with mathematics as a school boy until a teacher told him it was acceptable to think abstractly, he didn't necessarily need to deal with numbers.

In 1949 he graduated from the Gymnasium in Meppen, Germany, just 12 km from his home-town. He then studied at the University of Münster, where he was awarded his doctorate in 1954 after spending a year in 1953 at the ETH Zürich, where he was influenced by Beno Eckmann. His first paper "Métrique Kaehlérienne et domaines d'holomorphie" was published in French in 1954.

In September 1955 Grauert was appointed as an assistant at the University of Münster, submitting his habilitation thesis there in February 1957. His output of

published papers was quite remarkable, with 10 major papers published in 1956 and 1957. He spent the year 1957–58 at the Institute for Advanced Study in Princeton, then the spring semester of 1959 at the Institut des Hautes Études Scientifique in Bures-sur-Yvette, France.

In 1959 Grauert was appointed as an ordinary professor at the University of Göttingen to fill the chair which Carl Ludwig Siegel had occupied. He supervised there doctoral studies of 44 students, several of whom collaborated with him on major projects.

Grauert has been the leading mathematician in the theory of several complex variables in his generation. He not only solved several major problems but his work, along with the work of Henri Cartan, very much shaped the development of this field in the second half of 20th century. For example, the following results of Grauert play an important role in this book: Grauert's solution of the Levi problem for complex manifolds and his characterization of Stein manifolds in terms of J-convex functions (Sections 5.2 and 5.3); Grauert's Oka principle (Section 5.5); and his proof that complexifications of real analytic manifolds (*Grauert tubes*) are Stein (Section 5.7).

Grauert also wrote a large number of excellent textbooks, for example the classical books *Theory of Stein Spaces* (1979) and *Coherent Analytic Sheaves* (1984) with R. Remmert.

C.2. Differential and symplectic topology

Marston Morse (March 24, 1892 – June 22, 1977). Marston Morse was born in Waterville, Maine, USA. His mother was Ella Phoebe Marston and his father was Howard Calvin Morse, a farmer and real estate agent. The name "Marston" by which he wanted to be known was therefore his mother's maiden name and not a forename.

Morse received his B.A. from Colby College in Waterville in 1914, and his Ph.D from Harvard in 1917 for his thesis entitled "Certain types of geodesic motion on a surface of negative curvature" under the direction of G. D. Birkhoff. Morse taught briefly at Harvard before entering military service. For the duration of World War I he served as a private in the U.S. Army in France and for his outstanding work in the Ambulance Corps he was awarded the Croix de Guerre with Silver Star. After the war he resumed his academic career. After positions at Harvard (1919–20), Cornell (1920–25), Brown University (1925–26), and again Harvard (1926–35), he moved to the Institute for Advanced Study in Princeton where he remained until his retirement in 1962.

Morse was married twice and had four daughters and three sons.

In 1925 Morse published a paper entitled "Relations between the critical points of a real function of n independent variables" that would shape his mathematical life and that of generations of mathematicians to this day. In this paper he proves the famous *Morse inequalities* for *Morse functions* on a finite dimensional manifold, thus initiating what is now called *Morse theory* (see Chapter 9).

Realizing the power of this theory, Morse devoted a large part of his mathematical life to its extensions and applications. Almost from the beginning he also considered Morse theory on infinite dimensional spaces such as the loop space of a manifold. His groundbreaking work in this direction culminated in his famous book "The Calculus of Variations in the Large" from 1932, where he proved for

example the existence of infinitely many geodesics joining any two distinct points for an arbitrary Riemannian metric on a sphere.

Morse also developed topological versions of his theory for very general functions and found applications to other problems, such as the existence of minimal surfaces. Morse theory was not Morse's only contribution to mathematics — in all he wrote about 180 papers and eight books on a whole range of topics — but clearly the most influential one. It was the basis for many spectacular subsequent developments, from Smale's h-cobordism theorem and Bott's periodicity theorem to Floer homology in gauge theory and symplectic topology. Today, Morse theory is an indispensable tool in geometry and topology. Morse functions, and their J-convex analogues, are also the basic objects studied in this book.

Hassler Whitney (March 23, 1907 – May 10, 1989). Hassler Whitney was born in New York City, the son of Edward B. Whitney, a judge, and Josepha Newcomb. His grandfathers were the philologist William D. Whitney and the astronomer Simon Newcomb. Whitney received his first degree from Yale University in 1928, and his Ph.D. in mathematics from Harvard University in 1932 with the dissertation "The coloring of graphs" written under supervision of George D. Birkhoff. After spending the years 1931–33 as a National Research Council Fellow at Harvard and Princeton he returned to Harvard where he was successively promoted until he became full professor in 1946. From 1943 to 1945 he was a member of the Mathematics Panel of the National Defense Research Committee. In 1952 he joined the Institute for Advanced Study at Princeton, where he was professor of mathematics until his retirement in 1977.

Whitney was a keen mountaineer all his life. As an undergraduate in 1929, Whitney and his cousin Bradley Gilman made the first ascent of a 700-foot ridge in New Hampshire which is now known as the Whitney–Gilman ridge. Later climbing partners included the topologists James W. Alexander and Georges de Rham.

Whitney got married three times, the last time in 1986 at the age of 78, and had five children.

Whitney's work covers a wide range of subjects including graph theory, singularity theory, differential and algebraic topology, and geometric integration theory. In his work on graph theory in the early 1930s he made important contributions to the four color problem. In 1936 Whitney introduced the modern definition of a manifold of class C^r. In 1944 studied the self-intersection index of immersions of half dimension and proved the famous *Whitney embedding theorem* that any smooth manifold of dimension $n > 2$ can be embedded in R^{2n} (see Section 7.1). The *Whitney trick* used in this proof was the basis for much later work in differential topology, such as Smale's proof of the h-cobordism theorem. It also underlies all the flexibility results for Stein structures proved in this book.

In the late 1930s Whitney was one of the major developers of algebraic topology, in particular the theory of bundles and characteristic classes. The significance of his work is reflected in a large number of fundamental concepts that now carry his name such as the Whitney sum, Whitney product theorem, and Stiefel–Whitney classes.

In the 1950s Whitney studied the topology of singular spaces and singularities of maps. He introduced the notion of a Whitney stratification which became the basis for the modern theory of stratified spaces. His classification results for singularities of smooth maps (e.g., the Whitney umbrella) led to the new fields of singularity

theory and catastrophe theory. Whitney also did foundational work on analytic spaces, as a by-product of which he proved together with Bruhat that every real analytic manifold has a complexification (Theorem 5.41).

In the last two decades of his life Whitney became involved in mathematical education at elementary schools, vigorously opposing calls for more mathematics to be taught earlier in school.

Stephen Smale (born in 1930). Stephen Smale was born in Flint, Michigan, the home base of General Motors. From the age of five he lived on a farm while his father worked in the city for General Motors. Stephen attended an elementary school with only a single classroom about a mile from his farmhouse. In high school his favorite subject was chemistry. His interests had moved to physics by the time he entered the University of Michigan, Ann Arbor, in 1948, but after failing a physics course he turned to mathematics. He was awarded a BS in 1952 and an MS the following year. In 1957 Smale received his Ph.D. from the University of Michigan under the supervision of Raoul Bott. In his thesis he generalized a result proved by Whitney and Graustein in 1937 for curves in the plane to curves in arbitrary manifolds.

After postdoctoral years spent at the University of Chicago (1956–58), the Institute for Advanced Study in Princeton (1958–59), and the Instituto de Mathemática Pura e Aplicada (IMPA) in Rio de Janeiro, Smale was appointed an associate professor of mathematics at the University of California at Berkeley in 1960. After three years at Columbia University, New York, Smale returned in 1964 to a professorship at Berkeley where he remained until his retirement in 1995. After his retirement he took up a professor position at the City University of Hong Kong, a post he held until 2001 and again since 2009. Since 2002 he is also a professor at the Toyota Technological Institute in Chicago.

Smale's mathematical work is impressive both for its depth and its breadth. He made profound contributions to a whole range of subjects including differential topology, dynamical systems, mathematical economics, and theoretical computer science.

In the years after his Ph.D. Smale astounded the mathematical world with a number of breathtaking results in differential topology. In 1957 he found a general classification of immersions of spheres in Euclidean spaces (see Section 7.1), which implied as a special case that the standard 2-sphere in $\mathbb{R}^3$ can be turned inside out by immersions. His thesis advisor R. Bott at first did not believe this result because he could not picture such a *sphere eversion*, but Smale's proof withstood all scrutiny and was finally published in 1959. Only years later did mathematicians succeed in explicitly describing and visualizing a sphere eversion.

In 1961 Smale proved the generalized *Poincaré conjecture* in dimension > 4, followed in 1962 by the *h-cobordism theorem*. His proof, sketched in Section 9.8, is a beautiful application of Morse theory: beginning with an arbitrary Morse function, Smale successively removes critical points as far as the topology allows, crucially applying Whitney's trick in the process. The most startling aspect of these results was that differential topology suddenly looked *simpler* in higher dimensions than in dimensions 3 and 4. Indeed, in the decade following Smale's work many questions were settled for manifolds of higher dimensions (in a new field called *surgery theory*), while the corresponding questions in low dimensions either had negative answers (such as the existence of exotic smooth structures on $\mathbb{R}^4$), were only solved much

later (such as the 3-dimensional Poincaré conjecture), or still remain open (such as the 4-dimensional Poincaré conjecture). For his work on the generalized Poincaré conjecture Smale was awarded a Fields Medal at the International Congress of Mathematicians in Moscow in 1966.

In the 1960s Smale's main focus was the theory of dynamical systems where he introduced a number of new concepts such as his famous *horseshoe* and *Morse–Smale systems*, and proved foundational results such as his Ω-stability theorem. In the 1970s Smale applied his ideas on dynamical systems to questions in economics, and since the 1980s he has been mainly interested in theoretical computer science.

In the summer of 1965 Smale played an important role in the early protests against the Vietnam War in Berkeley. He was one of the main organizers of anti-war activities such as the Vietnam Day 1965, attempts to block trains transporting Vietnam troops, and a march to the Oakland Army Terminal. In early August 1966, the House Committee on Un-American Activities in Washington opened an investigation of radical anti-war protests by Smale and others. At that time Smale was in Europe on his way to Moscow for the Fields Medal Ceremony, which led to the following headline in the San Francisco Examiner on August 5, 1966: "UC Prof Dodges Subpoena, Skips U.S. for Moscow."

Mikhail Gromov (born in 1943). Mikhail Leonidovich (Misha) Gromov was born in Boksitogorsk, a town about 200 km east of St Petersburg (or Leningrad as it was then called). Misha did not speak until the war was over, but then began speaking with whole sentences. At the age of 6 he annoyed his first grade teacher by solving a problem given him by mistake and intended for the third graders. The teacher simply refused to believe that Misha solved it by himself. But when Misha was 10 years old the teacher told his mother that Misha will be a math professor, though at that time the future math professor found much more delight in playing with noxious chemicals.

From 1960 to 1969 Gromov studied at Leningrad University, receiving his masters degree in 1965 and the first doctoral ("candidate") degree in 1969 under the direction of V. A. Rokhlin, followed by his second doctoral degree in 1972. During his undergraduate years, he solved several open problems such as a problem of Banach on the characterization of Banach spaces all of whose k-dimensional subspaces are isometric. But his first major achievement was the far-reaching generalization in his Ph.D. dissertation of the Smale–Hirsch immersion theory, which laid the foundation for the area of mathematics that is now known under the name *h-principle* (see Chapter 7). Over the next four years he made several major advances in this theory, culminating in his theory of convex integration inspired by Nash and Kuiper's C^1-isometric embedding theorem.

The *h*-principle was the subject of Gromov's invited talk at the International Congress of Mathematicians 1970 in Nice (which he was not allowed to attend by the Soviet authorities). This was the first of a series of four invited ICM talks of Gromov, including two plenary addresses.

In 1974 Gromov left Russia and became a professor at the State University of New York in Stony Brook, USA. In 1981 Gromov moved to France and since that time has been a permanent member of the Institute des Hautes Études Scientifique in Bures-sur-Yvette. From 1991 until 1996 he also held a professor position at the University of Maryland, College Park, and since 1997 he has been a professor at New York University.

Gromov made revolutionary contributions to many branches of mathematics. His work transformed several classical areas and led to the creation of entirely new fields. In particular, his work shaped modern Riemannian geometry, and his introduction of new geometric methods into group theory led to the solution of many classical problems and the creation of the theory of hyperbolic groups. His fundamental paper on pseudo-holomorphic curves in symplectic manifolds essentially created the field of symplectic topology.

Alan Weinstein (born in 1943). Alan Weinstein was born in New York City. He received his undergraduate degree from the Massachusetts Institute of Technology, and his Ph.D. from University of California at Berkeley in 1967 under the direction of S.-S. Chern. After postdoctoral years at the Institute des Hautes Études Scientifique in Bures-sur-Yvette, MIT, and the University of Bonn, he joined the faculty at Berkeley in 1969, becoming full professor in 1976. On the occasion of Weinstein's 60th birthday his advisor S.-S. Chern wrote ([**127**]):

> Alan came to me in the early sixties as a graduate student at the University of California at Berkeley. At that time, a prevailing problem in our geometry group, and the geometry community at large, was whether on a Riemannian manifold the cut locus and the conjugate locus of a point can be disjoint. Alan immediately showed that this was possible. The result became a part of his PhD thesis, which was published in the *Annals of Mathematics*. He received his PhD degree in a short period of two years. I introduced him to IHES and the French mathematical community. He stays close with them and with the mathematical ideas of Charles Ehresmann. He is original and often came up with ingenious ideas. An example is his contribution to the solution of the Blaschke conjecture. I am very proud to count him as one of my students.

Weinstein became interested in symplectic geometry and its applications to mechanics already in the early years of his mathematical career. The Marsden–Weinstein reduction continues to play a fundamental role in classical and quantum mechanics and in the study of the geometry of moduli spaces. Weinstein did important work in the theory of periodic orbits of Hamitonian systems. The Weinstein conjecture about periodic orbits of Reeb vector fields, along with Arnold's fixed point conjectures, continues to be one of the driving forces in symplectic topology. Weinstein made fundamental contributions to Poisson geometry, such as the introduction of symplectic groupoids. Intertwined with his work on symplectic geometry and mechanics, Weinstein did extensive work on geometric partial differential equations, eigenvalues, the Schrödinger operator, and geometric quantization.

In [**187**] Weinstein introduced an object which in [**49**] was called a *Weinstein manifold*, and which is one of the main objects studied in this book.

Alan Weinstein is also an inspiring lecturer and a great teacher. Many of the 32 students who obtained a Ph.D. under his direction became themselves well-known mathematicians.